Electronic Journal of Differential Equations, Vol. 2006(2006), No. 25, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON SECOND ORDER PERIODIC BOUNDARY-VALUE PROBLEMS WITH UPPER AND LOWER SOLUTIONS IN THE REVERSED ORDER

HAIYIN GAO, SHIYOU WENG, DAQING JIANG, XUEZHANG HOU

Abstract. In this paper, we study the differential equation with the periodic boundary value

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{gathered}$$

The existence of solutions to the periodic boundary problem above with appropriate conditions is proved by using an upper and lower solution method.

1. Introduction and Main Results

In this paper, we study the second-order periodic boundary-value problem

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered} \tag{1.1}$$

where $f(t, u, v)$ is a Caratheodory function. A function $f : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a Carathrodary function if it possess the following three properties:

(i) For all $(u, v) \in \mathbb{R}^2$, the mapping $t \to f(t, u, v)$ is measurable on $[0, 2\pi]$.
(ii) For almost all $t \in [0, 2\pi]$, the mapping $(u, v) \to f(t, u, v)$ is continuous on $\mathbb{R}^2$.
(iii) For any given $N > 0$, there exists $g_N(t)$, a Lebesgue integrable function defined on $[0, 2\pi]$, such that

$$|f(t, u, v)| \le g_N(t) \quad \text{for a. e. } t \in [0, 2\pi],$$

whenever $|u|, |v| \le N$.

To develop upper and lower solutions method, we need the concepts of upper and lower solutions. We say that $\beta \in W^{2,1}[0, 2\pi]$ is an upper solution to (1.1), if it satisfies

$$\begin{gathered} \beta''(t) \le f(t, \beta(t), \beta'(t)), \quad t \in [0, 2\pi] \\ \beta(0) = \beta(2\pi), \quad \beta'(0) \le \beta'(2\pi). \end{gathered} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34B15, 34B16.
Key words and phrases. Periodic boundary value; existence; upper and lower solutions.

Submitted November 7, 2005. Published February 28, 2006.
Supported by grant 10171010 from the National Natural Science Foundation of China.

Similarly, a function $\alpha \in W^{2,1}[0,2\pi]$ is said to be a lower solution to (1.1), if it satisfies

$$\begin{gathered} \alpha''(t) \ge f(t,\alpha(t),\alpha'(t)), \quad t \in [0,2\pi] \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) \ge \alpha'(2\pi). \end{gathered} \tag{1.3}$$

We call a function $u \in W^{2,1}[0,2\pi]$ a solution to (1.1), if it is an upper and a lower solution to (1.1).

Under the classical assumption that $\alpha(t) \le \beta(t)$, a number of authors have studied the existence of the methods of lower and upper solutions or the monotone iterative technique [1, 3, 4, 5, 8, 6, 10, 11, 16, 17]. Only a few have study the case where $\alpha(t), \beta(t)$ satisfy the opposite ordering condition $\beta(t) \le \alpha(t)$; see [1, 2, 7, 9, 13, 14, 15, 18] Wang [18] has investigated a special case of (1.1) where $f(t,u,v) = -kv + F(t,u)$ and $F(t,u)$ is increasing with respect to u, in the presence of a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ with $\beta(t) \le \alpha(t)$. Rachunkova [15] has recently proved that (1.1) has at least one solution $u(t)$ under the case $\beta(t) \le \alpha(t)$. However, the proof of the result in [15] is not constructive and is not able to guarantee that $u(t)$ satisfies $\beta(t) \le u(t) \le \alpha(t)$. Recently, Jiang, Fan and Wan [7] have studied (1.1) by means of a monotone iterative technique in the presence of a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ with $\beta(t) \le \alpha(t)$. To develop a monotone method, the following hypotheses are needed in [7].

- (A1) For any given $\beta, \alpha \in C[0,2\pi]$ with $\beta(t) \le \alpha(t)$ on $[0,2\pi]$, there exist $0 < A \le B$ such that
$$A(v_2 - v_1) \le f(t,u,v_2) - f(t,u,v_1) \le B(v_2 - v_1)$$
or
$$-B(v_2 - v_1) \le f(t,u,v_2) - f(t,u,v_1) \le -A(v_2 - v_1)$$
for a.e. $t \in [0,2\pi]$ whenever $\beta(t) \le u \le \alpha(t), v_1, v_2 \in \mathbb{R}$, and $v_1 \le v_2$.
- (A2) Inequality
$$f(t,u_2,v) - f(t,u_1,v) \ge -\frac{A^2}{4}(u_2 - u_1)$$
holds for a.e. $t \in [0,2\pi]$, whenever $\beta(t) \le u_1 \le u_2 \le \alpha(t), v \in \mathbb{R}$.

The purpose of this paper is to prove the existence of solutions to (1.1) under the assumption that there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) with $\beta(t) \le \alpha(t)$ and $f(t,u,v)$ only satisfies one side Lipschitz condition. We use the upper and lower solutions method and prove that the solution $u(t)$ of (1.1) satisfies $\beta(t) \le u(t) \le \alpha(t)$. Our result extends and complements those in [18, 15, 7].

To develop upper and lower solutions method, we need one of the following hypotheses

- (H1) For any given $\beta, \alpha \in C[0,2\pi]$ with $\beta(t) \le \alpha(t)$ on $[0,2\pi]$, there exist $A > 0$ and $B > 0$ such that $B^2 \ge 4A$ and
$$f(t,u_2,v_2) - f(t,u_1,v_1) \ge -A(u_2 - u_1) + B(v_2 - v_1) \tag{1.4}$$
for a.e. $t \in [0,2\pi]$ whenever $\beta(t) \le u_1 \le u_2 \le \alpha(t)$, $v_1, v_2 \in \mathbb{R}$, and $v_1 \le v_2$.
- (H1') For any given $\beta, \alpha \in C[0,2\pi]$ with $\beta(t) \le \alpha(t)$ on $[0,2\pi]$, there exist $A > 0$ and $B > 0$ such that $B^2 \ge 4A$ and
$$f(t,u_2,v_2) - f(t,u_1,v_1) \ge -A(u_2 - u_1) + B(v_1 - v_2) \tag{1.5}$$
for a.e. $t \in [0,2\pi]$ whenever $\beta(t) \le u_1 \le u_2 \le \alpha(t)$, $v_1, v_2 \in \mathbb{R}$, and $v_1 \ge v_2$.

We remark that condition (H1') is equivalent to

(a1) For any given $\beta, \alpha \in C[0, 2\pi]$ with $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$, there exists $B > 0$ such that
$$f(t, u, v_1) - f(t, u, v_2) \le -B(v_1 - v_2)$$
for a.e. $t \in [0, 2\pi]$ whenever $\beta(t) \le u \le \alpha(t)$, $v_1, v_2 \in \mathbb{R}$, and $v_1 \ge v_2$.

(a2) There exists $A > 0$ such that $B^2 \ge 4A$ and
$$f(t, u_2, v) - f(t, u_1, v) \ge -A(u_2 - u_1)$$
holds for a.e. $t \in [0, 2\pi]$, whenever $\beta(t) \le u_1 \le u_2 \le \alpha(t)$, $v \in \mathbb{R}$.

Also we remark that (H1) or (H1') weaker than (A1)-(A2) in [7].

Let $m < 0$ and $M < 0$ be two real roots to the equation $x^2 + Bx + A = 0$, then
$$m + M = -B, \quad mM = A.$$
Let $m_0 > 0$ and $M_0 > 0$ are two roots to the equation $x^2 - Bx + A = 0$, then
$$m_0 + M_0 = B, \quad m_0 M_0 = A.$$
Let
$$A(t) := \alpha'(t) + m\alpha(t), \quad B(t) := \beta'(t) + m\beta(t), \tag{1.6}$$
and
$$A_0(t) := \alpha'(t) + m_0\alpha(t), \quad B_0(t) := \beta'(t) + m_0\beta(t). \tag{1.7}$$
The main results of this paper are stated as follows.

Theorem 1.1. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) such that $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$, and $f(t, u, v)$ is a Caratheodory function satisfying the hypothesis (H1). Then $A(t) \le B(t)$ on $[0, 2\pi]$ and (1.1) has a solution $u \in W^{2,1}[0, 2\pi]$ such that*
$$\beta(t) \le u(t) \le \alpha(t), \quad A(t) \le u'(t) + mu(t) \le B(t).$$

Theorem 1.2. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) such that $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$, and $f(t, u, v)$ is a Caratheodory function satisfying the hypothesis (H1'). Then $B_0(t) \le A_0(t)$ on $[0, 2\pi]$ and (1.1) has a solution $u \in W^{2,1}[0, 2\pi]$ such that*
$$\beta(t) \le u(t) \le \alpha(t), \quad B_0(t) \le u'(t) + m_0 u(t) \le A_0(t).$$

2. Proof of Theorems 1.1 and 1.2

To prove the validity of upper and lower solutions method, we use the following maximum-minimum principle, see [7].

Lemma 2.1. *Let $y \in W^{1,1}[0, 2\pi]$, and satisfy*
$$y'(t) + Ly(t) \ge 0 \quad \text{for a. e. } t \in [0, 2\pi],$$
$$y(0) \ge y(2\pi),$$
where $|L| > 0$. Then $Ly(t) \ge 0$ on $[0, 2\pi]$, i.e., when $L > 0$ the minimum of $y(t)$ is nonnegative; when $L < 0$ the maximum of $y(t)$ is nonpositive.

Lemma 2.2. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) such that $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$, and $f(t, u, v)$ is a Caratheodory function satisfying the hypothesis (H1). Then $A(t) \le B(t)$ on $[0, 2\pi]$.*

Proof. It follows from (1.2) and (1.3) that

$$A'(t)+MA(t)\ge f(t,\alpha(t),A(t)-m\alpha(t))+(m+M)A(t)-m^2\alpha(t),\quad t\in[0,2\pi]$$
$$A(0)\ge A(2\pi),$$

and

$$B'(t)+MB(t)\le f(t,\beta(t),B(t)-m\beta(t))+(m+M)B(t)-m^2\beta(t),\quad t\in[0,2\pi]$$
$$B(0)\le B(2\pi).$$

Let $y(t)=A(t)-B(t)$, then $y(0)\ge y(2\pi)$. Assume that $y(t)>0$ for some $t\in[0,2\pi]$. Indeed, if $y(t)>0$ on $[0,2\pi]$, then by (H1) we have

$$\begin{aligned}y'(t)+My(t)&\ge f(t,\alpha(t),A(t)-m\alpha(t))-f(t,\beta(t),B(t)-m\beta(t))\\&\quad+(m+M)y(t)-m^2(\alpha(t)-\beta(t))\\&\ge -(A+Bm+m^2)(\alpha(t)-\beta(t))+(B+m+M)y(t)\\&=0,\quad t\in[0,2\pi],\end{aligned}$$

then by Lemma 2.1, we have $y(t)\le 0$ on $[0,2\pi]$, which is a contradiction.

If $y(0)\le 0$ (then $y(2\pi)\le y(0)\le 0$), and hence there exists $s\in(0,2\pi)$ with $y(s)>0$, then there would be $0\le a<s<b\le 2\pi$ such that $y(t)>0$ in (a,b) with $y(a)=y(b)=0$. By (1.2) and (1.3), we have

$$y'(t)+My(t)\ge 0,\quad t\in[a,b],\quad y(a)=y(b)=0.$$

This leads to $y'(t)\ge -My(t)>0$ on $[a,b]$, which is again a contradiction.

If $y(0)>0$, then there exists $a\in(0,2\pi)$ such that $y(t)>0$ on $[0,a)$ with $y(a)=0$. So we have $y'(t)+My(t)\ge 0$ on $[0,a)$, hence $y'(t)>0$ in $[0,a)$, which implies that $y(0)<y(a)=0$, this is also a contradiction. The proof of Lemma 2.2 is completed. □

Similarly, we have the following result.

Lemma 2.3. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) such that $\beta(t)\le\alpha(t)$ on $[0,2\pi]$, and $f(t,u,v)$ is a Caratheodory function satisfying the hypothesis (H1'). Then $B_0(t)\le A_0(t)$ on $[0,2\pi]$.*

Proof. It follows from (1.2) and (1.3) that for $t\in[0,2\pi]$,

$$A_0'(t)+M_0A_0(t)\ge f(t,\alpha(t),A_0(t)-m_0\alpha(t))+(m_0+M_0)A_0(t)-m_0^2\alpha(t),$$
$$A_0(0)\ge A_0(2\pi)$$

and for $t\in[0,2\pi]$

$$B_0'(t)+MB(t)\le f(t,\beta(t),B_0(t)-m_0\beta(t))+(m_0+M_0)B_0(t)-m_0^2\beta(t),$$
$$B_0(0)\le B_0(2\pi).$$

Let $y(t)=A_0(t)-B_0(t)$, then $y(0)\ge y(2\pi)$. Assume that $y(t)<0$ for some $t\in[0,2\pi]$. Indeed, if $y(t)<0$ on $[0,2\pi]$, then by (H1') we have

$$\begin{aligned}y'(t)+M_0y(t)&\ge f(t,\alpha(t),A_0(t)-m_0\alpha(t))-f(t,\beta(t),B_0(t)-m_0\beta(t))\\&\quad+(m_0+M_0)y(t)-m_0^2(\alpha(t)-\beta(t))\\&\ge -(A-Bm_0+m_0^2)(\alpha(t)-\beta(t))+(-B+m_0+M_0)y(t)\\&=0,\quad t\in[0,2\pi],\end{aligned}$$

then by Lemma 2.1, we have $y(t) \geq 0$ on $[0, 2\pi]$, which is a contradiction.

If $y(2\pi) \geq 0$ (then $y(0) \geq y(2\pi) \geq 0$), and hence there exists $s \in (0, 2\pi)$ with $y(s) < 0$, then there would be $0 \leq a < s < b \leq 2\pi$ such that $y(t) < 0$ in (a, b) with $y(a) = y(b) = 0$. By (1.2) and (1.3), we have

$$y'(t) + M_0 y(t) \geq 0, \quad t \in [a, b], \quad y(a) = y(b) = 0.$$

This leads to $y'(t) \geq -M_0 y(t) > 0$ on $[a, b]$, which is again a contradiction.

If $y(2\pi) < 0$, then there exists $a \in (0, 2\pi)$ such that $y(t) < 0$ on $(a, 2\pi]$ with $y(a) = 0$. So we have $y'(t) + M_0 y(t) \geq 0$ on $(a, 2\pi]$, hence $y'(t) > 0$ in $(a, 2\pi]$, which implies that $y(2\pi) > y(a) = 0$, this is also a contradiction. The proof of Lemma 2.3 is complete. □

In the following arguments, we only give the proof of Theorem 1.1, since the proof of Theorem 1.2 can be treated in a similar way.

Let

$$p(t, x) = \begin{cases} A(t), & x < A(t), \\ x, & A(t) \leq x \leq B(t), \\ B(t), & x > B(t). \end{cases}$$

It is interesting to give an introduction to Lemma 2.4 and a reference where it can be found.

Lemma 2.4. *If $m > 0$, then for any $q(t) \in L^1[0, 2\pi]$, the problem*

$$\begin{gathered} u'(t) + mu(t) = q(t), \quad \text{for a.e. } t \in [0, 2\pi] \\ u(0) = u(2\pi), \end{gathered}$$

has a unique solution $u \in W^{1,1}[0, 2\pi]$, and

$$u(t) = L^{-1} q(t) = \int_0^{2\pi} G_m(t, s) q(s) ds,$$

where

$$G_m(t, s) := \begin{cases} \frac{e^{m(2\pi + s - t)}}{e^{2m\pi} - 1}, & 0 \leq s \leq t \leq 2\pi, \\ \frac{e^{m(s-t)}}{e^{2m\pi} - 1}, & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

By Lemma 2.1, we have

$$\alpha(t) = L^{-1} A(t), \quad \beta(t) = L^{-1} B(t), \quad \beta(t) \leq L^{-1} p(t, x) \leq \alpha(t).$$

Now we consider the modified problem

$$\begin{gathered} x'(t) + Mx(t) = f\big(t, L^{-1} p(t, x(t)), (I - mL^{-1}) p(t, x(t))\big) \\ + (m + M) p(t, x(t)) - m^2 L^{-1} p(t, x(t)), x(0) = x(2\pi). \end{gathered} \tag{2.1}$$

For each $x \in C[0, 2\pi]$, we define the mapping $\Phi : C[0, 2\pi] \to C[0, 2\pi]$,

$$(\Phi x)(t) = \int_0^{2\pi} G_M(t, s)(Fx)(s) ds, \tag{2.2}$$

where

$$\begin{aligned} (Fx)(t) := {} & f\big(t, L^{-1} p(t, x(t)), (I - mL^{-1}) p(t, x(t))\big) \\ & + (m + M) p(t, x(t)) - m^2 L^{-1} p(t, x(t)). \end{aligned}$$

Since $p(t, x(t))$ and $L^{-1}p(t, x(t))$ are bounded and $f(t, u, v)$ is a Caratheodary function, there exists $g(t)$, a Lebesgue integrable function defined on $[0, 2\pi]$ such that

$$|(Fx)(t)| \le g(t) \quad \text{for a. e. } t \in [0, 2\pi].$$

Thus $(\Phi x)(t)$ is also bounded.

We can easily prove that $\Phi : C[0, 2\pi] \to C[0, 2\pi]$ is completely continuous. Then Leray-Schauder fixed point Theorem assures that Φ has a fixed point $x \in C[0, 2\pi]$ and

$$x(t) = \int_0^{2\pi} G_M(t, s)(Fx)(s)ds, \tag{2.3}$$

thus the modified problem (2.1) has one solution $x \in W^{1,1}[0, 2\pi]$.

Lemma 2.5. *Suppose that (H1) holds. Assume that $\alpha(t)$ and $\beta(t)$ are lower and upper solutions to* (1.1) *and $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$. Let $x \in W^{1,1}[0, 2\pi]$ be a solution to* (2.1)*, then $A(t) \le x(t) \le B(t)$ on $[0, 2\pi]$.*

Remark 2.6. Lemma 2.4 implies $u(t) = L^{-1}x(t) = \int_0^{2\pi} G_m(t, s)x(s)ds$ is a solution to (1.1), since $u'(t) + mu(t) = x(t), u(0) = u(2\pi)$ and $A(t) \le x(t) \le B(t)$.

Proof of Lemma 2.5. Since $\alpha(t) = L^{-1}A(t), \beta(t) = L^{-1}B(t)$,

$$\begin{gathered}B'(t) + MB(t) \le f(t, L^{-1}B(t), (I - mL^{-1})B(t)) - m^2L^{-1}B(t) + (m + M)B(t),\\ B(0) \le B(2\pi)\end{gathered}$$

and

$$\begin{gathered}A'(t) + MA(t) \ge f(t, L^{-1}A(t), (I - mL^{-1})A(t)) - m^2L^{-1}A(t) + (m + M)A(t),\\ A(0) \ge A(2\pi).\end{gathered}$$

We shall prove only that $x(t) \le B(t)$ on $[0, 2\pi]$, because $A(t) \le x(t)$ can be proved by a similar manner. Let $y(t) = x(t) - B(t)$, then

$$y(0) \ge y(2\pi).$$

Assume that $y(t) > 0$ for some $t \in [0, 2\pi]$. Indeed, if $y(t) > 0$ on $[0, 2\pi]$, we have

$$\begin{aligned}x'(t) + Mx(t) &= f(t, L^{-1}B(t), (I - mL^{-1})B(t)) - m^2L^{-1}B(t) + (m + M)B(t)\\ &\ge B'(t) + MB(t),\end{aligned}$$

i.e., $y'(t) + My(t) \ge 0$ on $[0, 2\pi]$. Lemma 2.1 implies $y(t) \le 0$ on $[0, 2\pi]$, which is a contradiction. Therefore, there would be a point $s \in [0, 2\pi]$ with $y(s) \le 0$.

If $y(0) \le 0$ (then $y(2\pi) \le y(0) \le 0$), and hence there exist $0 \le a < s < b \le 2\pi$ such that $y(t) > 0$ in (a, b) with $y(a) = y(b) = 0$. Then $p(t, x(t)) = B(t)$ on $[a, b]$ and

$$\begin{aligned}&y'(t) + My(t)\\ &\ge f(t, L^{-1}p(t, x(t)), B(t) - mL^{-1}p(t, x(t))) + (m + M)B(t) - m^2L^{-1}p(t, x(t))\\ &\quad - [f(t, L^{-1}B(t), B(t) - mL^{-1}B(t)) + (m + M)B(t) - m^2L^{-1}B(t)]\\ &\ge (-A - Bm - m^2)(L^{-1}p(t, x(t)) - L^{-1}B(t))\\ &= 0, \quad t \in (a, b).\end{aligned}$$

This leads to $y'(t) \ge -My(t) > 0$ on (a, b), which is again a contradiction.

If $y(0) > 0$, there exists $a \in (0, 2\pi)$ such that $y(t) > 0$ on $[0, a)$ with $y(a) = 0$. So we have $y'(t) + My(t) \geq 0$, hence $y'(t) > 0$ in $[0, a)$, which implies that $y(0) < y(a) = 0$, this is also a contradiction. The proof is complete. □

By Remark 2.6, we have obtained the results of Theorem 1.1.

3. Example

In this section, we consider the periodic boundary-value problem

$$\begin{gathered} u''(t) + ku'(t) = F(t, u), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi), u'(0) = u'(2\pi), \end{gathered} \tag{3.1}$$

where $F(t, u)$ is a Caratheodory function, $k > 0$ or $k < 0$.

We say that $\beta \in W^{2,1}[0, 2\pi]$ is an upper solution to the problem (3.1), if it satisfies

$$\begin{gathered} \beta''(t) + k\beta'(t) \leq F(t, \beta(t)), \quad t \in [0, 2\pi] \\ \beta(0) = \beta(2\pi), \beta'(0) \leq \beta'(2\pi). \end{gathered} \tag{3.2}$$

Similarly, a function $\alpha \in W^{2,1}[0, 2\pi]$ is said to be a lower solution to (3.1), if it satisfies

$$\begin{gathered} \alpha''(t) + k\alpha'(t) \geq F(t, \alpha(t)), \quad t \in [0, 2\pi] \\ \alpha(0) = \alpha(2\pi), \alpha'(0) \geq \alpha'(2\pi). \end{gathered} \tag{3.3}$$

To develop the upper and lower solutions method, we also need the following hypothesis:

(H) For any given $\beta, \alpha \in C[0, 2\pi]$ with $\beta(t) \leq \alpha(t)$ on $[0, 2\pi]$, the inequality

$$F(t, u_2) - F(t, u_1) \geq -\frac{k^2}{4}(u_2 - u_1)$$

holds for a.e. $t \in [0, 2\pi]$, whenever $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$.

Let $A = k^2/4$, $B = |k|$, then (H1) holds when $k < 0$, and (H1') holds when $k > 0$. Hence the conclusions of Theorem 1.1 hold when $k < 0$, thus $\alpha'(t) + \frac{k}{2}\alpha(t) \leq \beta'(t) + \frac{k}{2}\beta(t)$ and problem (3.1) has one solution $u \in W^{2,1}[0, 2\pi]$ such that

$$\beta(t) \leq u(t) \leq \alpha(t), \alpha'(t) + \frac{k}{2}\alpha(t) \leq u'(t) + \frac{k}{2}u(t) \leq \beta'(t) + \frac{k}{2}\beta(t).$$

The conclusions of Theorem 1.2 hold when $k > 0$, thus $\alpha'(t) + \frac{k}{2}\alpha(t) \geq \beta'(t) + \frac{k}{2}\beta(t)$ and problem (3.1) has one solution $u \in W^{2,1}[0, 2\pi]$ such that

$$\beta(t) \leq u(t) \leq \alpha(t),\ \beta'(t) + \frac{k}{2}\beta(t) \leq u'(t) + \frac{k}{2}u(t) \leq \alpha'(t) + \frac{k}{2}\alpha(t).$$

In [7, 18], the authors obtained one solution $u \in W^{2,1}[0, 2\pi]$ of (3.1) such that $\beta(t) \leq u(t) \leq \alpha(t)$. Here we have improved the results of [7, 18].

References

[1] A. Cabada. *The method of lower and upper solutions for second,third, fourth, and higher order boundary value problems*, J. Math. Anal. Appl., **185**(1994),302-320.

[2] A. Cabada, *The Monotone Method for Boundary Value Problems, doctoral thesis*, Universidad de Santiago de Compostela, 1992. [in Spanish]

[3] A. Cabada and J. J. Nieto, *A generalization of the monotone iterative technique for linear second order periodic boundary value problems*, J. Math. Anal., **151** (1990), 181-189.

[4] A. Cabada and J. J. Nieto, *Extemal solutions of second order nonlinear periodic boundary value problems*, Appl. Math. Comput., **40**(1990), 135-145.

[5] W. Gao and J. Wang, *On a nonlinear second order periodic boundary value problem with Caratheodory functions*, Ann. Polon. Math., **62**(1995), 283-291.
[6] D. Jiang and J. Wang, *A generalized periodic boundary value problem for the one-dimensional p-Laplacian*, Ann. Polon. Math., **65**(1997), 265-270.
[7] D. Jiang, M. Fan and A. Wan, *A monotone method for constructing extremal solutions to second-order periodic boundary value problems*, J. Comput. Appl. Math., **136** (2001), 189-197.
[8] G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Boston, 1985.
[9] J. J. Nieto, *Nonlinear second-order periodic boundary value problems*, J. Math. Anal. Appl. **30**(1988), 22-29.
[10] J. J. Nieto, *Nonlinear second-order periodic boundary value problems with Caratheodory functions*, Appl. Anal., **34**(1989), 111-128.
[11] J. J. Nieto and A. Cabada, *A generalized upper and lower solution method for nonlinear second order ordinary differential equations*, J. Appl. Math. Stochastic Anal., **5**(1992), 157-166.
[12] P. Omari, *A monotone method for constructing extremal soltions of second order scalar boundary value problems*, Appl. Math. Comput., **18**(1986), 257-275.
[13] P. Omari, *Nonordered lower and upper solutions and solvability of the periodic problem for the Lienard and the Rayleigh equations*, Rend. Istit. Mat. Univ. Trieste, **20**(1991), 54-64.
[14] B. Rudolf and Z. Kubacek, *Remarks on J. J. Niteo's paper: Nolinear second-order periodic boundary value problems*, J. Math. Anal. Appl., **146**(1990), 203-206.
[15] I. Rachunkova, *Upper and lower solutions satisfying the inverse inequality*, Ann. Polon. Math., **65**(1997), 235-244.
[16] V. Seda, J. J. Nieto, and M. Gera, *Periodic boundary value problems for nonlinear higher order ordinary differential equations*, Appl. Math. Comput., **48**(1992), 71-82.
[17] M. X. Wang, A. Cabada, and J. J. Nieto, *Monotone method for nonlinear second order periodic boundary value problemws with Caratheodory functions*, Ann. Polon. Math., **58**(1993), 221-235.
[18] C. Wang, *Generalized upper and lower solution method for the forced duffing equation*, Proc. Amer. Math. Soc., **125**(1997), 397-406.

Haiyin Gao
School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
E-mail address: gaohaiyinhealthy@yahoo.com.cn

Shiyou Weng
Applied Science College, Changchun University, Jilin 130022, China
E-mail address: wengshiyou2001@yahoo.com.cn

Daqing Jiang
School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
E-mail address: daqingjiang@vip.163.com

Xuezhang Hou
Mathematics Department, Towson University, Baltimore, MD 21252, USA
E-mail address: xhou@towson.edu