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EXISTENCE AND EXPONENTIAL STABILITY OF PERIODIC SOLUTION FOR CONTINUOUS-TIME AND DISCRETE-TIME GENERALIZED BIDIRECTIONAL NEURAL NETWORKS

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Abstract. We study the existence and global exponential stability of positive periodic solutions for a class of continuous-time generalized bidirectional neural networks with variable coefficients and delays. Discrete-time analogues of the continuous-time networks are formulated and the existence and global exponential stability of positive periodic solutions are studied using the continuation theorem of coincidence degree theory and Lyapunov functionals. It is shown that the existence and global exponential stability of positive periodic solutions of the continuous-time networks are preserved by the discrete-time analogues under some restriction on the discretization step-size. An example is given to illustrate the results obtained.

1. Introduction

In recent years, the stability of the following bidirectional associative neural networks with or without delays has been extensively studied:

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ji} f_j(y_j(t-\tau_{ji})) + I_i, \quad i = 1, 2, \dots, l, \\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -b_j y_j(t) + \sum_{i=1}^{l} q_{ij} g_i(x_i(t-\sigma_{ij})) + J_j, \quad j = 1, 2, \dots, m. \end{aligned} \tag{1.1}$$

Also some of its generalizations have been studied and various stability conditions have been obtained [3, 4, 5, 6, 11, 12, 14, 15, 16, 17]. Here p_{ji}, q_{ij}, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$ are the connection weights through the neurons in two layers: I-layer and J-layer. On I-layer, the neurons whose states denoted by $x_i(t)$ receive the inputs I_i and the inputs outputted by those neurons in J-layer via activation functions (output-input functions) f_j, while on J-layer, the neurons whose associated states denoted by $y_j(t)$ receive the inputs J_i and the inputs outputted from those neurons in I-layer via activation functions (output-input functions) g_i. And $\tau_{ji}, \sigma_{ij}, i =$

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$1, 2, \dots, l, j = 1, 2, \dots, m$ are the associated delays due to the finite transmission speed among neurons in different layers.

When there is no delay present, (1.1) reduces to a system of ordinary differential equations which was investigated by Kosko [7, 8, 9] and it produces many nice properties due to the special structure of connection weights and has practical applications in storing paired patterns or memories and the ability to search the desired patterns via both directions: forward and backward directions. See [3, 14, 7, 8, 9] for details about the applications on learning and associative memories.

It is well known that in the study of neural dynamical systems, the periodic oscillatory behavior of the systems is an important aspect. In this paper, we are concerned with the following generalized BAM networks with variable coefficients and delays

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i(t)x_i(t) + \sum_{j=1}^{m} a_{ji}(t)f_j(y_j(t)) + \sum_{j=1}^{m} p_{ji}(t)g_j(y_j(t-\tau_{ji}(t))) + I_i(t),\\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -b_j(t)y_j(t) + \sum_{i=1}^{l} b_{ij}(t)\hat{f}_i(x_i(t)) + \sum_{i=1}^{l} q_{ij}(t)\hat{g}_i(x_i(t-\sigma_{ij}(t))) + J_j(t),\end{aligned} \tag{1.2}$$

for $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$.

In this paper, we assume that

(S1) $a_i, b_j \in C(\mathbb{R}, (0, \infty))$, $\tau_{ji}, \sigma_{ij} \in C(\mathbb{R}, [0, \infty))$, $I_i, J_j, p_{ji}, q_{ij} \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$ are all ω-periodic functions.

(S2) $f_j, g_j, \hat{f}_i, \hat{g}_i \in C(\mathbb{R}, \mathbb{R})$ $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$ are bounded on $\mathbb{R}$.

(S3) There exist positive number $L_j^f, L_j^g, L_i^{\hat{f}}, L_j^{\hat{g}}$ such that

$$\begin{aligned} |f_j(x) - f_j(y)| &\le L_j^f |x - y| \quad \text{for all } x, y \in \mathbb{R},\ j = 1, 2, \dots, m,\\ |g_j(x) - g_j(y)| &\le L_j^g |x - y| \quad \text{for all } x, y \in \mathbb{R},\ j = 1, 2, \dots, m,\\ |\hat{f}_i(x) - \hat{f}_i(y)| &\le L_i^{\hat{f}} |x - y| \quad \text{for all } x, y \in \mathbb{R},\ i = 1, 2, \dots, l,\\ |\hat{g}_i(x) - \hat{g}_i(y)| &\le L_i^{\hat{g}} |x - y| \quad \text{for all } x, y \in \mathbb{R},\ i = 1, 2, \dots, l.\end{aligned}$$

Our purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [2, 13] and by constructing suitable Lyapunov functions to investigate the stability and existence of periodic solutions of (1.2); then, we shall use a novel method in formulating discrete-time analogues of the continuous time networks. It is shown that the existence and global exponential stability of positive periodic solutions of the continuous-time networks are preserved by the discrete-time analogues under some restriction on the discretization step-size.

2. Existence of periodic solutions

In this section, based on the Mawhin's continuation theorem, we shall study the existence of at least one positive periodic solution of (1.2). First, we shall make some preparations.

Let $\mathbb{X}, \mathbb{Y}$ be normed vector spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be a linear mapping, and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$

and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P = \ker L, \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that mapping $L|_{\operatorname{Dom} L \bigcap \ker P} : (I - P)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L- compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

For convenience of use, we introduce Mawhin's continuation theorem [2, P. 40] as follows.

Lemma 2.1. *Let $\Omega \subset \mathbb{X}$ be an open bounded set and let $N : \mathbb{X} \to \mathbb{Y}$ be a continuous operator which is L-compact on $\bar{\Omega}$ (i.e., $QN : \bar{\Omega} \to \mathbb{Y}$ and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{Y}$ are compact). Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \bigcap \operatorname{Dom} L Lx \neq \lambda Nx$,*
(b) *for each $x \in \partial\Omega \bigcap \ker L$. $QNx \neq 0$,*
(c) $\deg(JNQ, \Omega \bigcap \ker L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \bigcap \operatorname{Dom} L$.

Our result about the existence of periodic solutions of (1.2) is as follows.

Theorem 2.2. *Assume that (S1) and (S2) hold. Then system* (1.2) *has at least one ω-periodic solution.*

Proof. To use the continuation theorem of coincidence degree theory to establish the existence of an ω-periodic solution of (1.2), we take $\mathbb{X} = \mathbb{Y} = \{u \in C(\mathbb{R}, \mathbb{R}^{l+m}) : u(t + \omega) = u(t)\}$ and $\|u\| = \sum_{i=1}^{l+m} \max_{t\in[0,\omega]} |u_i(t)|$, then $\mathbb{X}$ is a Banach space. Set

$$L : \operatorname{Dom} L \cap \mathbb{X}, \quad Lu = \dot{u}(t), \quad u \in \mathbb{X},$$

where $\operatorname{Dom} L = \{u \in C^1(\mathbb{R}, \mathbb{R}^{l+m})\}$ and $N : \mathbb{X} \to \mathbb{X}$, $N[x_1, \dots, x_l, y_1, \dots, y_m]^T =$

$$\begin{bmatrix} -a_1(t)x_1(t) + \sum_{j=1}^m a_{j1}(t)f_j(y_j(t)) + \sum_{j=1}^m p_{j1}(t)g_j(y_j(t - \tau_{j1}(t))) + I_1(t) \\ \vdots \\ -a_l(t)x_l(t) + \sum_{j=1}^m a_{jl}(t)f_j(y_j(t)) + \sum_{j=1}^m p_{jl}(t)g_j(y_j(t - \tau_{jl}(t))) + I_l(t) \\ -b_1(t)y_1(t) + \sum_{i=1}^l b_{i1}(t)\hat{f}_i(x_i(t)) + \sum_{i=1}^l q_{i1}(t)\hat{g}_i(x_i(t - \sigma_{i1}(t))) + J_1(t) \\ \vdots \\ -b_m(t)y_m(t) + \sum_{i=1}^l b_{im}(t)\hat{f}_i(x_i(t)) + \sum_{i=1}^l q_{im}(t)\hat{g}_i(x_i(t - \sigma_{im}(t))) + J_m(t) \end{bmatrix}.$$

Define two projectors P and Q as

$$Pu = Qu = \frac{1}{\omega}\int_0^\omega u(t)\,\mathrm{d}t, u \in \mathbb{X}.$$

Clearly, $\ker L = R^{l+m}$, $\operatorname{Im} L = \{(x_1, x_2, \dots, x_l, y_1, \dots, y_m)^T \in \mathbb{X} : \int_0^\omega x_i(t)\,\mathrm{d}t = 0, \int_0^\omega y_j(t)\,\mathrm{d}t = 0, i = 1, 2, \dots, l, j = 1, 2, \dots, m\}$ is closed in $\mathbb{X}$ and $\dim\ker L = \operatorname{codim}\operatorname{Im} L = l + m$. Hence, L is a Fredholm mapping of index zero. Furthermore, similar to the proof of [10, Theorem 1], one can easily show that N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset \mathbb{X}$. Corresponding to operator equation

$Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= \lambda\Big[-a_i(t)x_i(t) + \sum_{j=1}^{m} a_{ji}(t)f_j(y_j(t)) + \sum_{j=1}^{m} p_{ji}(t)g_j(y_j(t-\tau_{ji}(t))) + I_i(t)\Big], \\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= \lambda\Big[-b_j(t)y_j(t) + \sum_{i=1}^{l} b_{ij}(t)\hat{f}_i(x_i(t)) + \sum_{i=1}^{l} q_{ij}(t)\hat{g}_i(x_i(t-\sigma_{ij}(t))) + J_j(t)\Big] \end{aligned} \tag{2.1}$$

for $i = 1,2,\dots,l, j = 1,2,\dots,m$. Suppose that $(x_1,x_2,\dots,x_l,y_1,y_2,\dots,y_m) \in \mathbb{X}$ is a solution of (2.1) for some $\lambda \in (0,1)$. Let $\bar{\xi}_i, \bar{\zeta}_j \in [0,\omega]$ such that $x_i(\bar{\xi}_i) = \max_{t\in[0,\omega]} x_i(t)$ and $y_j(\bar{\zeta}_j) = \max_{t\in[0,\omega]} y_j(t), i = 1,2,\dots,l, j = 1,2\dots,m$, then

$$a_i(\bar{\xi}_i)x_i(\bar{\xi}_i) = \sum_{j=1}^{m} a_{ji}(\bar{\xi}_i)f_j(y_j(\bar{\xi}_i)) + \sum_{j=1}^{m} p_{ji}(\bar{\xi}_i)g_j(y_j(\bar{\xi}_i-\tau_{ji}(\bar{\xi}_i))) + I_i(\bar{\xi}_i)$$

and

$$b_j(\bar{\zeta}_j)y_j(\bar{\zeta}_j) = \sum_{i=1}^{l} b_{ij}(\bar{\zeta}_j)\hat{f}_i(x_i(\bar{\zeta}_j)) + \sum_{i=1}^{l} q_{ij}(\bar{\zeta}_j)\hat{g}_i(x_i(\bar{\zeta}_j-\sigma_{ij}(\bar{\zeta}_j))) + J_j(\bar{\zeta}_j),$$

for $i = 1,2,\dots,l$, $j = 1,2,\dots,m$. Hence,

$$\begin{aligned} a_i(\bar{\xi}_i)x_i(\bar{\xi}_i) &\le \sum_{j=1}^{m} |a_{ji}(\bar{\xi}_i)||f_j(y_j(\bar{\xi}_i))| + \sum_{j=1}^{m} |p_{ji}(\bar{\xi}_i)||g_j(y_j(\bar{\xi}_i-\tau_{ji}(\bar{\xi}_i)))| + |I_i(\bar{\xi}_i)|, \\ b_j(\bar{\zeta}_j)y_j(\bar{\zeta}_j) &\le \sum_{i=1}^{l} |b_{ij}(\bar{\zeta}_j)||\hat{f}_i(x_i(\bar{\zeta}_j))| + \sum_{i=1}^{l} |q_{ij}(\bar{\zeta}_j)||\hat{g}_i(x_i(\bar{\zeta}_j-\sigma_{ij}(\bar{\zeta}_j)))| + |J_j(\bar{\zeta}_j)| \end{aligned}$$

for $i = 1,2,\dots,l$, $j = 1,2,\dots,m$. Therefore,

$$\begin{aligned} x_i(\bar{\xi}_i) &\le \frac{1}{\underline{a}}[mM_f a^M + mM_g p^M + I^M], \quad i = 1,2,\dots,l, \\ y_j(\bar{\zeta}_j) &\le \frac{1}{\underline{b}}[lM_{\hat{f}} b^M + lM_{\hat{g}} q^M + J^M], \quad j = 1,2,\dots,m, \end{aligned}$$

where

$$\begin{gathered} \underline{a} = \min_{t\in[0,\omega]}\{a_i(t),\, i = 1,2,\dots,n\}, \quad \underline{b} = \min_{t\in[0,\omega]}\{b_j(t),\, i = 1,2,\dots,l\}, \\ M_f = \sup_{u\in R}\{|f_j(u)|,\, j = 1,2,\dots,m\}, \quad M_g = \sup_{u\in R}\{|g_j(u)|,\, j = 1,2,\dots,m\}, \\ M_{\hat{f}} = \sup_{u\in R}\{|\hat{f}_i(u)|,\, i = 1,2,\dots,l\}, \quad M_{\hat{g}} = \sup_{u\in R}\{|\hat{g}_i(u)|, i = 1,2,\dots,l\}, \\ a^M = \max_{t\in[0,\omega]}\{|a_{ji}(t)|,\, i = 1,2,\dots,l,\ j = 1,2,\dots,m\}, \\ b^M = \max_{t\in[0,\omega]}\{|b_{ij}(t)|, \quad i = 1,2,\dots,l,\ j = 1,2,\dots,m\}, \\ p^M = \max_{t\in[0,\omega]}\{|p_{ji}(t)|,\, i = 1,2,\dots,l, j = 1,2,\dots,m\}, \\ q^M = \max_{t\in[0,\omega]}\{|q_{ij}(t)|,\, i = 1,2,\dots,l, j = 1,2,\dots,m\}, \\ I^M = \max_{t\in[0,\omega]}\{|I_i(t)|,\, i = 1,2,\dots,l\}, \quad J^M = \max_{t\in[0,\omega]}\{|J_i(t)|, j = 1,2,\dots,m\}. \end{gathered}$$

Let $\underline{\xi}_i, \underline{\zeta}_j \in [0,\omega]$ be such that $x_i(\underline{\xi}_i) = \min_{t\in[0,\omega]} x_i(t)$ and $y_j(\underline{\zeta}_j) = \min_{t\in[0,\omega]} y_j(t)$, $i = 1,2,\dots,l$, $j = 1,2,\dots,m$, then

$$a_i(\underline{\xi}_i)x_i(\underline{\xi}_i) = \sum_{j=1}^{m} a_{ji}(\underline{\xi}_i) f_j(y_j(\underline{\xi}_i)) + \sum_{j=1}^{m} p_{ji}(\underline{\xi}_i) g_j(y_j(\underline{\xi}_i - \tau_{ji}(\underline{\xi}_i))) + I_i(\underline{\xi}_i)$$

$$b_j(\underline{\zeta}_j)y_j(\underline{\zeta}_j) = \sum_{i=1}^{n} b_{ij}(\underline{\zeta}_j)\hat{f}_i(x_i(\underline{\zeta}_j)) + \sum_{i=1}^{n} q_{ij}(\underline{\zeta}_j)\hat{g}_i(x_i(\underline{\zeta}_j - \sigma_{ij}(\underline{\zeta}_j))) + J_j(\underline{\zeta}_j)$$

for $i = 1,2,\dots,l, j = 1,2,\dots,m$. Thus,

$$\begin{aligned}
&a_i(\underline{\xi}_i)x_i(\underline{\xi}_i)\\
&\geq -\sum_{j=1}^{m} |a_{ji}(\underline{\xi}_i)||f_j(y_j(\underline{\xi}_i))| - \sum_{j=1}^{m} |p_{ji}(\underline{\xi}_i)||g_j(y_j(\underline{\xi}_i - \tau_{ji}(\underline{\xi}_i)))| - |I_i(\underline{\xi}_i)|,\\
&b_j(\underline{\zeta}_j)y_j(\underline{\zeta}_j)\\
&\geq -\sum_{i=1}^{n} |b_{ij}(\underline{\zeta}_j)||\hat{f}_i(x_i(\underline{\zeta}_j))| - \sum_{i=1}^{n} |q_{ij}(\underline{\zeta}_j)||\hat{g}_i(x_i(\underline{\zeta}_j - \sigma_{ij}(\underline{\zeta}_j)))| - |J_j(\underline{\zeta}_j)|
\end{aligned}$$

for $i = 1,2,\dots,l, j = 1,2,\dots,m$. Therefore,

$$\begin{gathered}
x_i(\bar{\xi}_i) \geq -\frac{1}{\underline{a}}[mM_f a^M + mM_g p^M + I^M],\\
y_j(\bar{\zeta}_j) \geq -\frac{1}{\underline{b}}[lM_{\hat{f}} b^M + lM_{\hat{g}} q^M + J^M]
\end{gathered}$$

for $i = 1,2,\dots,l$, $j = 1,2,\dots,m$. Denote $C = \frac{l}{\underline{a}}[mM_f a^M + mM_g p^M + I^M] + \frac{m}{\underline{b}}[lM_{\hat{f}} b^M + lM_{\hat{g}} q^M + J^M] + D$, where D is a positive constant. Then it is clear that C is independent of λ. Now we take $\Omega = \{u \in \mathbb{X} : \|u\| < C\}$. This Ω satisfies condition (a) in Lemma 2.1. When $u \in \partial\Omega \bigcap \ker L = \partial\Omega \bigcap \mathbb{R}^{l+m}$, u is a constant vector in R^{m+n} with $\|u\| = C$. Then

$$\begin{aligned}
u^T QNu = &\sum_{i=1}^{l} \Big\{ -\bar{a}_i x_i^2 + \sum_{j=1}^{m} \bar{a}_{ji} x_i f_j(y_j) + \sum_{j=1}^{m} \bar{p}_{ji} x_i g_j(y_j) + x_i \bar{I}_i \Big\}\\
&+ \sum_{j=1}^{m} \Big\{ -\bar{b}_j y_j^2 + \sum_{i=1}^{l} \bar{b}_{ij} y_j \hat{f}_i(x_i) + \sum_{i=1}^{l} \bar{q}_{ij} y_j \hat{g}_i(x_i) + y_j \bar{J}_j \Big\} < 0,
\end{aligned}$$

where $u = (x_1, x_2, \dots, x_l, y_1, y_2, \dots, y_m)$. If necessary, we can let C be large such that

$$\begin{aligned}
&\sum_{i=1}^{l} \Big\{ -\bar{a}_i x_i^2 + \sum_{j=1}^{m} \bar{a}_{ji} x_i f_j(y_j) + \sum_{j=1}^{m} \bar{p}_{ji} x_i g_j(y_j) + x_i \bar{I}_i \Big\}\\
&+ \sum_{j=1}^{m} \Big\{ -\bar{b}_j y_j^2 + \sum_{i=1}^{l} \bar{b}_{ij} y_j \hat{f}_i(x_i) + \sum_{i=1}^{l} \bar{q}_{ij} y_j \hat{g}_i(x_i) + y_j \bar{J}_j \Big\} < 0.
\end{aligned}$$

So for any $u \in \partial\Omega \bigcap \ker L$, $QNu \neq 0$. This proves that condition (b) in Lemma 2.1 is satisfied.

Furthermore, let $\Psi(\gamma; u) = -\gamma u + (1-\gamma)QNu$, then for any $x \in \partial\Omega \cap \ker L$, $u^T\Psi(\gamma; u) < 0$, we get

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{-u, \Omega \cap \ker L, 0\} \neq 0,$$

hence condition (c) of Lemma 2.1 is also satisfied. Thus, by Lemma 2.1 we conclude that $Lu = Nu$ has at least one solution in $\mathbb{X}$, that is, (1.2) has at least one ω-periodic solution. The proof is complete. $\square$

Next, we shall construct some suitable Lyapunov functionals to derive the sufficient conditions which ensure that the global exponential stability of periodic solutions of the system (1.2) associated with the initial conditions

$$\begin{gathered}
x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad \tau = \max_{t\in[0,\omega]}\{\tau_{ji}(t), i = 1, 2, \dots, l, j = 1, 2, \dots, m\}, \\
y_j(s) = \psi_j(s), \quad s \in [-\sigma, 0], \quad \sigma = \max_{t\in[0,\omega]}\{\sigma_{ij}(t), i = 1, 2, \dots, l, j = 1, 2, \dots, m\},
\end{gathered}$$

where $\varphi_i \in C([-\tau, 0], \mathbb{R}), \psi_i \in C([-\sigma, 0], \mathbb{R}), i = 1, 2, \dots, l, j = 1, 2, \dots, m$. In the sequel, we will use the following notation:

$$\begin{gathered}
a_i^m = \min_{t\in[0,\omega]} a_i(t), \quad b_j^m = \min_{t\in[0,\omega]} b_j(t), \quad a_{ji}^M = \max_{t\in[0,\omega]} |p_{ji}(t)|, \\
b_{ij}^M = \max_{t\in[0,\omega]} |q_{ij}(t)|, \quad \tau_{ji}^M = \max_{t\in[0,\omega]} |\tau_{ji}(t)|, \quad \sigma_{ij}^M = \max_{t\in[0,\omega]} |\sigma_{ij}(t)|, \\
p_{ji}^M = \max_{t\in[0,\omega]} |p_{ji}(t)|, \quad q_{ij}^M = \max_{t\in[0,\omega]} |q_{ij}(t)|,
\end{gathered}$$

where $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$.

Our result about the global exponential stability of periodic solutions of (1.2) is as follows.

Theorem 2.3. *Assume that (S1)-(S3) hold. Furthermore, assume*

(P1) *For $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, m$, $\sigma_{ij}, \tau_{ji} \in C^1(R, [0, \infty))$ satisfy*

$$\sigma'^M_{ij} = \max_{t\in[0,\omega]} \sigma'_{ij}(t) < 1, \quad \tau'^M_{ji} = \max_{t\in[0,\omega]} \tau'_{ji}(t) < 1,$$

(P2)

$$\begin{gathered}
a_i^m > \sum_{j=1}^m \Big(b_{ij}^M L_i^{\hat f} + \frac{q_{ij}^M L_i^{\hat g}}{1-\sigma'^M_{ij}}\Big), \quad i = 1, 2, \dots, l, \\
b_j^m > \sum_{i=1}^l \Big(a_{ji}^M L_j^f + \frac{p_{ji}^M L_j^g}{1-\tau'^M_{ji}}\Big), \quad j = 1, 2, \dots, m,
\end{gathered}$$

then (1.2) has a unique ω-periodic solution $(x_1^*, x_2^*, \dots, x_l^*, y_1^*, y_2^*, \dots, y_m^*)^T$ and, moreover, there exist constants $\eta > 0$ and $\Lambda \geq 1$ such that for $t > 0$,

$$\begin{aligned}
&\sum_{i=1}^l |x_i(t) - x_i^*(t)| + \sum_{j=1}^m |y_j(t) - y_j^*(t)| \\
&\leq \Lambda e^{-\eta t}\Big[\sum_{i=1}^l \sup_{s\in[-\tau,0]} |x_i(s) - x_i^*(s)| + \sum_{j=1}^m \sup_{s\in[-\sigma,0]} |y_j(s) - y_j^*(s)|\Big].
\end{aligned}$$

Proof. Let $x(t)=\{x_1(t),x_2(t),\dots,x_l(t),y_1(t),y_2(t),\dots,y_m(t)\}$ be an arbitrary solution of (1.2), and $x^*(t)=\{x_1^*(t),x_2^*(t),\dots,x_l^*(t),y_1^*(t),y_2^*(t),\dots,y_m^*(t)\}$ be an ω-periodic solution of (1.2). Then

$$
\begin{aligned}
\frac{\mathrm{d}^+(x_i(t)-x_i^*(t))}{\mathrm{d}t} &\le -a_i^m|x_i(t)-x_i^*(t)|+\sum_{j=1}^m a_{ji}^M L_j^f|y_j(t)-y_j^*(t)|\\
&\quad+\sum_{j=1}^m p_{ji}^M L_j^g|y_j(t-\tau_{ji}(t))-y_j^*(t-\tau_{ji}(t))|,\\
\frac{\mathrm{d}^+(y_j(t)-y_j^*(t))}{\mathrm{d}t} &\le -b_j^m|y_j(t)-y_j^*(t)|+\sum_{i=1}^n b_{ij}^M L_i^{\hat f}|x_i(t)-x_i^*(t)|\\
&\quad+\sum_{i=1}^n q_{ij}^M L_i^{\hat g}|x_i(t-\sigma_{ij}(t))-x_i^*(t-\sigma_{ij}(t))|,
\end{aligned}
\tag{2.2}
$$

for$i=1,2,\dots,l$, $j=1,2,\dots,m$. Let F_i and G_j be defined by

$$
\begin{gathered}
F_i(\varepsilon_i)=a_i^m-\varepsilon_i-\sum_{j=1}^m\Big(b_{ij}^M L_i^{\hat f}+\frac{q_{ij}^M L_i^{\hat g}e^{\varepsilon_i\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\Big),\quad i=1,2,\dots,l,\\
G_j(\zeta_j)=b_j^m-\zeta_j-\sum_{i=1}^l\Big(a_{ji}^M L_j^f+\frac{p_{ji}^M L_j^g e^{\zeta_j\tau_{ji}^M}}{1-\tau_{ji}'^M}\Big),\quad j=1,2,\dots,m,
\end{gathered}
$$

where $\varepsilon_i,\zeta_j\in[0,\infty)$. It is clear that

$$
\begin{gathered}
F_i(0)=a_i^m-\sum_{j=1}^m\Big(b_{ij}^M L_i^{\hat f}+\frac{q_{ij}^M L_i^{\hat g}}{1-\sigma_{ij}'^M}\Big)>0,\quad i=1,2,\dots,l,\\
G_j(0)=b_j^m-\sum_{i=1}^l\Big(a_{ji}^M L_j^f+\frac{p_{ji}^M L_j^g}{1-\tau_{ji}'^M}\Big)>0,\quad j=1,2,\dots,m.
\end{gathered}
$$

Since $F_i(\cdot)$ and $G_j(\cdot)$ are continuous on $[0,\infty)$ and $F_i(\varepsilon_i),G_j(\zeta_j)\to\infty$ as $\varepsilon_i,\zeta_j\to\infty$, there exist $\varepsilon_i^*,\zeta_j^*>0$ such that $F_i(\varepsilon_i)=0,G_j(\zeta_j)=0$ for $\varepsilon_i\in(0,\varepsilon_i^*)$ and $F_i(\varepsilon_i)>0,G_j(\zeta_j)>0$ for $\zeta_j\in(0,\zeta_j^*)$. By choosing

$$
\eta=\min\{\varepsilon_1^*,\varepsilon_2^*,\dots,\varepsilon_l^*,\zeta_1^*,\zeta_2^*,\dots,\zeta_m^*\},
$$

we obtain

$$
\begin{gathered}
F_i(\eta)=a_i^m-\eta-\sum_{j=1}^m\Big(b_{ij}^M L_i^{\hat f}+\frac{q_{ij}^M L_i^{\hat g}e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\Big)\ge 0,\quad i=1,2,\dots,l,\\
G_i(\eta)=b_j^m-\eta-\sum_{i=1}^l\Big(a_{ji}^M L_j^f+\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}\Big)\ge 0,\quad j=1,2,\dots,m.
\end{gathered}
$$

Now let us define

$$
\begin{gathered}
u_i(t)=e^{\eta t}|x_i(t)-x_i^*(t)|,\quad t\in[-\tau,\infty),i=1,2,\dots,l,\\
v_j(t)=e^{\eta t}|y_j(t)-y_j^*(t)|,\quad t\in[-\tau,\infty),j=1,2,\dots,m.
\end{gathered}
\tag{2.3}
$$

Then it follows from (2.2) and (2.3) that

$$\begin{aligned}\frac{\mathrm{d}^+u_i(t)}{\mathrm{d}t} &\le -(a_i^m-\eta)u_i(t)+\sum_{j=1}^m a_{ji}^M L_j^f v_j(t)+\sum_{j=1}^m p_{ji}^M L_j^g e^{\eta\tau_{ji}(t)}v_j(t-\tau_{ji}(t)),\\ \frac{\mathrm{d}^+v_j(t)}{\mathrm{d}t} &\le -(b_j^m-\eta)v_j(t)+\sum_{i=1}^n b_{ij}^M L_i^{\hat f} u_i(t)+\sum_{i=1}^n q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}(t)}u_i(t-\sigma_{ij}(t)),\end{aligned} \tag{2.4}$$

$i=1,2,\dots,n$, $j=1,2,\dots,m$. Consider a Lyapunov function defined by

$$\begin{aligned}V(t)=&\sum_{i=1}^l\Big(u_i(t)+\sum_{j=1}^m\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}\int_{t-\tau_{ji}(t)}^t v_j(s)\,\mathrm{d}s\Big)\\ &+\sum_{j=1}^m\Big(v_j(t)+\sum_{i=1}^l\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\int_{t-\sigma_{ij}(t)}^t u_i(s)\,\mathrm{d}s\Big),\end{aligned} \tag{2.5}$$

and we note that $V(t)>0$ for $t>0$ and $V(0)$ is positive and finite. Calculating the derivatives of V along the solutions of (2.4), we get

$$\begin{aligned}\frac{\mathrm{d}V^+}{\mathrm{d}t} &\le \sum_{i=1}^l\Big[-(a_i^m-\eta)u_i(t)+\sum_{j=1}^m a_{ji}^M L_j^f v_j(t)+\sum_{j=1}^m\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}v_j(t)\Big]\\ &\quad+\sum_{j=1}^m\Big[-(b_j^m-\eta)v_j(t)+\sum_{i=1}^l b_{ij}^M L_i^{\hat f}u_i(t)+\sum_{i=1}^l\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}u_i(t)\Big]\\ &\le -\sum_{i=1}^l\Big[\Big(a_i^m-\eta-\sum_{j=1}^m b_{ij}^M L_i^{\hat f}-\sum_{j=1}^m\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\Big)u_i(t)\Big]\\ &\quad-\sum_{j=1}^m\Big[\Big(b_j^m-\eta-\sum_{i=1}^l a_{ji}^M L_j^f-\sum_{i=1}^l\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}\Big)v_j(t)\Big]\\ &= -\sum_{i=1}^l F_i(\eta)u_i(t)-\sum_{j=1}^m G_j(\eta)v_j(t)\le 0,\quad t>0.\end{aligned}$$

It follows that $V(t)\le V(0)$ for $t>0$ and hence from (2.3) and (2.5) we obtain

$$\begin{aligned}\sum_{i=1}^l u_i(t)+\sum_{j=1}^m v_j(t) &\le \sum_{i=1}^l\Big(u_i(0)+\sum_{j=1}^m\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}\int_{-\tau_{ji}(0)}^0 v_j(s)\,\mathrm{d}s\Big)\\ &\quad+\sum_{j=1}^m\Big(v_j(0)+\sum_{i=1}^l\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\int_{-\sigma_{ij}(0)}^0 u_i(s)\,\mathrm{d}s\Big).\end{aligned} \tag{2.6}$$

It follows from (2.3) and (2.6) that

$$
\begin{aligned}
&\sum_{i=1}^{l}|x_i(t)-x_i^*(t)|+\sum_{j=1}^{m}|y_j(t)-y_j^*(t)| \\
&\le e^{-\eta t}\sum_{i=1}^{l}\Big(1+\sum_{j=1}^{m}\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M}\Big)\sup_{s\in[-\tau,0]}|x_i(s)-x_i^*(s)| \\
&\quad+\sum_{j=1}^{m}\Big(1+\sum_{i=1}^{l}\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\Big)\sup_{s\in[-\sigma,0]}|y_j(s)-y_j^*(s)| \\
&\le \Lambda e^{-\eta t}\Big[\sum_{i=1}^{l}\sup_{s\in[-\tau,0]}|x_i(s)-x_i^*(s)|+\sum_{j=1}^{m}\sup_{s\in[-\sigma,0]}|y_j(s)-y_j^*(s)|\Big],
\end{aligned} \tag{2.7}
$$

where $t>0$ and

$$
\Lambda=\max_{1\le i\le l,\,1\le j\le m}\Big\{1+\sum_{j=1}^{m}\frac{p_{ji}^M L_j^g e^{\eta\tau_{ji}^M}}{1-\tau_{ji}'^M},1+\sum_{i=1}^{l}\frac{q_{ij}^M L_i^{\hat g} e^{\eta\sigma_{ij}^M}}{1-\sigma_{ij}'^M}\Big\}\ge 1.
$$

The uniqueness of the periodic solution is follows from (2.7). This completes the proof. $\square$

3. Discrete-time analogues

In this section, we shall use a semi-discretization technique to obtain the discrete-time analogue of (1.2). For convenience, we use the following notations. Let $\mathbb{Z}$ denote the set of integers; $\mathbb{Z}_0^+=\{0,1,2,\dots\}$; $[a,b]_{\mathbb{Z}}=\{a,a+1,\dots,b-1,b\}$, where $a,b\in\mathbb{Z}, a\le b$; $[a,\infty)_{\mathbb{Z}}=\{a,a+1,a+2,\dots\}$, where $a\in\mathbb{Z}$. While there is no unique way of obtaining a discrete-time analogue from the continuous-time network (1.2), we begin by approximating the network (1.2) by equations with piecewise constant arguments of the form

$$
\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}&=-a_i\big(\big[\frac{t}{h}\big]h\big)x_i(t)+\sum_{j=1}^{m}a_{ji}\big(\big[\frac{t}{h}\big]h\big)f_j(y_j(t)) \\
&\quad+\sum_{j=1}^{m}p_{ji}\big(\big[\frac{t}{h}\big]h\big)g_j\Big(y_j\Big(\big(\big[\frac{t}{h}\big]h\big)-\tau_{ji}\big(\big[\frac{t}{h}\big]h\big)\Big)\Big)+I_i\big(\big[\frac{t}{h}\big]h\big), \\
\frac{\mathrm{d}y_j(t)}{\mathrm{d}t}&=-b_j\big(\big[\frac{t}{h}\big]h\big)y_j(t)+\sum_{i=1}^{n}b_{ij}\big(\big[\frac{t}{h}\big]h\big)\hat f_i(x_i(t)) \\
&\quad+\sum_{i=1}^{n}q_{ij}\big(\big[\frac{t}{h}\big]h\big)\hat g_i\Big(x_i\Big(\big(\big[\frac{t}{h}\big]h\big)-\sigma_{ij}\big(\big[\frac{t}{h}\big]h\big)\Big)\Big)+J_j\big(\big[\frac{t}{h}\big]h\big),
\end{aligned} \tag{3.1}
$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$, $t\in[nh,(n+1)h)$, $n\in\mathbb{Z}_0^+$ and h is a fixed positive real number denoting a uniform discretization step-size and $[r]$ denotes the integer part of $r\in\mathbb{R}$. We note that $[t/h]=n$ for $t\in[nh,(n+1)h)$. For convenience, we use the notations $a_i(n)=a_i(nh)$, $b_i(n)=b_i(nh)$, $a_{ji}(n)=a_{ji}(nh)$, $p_{ji}(n)=p_{ji}(nh)$, $b_{ij}(n)=b_{ij}(nh)$, $q_{ij}(n)=q_{ij}(nh)$, $\tau_{ji}(n)=\tau_{ji}(nh)$, $\sigma_{ij}(n)=\sigma_{ij}(nh)$, $I_i(n)=I_i(nh)$, $J_j(n)=J_j(nh)$, $x_i(n)=x_i(nh)$, and $y_j(n)=y_j(nh)$. With these preparations, (3.1) can be rewritten as

$$\begin{aligned}
&\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\\
&= -a_i(n)x_i(t) + \sum_{j=1}^{m} a_{ji}(n)f_j(y_j(n)) + \sum_{j=1}^{m} p_{ji}(t)g_j(y_j(n-\tau_{ji}(n))) + I_i(n),\\
&\frac{\mathrm{d}y_j(t)}{\mathrm{d}t}\\
&= -b_j(n)y_j(n) + \sum_{i=1}^{l} b_{ij}(n)\hat{f}_i(x_i(n)) + \sum_{i=1}^{l} q_{ij}(n)\hat{g}_i(x_i(n-\sigma_{ij}(n))) + J_j(n),
\end{aligned} \tag{3.2}$$

where $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$, $t \in [nh, (n+1)h)$, $n \in \mathbb{Z}_0^+$. The initial values of (3.2) will be given below in (3.6). Integrating (3.2) over the interval $[nh, t)$, where $t < (n+1)h$, we get

$$\begin{aligned}
x_i(t)e^{a_i(n)t} - x_i(n)e^{-a_i(n)nh} &= \Big(\frac{e^{a_i(n)t} - e^{a_i(n)nh}}{a_i(n)}\Big)\Big\{\sum_{j=1}^{m} a_{ji}(n)f_j(y_j(n))\\
&\quad + \sum_{j=1}^{m} p_{ji}(t)g_j(y_j(n-\tau_{ji}(n))) + I_i(n)\Big\},\\
y_j(t)e^{b_j(n)t} - y_j(n)e^{b_j(n)nh} &= \Big(\frac{e^{b_j(n)t} - e^{b_j(n)nh}}{b_j(n)}\Big)\Big\{\sum_{i=1}^{l} b_{ij}(n)\hat{f}_i(x_i(n))\\
&\quad + \sum_{i=1}^{l} q_{ij}(n)\hat{g}_i(x_i(n-\sigma_{ij}(n))) + J_j(n)\Big\},
\end{aligned} \tag{3.3}$$

$i = 1, 2, \dots, l$, $j = 1, 2, \dots, m$. By allowing $t \to (n+1)h$ in the above expression, we obtain

$$\begin{aligned}
x_i(n+1) &= x_i(n)e^{-a_i(n)h} + \alpha_i(h)\sum_{j=1}^{m} a_{ji}(n)f_j(y_j(n))\\
&\quad + \alpha_i(h)\sum_{j=1}^{m} p_{ji}(n)g_j(y_j(n-\tau_{ji}(n))) + \alpha_i(h)I_i(n), \quad i = 1, 2, \dots, l,\\
y_j(n+1) &= y_j(n)e^{-b_j(n)h} + \beta_j(h)\sum_{i=1}^{l} b_{ij}(n)\hat{f}_i(x_i(n))\\
&\quad + \beta_j(h)\sum_{i=1}^{l} q_{ij}(n)\hat{g}_i(x_i(n-\sigma_{ij}(n))) + \beta_j(h)J_j(n), \quad j = 1, 2, \dots, m,
\end{aligned} \tag{3.4}$$

where

$$\alpha_i(h) = \frac{1 - e^{-a_i(n)h}}{a_i(n)}, \beta_j(h) = \frac{1 - e^{-b_j(n)h}}{b_j(n)},$$

$i = 1, 2, \dots, l$, $i = 1, 2, \dots, m$, $n \in \mathbb{Z}_0^+$. It is not difficult to verify that $\alpha_i(h) > 0$, $\beta_i(h) > 0$ if $a_i, b_i, h > 0$ and $\alpha_i(h) \approx h + O(h^2)$, $\beta_i(h) \approx h + O(h^2)$ for small $h > 0$. Also, one can show that (3.4) converges towards (1.2) when $h \to 0^+$. In studying the discrete-time analogue (3.4), we assume that

$$\begin{gathered} h \in (0,\infty), \quad a_i, b_i : \mathbb{Z} \to (0,\infty), \quad a_{ji}, p_{ji}, b_{ij}, q_{ij}, \quad I_i : \mathbb{Z} \to \mathbb{R}, \\ \tau_{ji}, \sigma_{ij} : \mathbb{Z} \to \mathbb{Z}_0^+, \quad i = 1,2,\dots,l, \ j = 1,2,\dots,m \end{gathered} \tag{3.5}$$

and the nonlinear activation functions $f_j, g_j, \hat{f}_i, \hat{g}_i$ satisfy (S2) and (S3). The system (3.4) is supplemented with initial values given by

$$\begin{gathered} x_i(s) = \varphi_i(s), s \in [-\tau, 0]_{\mathbb{Z}}, \tau = \max_{1\le i\le l, 1\le j\le m} \sup\{\tau_{ji}(n), n \in \mathbb{Z}\}, i = 1,2,\dots,l, \\ y_j(s) = \psi_j(s), s \in [-\sigma, 0]_{\mathbb{Z}}, \sigma = \max_{1\le i\le l, 1\le j\le m} \sup\{\sigma_{ij}(n), n \in \mathbb{Z}\}, j = 1,2,\dots,m, \end{gathered} \tag{3.6}$$

where $\varphi_i(\cdot)$ and $\psi_j(\cdot)$ denote real-valued continuous functions defined on $[-\tau, 0]$ and $[-\sigma, 0]$, respectively.

In what follows, for convenience, we will use the following notation:

$$\begin{gathered} a_i^m = \min_{n\in[0,\omega-1]_{\mathbb{Z}}} \{a_i(n)\}, \quad b_j^m = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{b_j(n)\}, \\ a_{ji}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|a_{ji}(n)|\}, \quad b_{ij}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|b_{ij}(n)|\}, \\ p_{ji}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|p_{ji}(n)|\}, \quad q_{ij}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|q_{ij}(n)|\}, \\ \tau_{ji}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|\tau_{ji}(n)|\}, \quad \sigma_{ij}^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|\sigma_{ij}(n)|\}, \end{gathered}$$

for $i = 1,2,\dots,l$ and $j = 1,2,\dots,m$. Also define

$$\begin{gathered} a^M = \max_{1\le i\le l,\, 1\le j\le m} \{a_{ji}^M(n)\}, \quad b^M = \max_{1\le i\le l,\, 1\le j\le m} \{b_{ij}^M\}, \\ M_f = \sup_{u\in\mathbb{R}} \{|f_j(u)|, j = 1,2,\dots,m\}, \quad M_g = \sup_{u\in\mathbb{R}} \{|g_j(u)|, j = 1,2,\dots,m\}, \\ M_{\hat{f}} = \sup_{u\in\mathbb{R}} \{|\hat{f}_i(u)|, i = 1,2,\dots,l\}, \quad M_{\hat{g}} = \sup_{u\in\mathbb{R}} \{|\hat{g}_i(u)|, i = 1,2,\dots,l\}, \\ I^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|I_i(n)|, i = 1,2,\dots,l\}, \quad J^M = \max_{n\in[0,\omega-1]_{\mathbb{Z}}} \{|J_j(n)|, j = 1,2,\dots,m\}. \end{gathered}$$

The following result was given in [1, Lemma 3.2].

Lemma 3.1. *Let $f : \mathbb{Z} \to \mathbb{R}$ be ω-periodic, i.e., $f(k+\omega) = f(k)$. Then for any fixed $n_1, n_2 \in I_\omega$, and any $k \in \mathbb{Z}$, one has*

$$\begin{gathered} f(n) \le f(n_1) + \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|, \\ f(n) \ge f(n_2) - \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|. \end{gathered}$$

Using Lemma 2.1, we shall show the following result about the existence of at least one periodic solution of (3.4).

Theorem 3.2. *Assume that (S2), (S3) and* (3.5) *hold. Furthermore, assume that*

- (S4) $a_i, a_{ji}, p_{ji}, I_i, b_j, b_{ij}, q_{ij}, J_j, i = 1,2,\dots,l, j = 1,2,\dots,m$ *are all ω-periodic functions, where $\omega > 1$ is a positive integer.*
- (S5)
$$h \le \min_{1\le i\le l,\, 1\le j\le m} \Big\{ -\frac{1}{a_i^m} \ln\Big(\frac{\omega-1}{\omega}\Big), -\frac{1}{b_j^m} \ln\Big(\frac{\omega-1}{\omega}\Big) \Big\}.$$

Then system (3.4) *has at least one* ω*-periodic solution.*

Proof. Define
$$l_d = \{u = \{u(n)\} : u(n) \in \mathbb{R}^{l+m}, n \in \mathbb{Z}\}.$$
Let $l^\omega \subset l_d$ denotes the subspace of all ω periodic sequences equipped with the norm $||\cdot||$, i.e.,
$$||u|| = ||(u_1, u_2 \dots, u_{l+m})^T|| = \sum_{i=1}^{l+m} \max_{n\in[0,\omega-1]_{\mathbb{Z}}} |u_i(n)|,$$
where $u = \{(u_1(n), u_2(n), \dots, u_{l+m}(n)), n \in \mathbb{Z}\} \in l^\omega$. It is not difficult to show that l^ω is a finite-dimensional Banach space. Let
$$l_0^\omega = \{u = \{u(n)\} \in l^\omega : \sum_{n=0}^{\omega-1} u(n) = 0\},$$
$$l_c^\omega = \{u = \{u(n)\} \in l^\omega : u(n) = c \in \mathbb{R}^m, n \in \mathbb{Z}\}.$$
Then it is easy to check that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and
$$l^\omega = l_0^\omega \oplus l_c^\omega, \dim l_c^\omega = l + m.$$
Take $\mathbb{X} = \mathbb{Y} = l^\omega$ and let
$$N\begin{bmatrix} x_1 \\ \vdots, \\ x_l \\ y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1(n)(e^{-a_1(n)h} - 1) + \alpha_1(h)\sum_{j=1}^m a_{j1}(n) f_j(y_j(n)) \\ \vdots \\ x_l(n)(e^{-a_l(n)h} - 1) + \alpha_l(h)\sum_{j=1}^m a_{jl}(n) f_j(y_j(n)) \\ y_1(n)(e^{-b_1(n)h} - 1) + \beta_1(h)\sum_{i=1}^l b_{i1}(n) \hat{f}_i(x_i(n)) \\ \vdots \\ y_m(n)(e^{-b_m(n)h} - 1) + \beta_m(h)\sum_{i=1}^l b_{im}(n) \hat{f}_i(x_i(n)) \end{bmatrix}$$
$$+ \begin{bmatrix} \alpha_1(h)\sum_{j=1}^m p_{j1}(t) g_j(y_j(n - \tau_{j1}(n))) + \alpha_1(h) I_1(n) \\ \vdots \\ \alpha_l(h)\sum_{j=1}^m p_{jl}(t) g_j(y_j(n - \tau_{jl}(n))) + \alpha_l(h) I_l(n) \\ \beta_1(h)\sum_{i=1}^l q_{i1}(n) \hat{g}_i(x_i(n - \sigma_{i1}(n))) + \beta_1(h) J_1(n) \\ \vdots \\ \beta_m(h)\sum_{i=1}^l q_{im}(n) \hat{g}_i(x_i(n - \sigma_{im}(n))) + \beta_m(h) J_m(n) \end{bmatrix},$$
$x \in \mathbb{X}$, $n \in \mathbb{Z}$,
$$(Lu)(n) = u(n+1) - u(n), u \in \mathbb{X}, n \in \mathbb{Z}.$$
It is easy to see that L is a bounded linear operator with
$$\ker\ L = l_c^\omega, \operatorname{Im}\ L = l_0^\omega, \dim\ \ker\ L = l + m = \operatorname{codim} \operatorname{Im} L,$$
then it follows that L is a Fredholm mapping of index zero. Define
$$Pu = \frac{1}{\omega}\sum_{s=0}^{\omega-1} u(s), \quad u \in \mathbb{X}, \quad Qv = \frac{1}{\omega}\sum_{s=0}^{\omega-1} v(s), \quad v \in \mathbb{Y}.$$
It is not difficult to show that P and Q are continuous projectors such that
$$\operatorname{Im} P = \ker L, \quad \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \ker P \bigcap \operatorname{Dom} L$ exists, which is given by

$$K_P(y) = \sum_{s=0}^{\omega-1} y(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)y(s).$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since $\mathbb{X}$ is a finite-dimensional Banach space, one can easily show that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset \mathbb{X}$. Moreover, $QN(\bar{\Omega})$ is bounded, and hence N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset \mathbb{X}$. We now are in a position to search for an appropriate open, bounded subset $\Omega \subset \mathbb{X}$ for the continuation theorem.

Corresponding to operator equation $Lu = \lambda N u$, $\lambda \in (0,1)$, we have

$$\begin{aligned} x_i(n+1) - x_i(n) &= \lambda\Big\{ - x_i(n)(1-e^{-a_i(n)h}) + \alpha_i(h)\sum_{j=1}^{m} a_{ji}(n) f_j(y_j(n)) \\ &\quad + \alpha_i(h)\sum_{j=1}^{m} p_{ji}(n) g_j(y_j(n-\tau_{ji}(n))) + \alpha_i(h) I_i(n)\Big\}, \\ y_j(n+1) - y_j(n) &= \lambda\Big\{ - y_j(n)(1-e^{-b_j(n)h}) + \beta_j(h)\sum_{i=1}^{l} b_{ij}(n) \hat{f}_i(x_i(n)) \\ &\quad + \beta_j(h)\sum_{i=1}^{l} q_{ij}(n) \hat{g}_i(x_i(n-\sigma_{ij}(n))) + \beta_j(h) J_j(n)\Big\}, \end{aligned} \tag{3.7}$$

where $i = 1,2,\dots,l$, $j = 1,2,\dots,m$.

Suppose that $\{(x_1(n), x_2(n), \dots, x_l(n), y_1(n), y_2(n), \dots, y_m(n))^T\} \in \mathbb{X}$ is a solution of system (3.7) for a certain $\lambda \in (0,1)$. Summing on both sides of (3.7) from 0 to $\omega - 1$ with respect to n, we obtain

$$\begin{aligned} \sum_{s=0}^{\omega-1} x_i(s)(1-e^{-a_i(s)h}) &= \sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m} a_{ji}(s) f_j(y_j(s)) \\ &\quad + \sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m} p_{ji}(s) g_j(y_j(s-\tau_{ji}(s))) + \sum_{s=0}^{\omega-1}\alpha_i(h) I_i(s), \end{aligned} \tag{3.8}$$

for $i = 1,2,\dots,l$, and

$$\begin{aligned} \sum_{s=0}^{\omega-1} y_j(s)(1-e^{-b_j(s)h}) &= \sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l} b_{ij}(s) \hat{f}_i(x_i(s)) \\ &\quad + \sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l} q_{ij}(s) \hat{g}_i(x_i(s-\sigma_{ij}(s))) + \sum_{s=0}^{\omega-1}\beta_j(h) J_j(s), \end{aligned} \tag{3.9}$$

for $j = 1,2,\dots,m$. Let $n_i, \bar{n}_j \in [0, \omega-1]_{\mathbb{Z}}$ such that

$$x_i(n_i) = \max_{n\in[0,\omega-1]_{\mathbb{Z}}}\{x_i(n)\}, \quad y_j(\bar{n}_j) = \max_{n\in[0,\omega-1]_{\mathbb{Z}}}\{y_j(n)\},$$

for $i = 1,2,\dots,l$, $j = 1,2,\dots,m$, then it follows from (3.8) and (3.9) that

$$x_i(n_i)\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h}) \ge \sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m} a_{ji}(s) f_j(y_j(s))$$

$$+\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}p_{ji}(s)g_j(y_j(s-\tau_{ji}(s)))+\sum_{s=0}^{\omega-1}\alpha_i(h)I_i(s),$$

for $i=1,2,\dots,l$ and

$$\begin{aligned}
y_j(\bar{n}_j)\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h}) &\geq \sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}b_{ij}(s)\hat{f}_i(x_i(s))\\
&\quad+\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}q_{ij}(s)\hat{g}_i(x_i(s-\sigma_{ij}(s)))+\sum_{s=0}^{\omega-1}\beta_j(h)J_j(s),
\end{aligned}$$

for $j=1,2,\dots,m$. Hence

$$\begin{aligned}
x_i(n_i) &\geq \Big[\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\Big]^{-1}\Big[\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}a_{ji}(s)f_j(y_j(s))\\
&\quad+\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}p_{ji}(s)g_j(y_j(s-\tau_{ji}(s)))+\sum_{s=0}^{\omega-1}\alpha_i(h)I_i(s)\Big]\\
&\geq -\Big[\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\Big]^{-1}\Big[\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}|a_{ji}(s)||f_j(y_j(s))|\\
&\quad+\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}|p_{ji}(s)||g_j(y_j(s-\tau_{ji}(s)))|+\sum_{s=0}^{\omega-1}\alpha_i(h)|I_i(s)|\Big]\\
&\geq -\alpha_i(h)\omega\Big[\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\Big]^{-1}[ma^M M_f+mp^M M_g+I^M]\\
&:= -A_i,\quad i=1,2,\dots,l
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
y_j(\bar{n}_j) &\geq \Big[\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h})\Big]^{-1}\Big[\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}b_{ij}(s)\hat{f}_i(x_i(s))\\
&\quad+\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}q_{ij}(s)\hat{g}_i(x_i(s-\sigma_{ij}(s)))+\sum_{s=0}^{\omega-1}\beta_j(h)J_j(s)\Big]\\
&\geq -\beta_j(h)\omega\Big[\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h})\Big]^{-1}[lb^M M_{\hat{f}}+lq^M M_{\hat{g}}+J^M]\\
&:= -B_j,\quad j=1,2,\dots,m.
\end{aligned}\tag{3.11}$$

Let $n_i^*,\underline{n}_j^*\in[0,\omega-1]_{\mathbb{Z}}$ such that

$$x_i(n_i^*)=\min_{n\in[0,\omega-1]_Z}\{x_i(n)\},\quad y_j(\underline{n}_j^*)=\min_{n\in[0,\omega-1]_Z}\{y_j(n)\},$$

for $i=1,2,\dots,l$, $j=1,2,\dots,m$. Again, it follows from (3.8) and (3.9) that

$$x_i(n_i^*)\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\leq\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}a_{ji}(s)f_j(y_j(s))$$

$$+\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}p_{ji}(s)g_j(y_j(s-\tau_{ji}(s)))+\sum_{s=0}^{\omega-1}\alpha_i(h)I_i(s),$$

for $i=1,2,\dots,l$ and

$$\begin{aligned}y_j(\underline{n}_j^*)\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h}) &\le \sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}b_{ij}(s)\hat{f}_i(x_i(s))\\ &\quad+\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}q_{ij}(s)\hat{g}_i(x_i(s-\sigma_{ij}(s)))+\sum_{s=0}^{\omega-1}\beta_j(h)J_j(s),\end{aligned}$$

for $j=1,2,\dots,m$. Hence

$$\begin{aligned}x_i(n_i^*) &\le \Big[\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\Big]^{-1}\Big[\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}a_{ji}(s)f_j(y_j(s))\\ &\quad+\sum_{s=0}^{\omega-1}\alpha_i(h)\sum_{j=1}^{m}p_{ji}(s)g_j(y_j(s-\tau_{ji}(s)))+\sum_{s=0}^{\omega-1}\alpha_i(h)I_i(s)\Big]\\ &\le \omega\alpha_i(h)\Big[\sum_{s=0}^{\omega-1}(1-e^{-a_i(s)h})\Big]^{-1}[ma^MM_f+mp^MM_g+I^M]\\ &= A_i,\quad i=1,2,\dots,l\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}y_j(\underline{n}_j^*) &\le \Big[\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h})\Big]^{-1}\Big[\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}b_{ij}(s)\hat{f}_i(x_i(s))\\ &\quad+\sum_{s=0}^{\omega-1}\beta_j(h)\sum_{i=1}^{l}q_{ij}(s)\hat{g}_i(x_i(s-\sigma_{ij}(s)))+\sum_{s=0}^{\omega-1}\beta_j(h)J_j(s)\Big]\\ &\le \omega\beta_j(h)\Big[\sum_{s=0}^{\omega-1}(1-e^{-b_j(s)h})\Big]^{-1}[lb^MM_{\hat{f}}+lq^MM_{\hat{g}}+M_J]\\ &= B_j,\quad j=1,2,\dots,m.\end{aligned}\tag{3.13}$$

According to (3.7), we have

$$\begin{aligned}|x_i(n+1)-x_i(n)| &\le \lambda\Big\{|x_i(n)|(1-e^{-a_i(n)h})+\alpha_i(h)\sum_{j=1}^{m}|a_{ji}(n)||f_j(y_j(n))|\\ &\quad+\alpha_i(h)\sum_{j=1}^{m}|p_{ji}(n)||g_j(y_j(n-\tau_{ji}(n)))|+\alpha_i(h)|I_i(n)|\Big\}\\ &\le |x_i(n)|(1-e^{-a_i^mh})+\alpha_i(h)[ma^MM_f+mp^MM_g+I^M]\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}|y_j(n+1)-y_j(n)| &\le \lambda\Big\{|y_j(n)|(1-e^{-b_j(n)h})+\beta_j(h)\sum_{i=1}^{l}|b_{ij}(n)||\hat{f}_i(x_i(n))|\\ &\quad+\beta_j(h)\sum_{i=1}^{l}|q_{ij}(n)||\hat{g}_i(x_i(n-\sigma_{ij}(n)))|+\beta_j(h)|J_j(n)|\Big\}\\ &\le |y_j(n)|(1-e^{-b_i^m h})+\beta_j(h)[lb^M M_{\hat{f}}+lp^M M_{\hat{g}}+J^M],\end{aligned} \tag{3.15}$$

where $i=1,2,\dots,l$, $j=1,2,\dots,m$. It follows from (3.10), (3.12), (3.14) and Lemma 3.1 that for $i=1,2,\dots,l$,

$$\begin{aligned}x_i(n) &\le x_i(n_i^*)+\sum_{s=0}^{\omega-1}|x_i(s+1)-x_i(s)|\\ &\le A_i+(1-e^{-a_i^m h})\sum_{s=0}^{\omega-1}|x_i(s)|+\omega\alpha_i(h)[ma^M M_f+mp^M M_g+I^M]\end{aligned}$$

and

$$\begin{aligned}x_i(n) &\ge x_i(n_i)-\sum_{s=0}^{\omega-1}|x_i(s+1)-x_i(s)|\\ &\ge -A_i-(1-e^{-a_i^m h})\sum_{s=0}^{\omega-1}|x_i(s)|-\omega\alpha_i(h)[ma^M M_f+mp^M M_g+I^M].\end{aligned}$$

Thus,

$$|x_i(n)|\le A_i+(1-e^{-a_i^m h})\sum_{s=0}^{\omega-1}|x_i(s)|+\omega\alpha_i(h)[ma^M M_f+mp^M M_g+I^M] \tag{3.16}$$

for $i=1,2,\dots,l$. Summing on both sides of (3.16) from 0 to $\omega-1$ with respect to n, we obtain

$$\begin{aligned}&\sum_{s=0}^{\omega-1}|x_i(s)|\\ &\le \omega A_i+\omega(1-e^{-a_i^m h})\sum_{s=0}^{\omega-1}|x_i(s)|+\omega^2\alpha_i(h)[ma^M M_f+mp^M M_g+I^M]\end{aligned}$$

for $i=1,2,\dots,l$. Since $0<h\le -\frac{1}{a_i^m}\ln\left(\frac{\omega-1}{\omega}\right)$ for $i=1,2,\dots,l$, we have

$$\begin{aligned}&\sum_{s=0}^{\omega-1}|x_i(s)|\\ &\le [1-\omega(1-e^{-a_i^m h})]^{-1}\Big[\omega A_i+\omega^2\alpha_i(h)(ma^M M_f+mp^M M_g+I^M)\Big]\\ &:= C_i,\quad i=1,2,\dots,l.\end{aligned} \tag{3.17}$$

In view of (3.16) and (3.17), we get

$$|x_i(n)|\le A_i+(1-e^{-a_i^m h})C_i+\omega\alpha_i(h)[ma^M M_f+mp^M M_g+I^M] := E_i,$$

for $i = 1, 2, \dots, l$. Similarly, it follows from (3.11), (3.13), (3.15) and Lemma 3.1 that

$$|y_j(n)| \le B_i + (1 - e^{-b_j^m h})D_j + \omega\beta_j(h)[lb^M M_{\hat{f}} + lq^M M_{\hat{g}} + J^M] := F_j,$$

for $j = 1, 2, \dots, m$. where

$$D_j = [1 - \omega(1 - e^{-b_j^m h})]^{-1}\Big[\omega B_j + \omega^2\beta_j(h)(lb^M M_{\hat{f}} + lq^M M_{\hat{g}} + J^M)\Big],$$

for $j = 1, 2, \dots, m$. Denote $C = \sum_{i=1}^{l} E_i + \sum_{j=1}^{m} F_j + H$, where $H > 0$ is a constant. Clearly, C is independent of λ. Now we take $\Omega = \{x \in \mathbb{X} : ||x|| < C\}$. The rest of the proof is similar to that of the proof of Theorem 2.2 and will be omitted. The proof is complete. □

Theorem 3.3. *Assume that (S2)-(S3) and* (3.5) *hold. Furthermore, assume that* $\tau_{ji}(n) \equiv \tau_{ji}$, $\sigma_{ij}(n) \equiv \sigma_{ij} \in Z^+$, $n \in Z$, $i, j = 1, 2, \dots, m$ *are constants and*

(P3)

$$a_i^m > \sum_{j=1}^{m}\Big(b_{ij}^M L_i^{\hat{f}} + q_{ij}^M L_j^{\hat{g}}\Big), \quad i = 1, 2, \dots, l,$$
$$b_j^m > \sum_{i=1}^{l}\Big(a_{ji}^M L_j^{f} + p_{ji}^M L_i^{g}\Big), \quad j = 1, 2, \dots, m,$$

then the ω*-periodic solution* $\{(x_1^*(n), x_2^*(n), \dots, x_l^*(n), y_1^*(n), y_2^*(n), \dots, y_m^*(n))^T\}$ *of* (3.4) *is unique and is globally exponentially stable in the sense that there exist constants* $\lambda > 1$ *and* $\delta \ge 1$ *such that*

$$\begin{aligned} &\sum_{i=1}^{l}|x_i(n) - x_i^*(n)| + \sum_{i=1}^{m}|y_j(n) - y_j^*(n)| \\ &\le \delta\Big(\frac{1}{\lambda}\Big)^n\Big\{\sum_{i=1}^{l}\sup_{s\in[-\tau,0]_Z}|x_i(s) - x_i^*(s)| + \sum_{j=1}^{m}\sup_{s\in[-\sigma,0]_Z}|y_j(s) - y_j^*(s)|\Big\}, \end{aligned} \tag{3.18}$$

for $n \in \mathbb{Z}^+$.

Proof. Let $x(n) = \{(x_1(n), x_2(n), \dots, x_l(n), y_1(n), y_2(n), \dots, y_m(n))^T\}$ be an arbitrary solution of (3.4), and

$$x^*(n) = \{(x_1^*(n), x_2^*(n), \dots, x_m^*(n), y_1^*(n), y_2^*(n), \dots, y_m^*(n))^T\}$$

be an ω-periodic solution of (3.4). Then

$$
\begin{aligned}
&|x_i(n+1)-x_i^*(n+1)| \\
&\le |x_i(n)-x_i^*(n)|e^{-a_i^m h}+\alpha_i(h)\sum_{j=1}^m a_{ji}^M L_j^f |y_j(n)-y_j^*(n)| \\
&\quad + \alpha_i(h)\sum_{j=1}^m p_{ji}^M L_j^g |y_j(n-\tau_{ji}(n)))-y_j^*(n-\tau_{ji}(n)))|, \\
&|y_j(n+1)-y_j^*(n+1)| \le |y_j(n)-y_j^*(n)|e^{-b_j^m h}+\beta_j(h)\sum_{i=1}^l b_{ij}^M L_i^{\hat{f}} |x_i(n)-x_i^*(n)| \\
&\quad + \beta_j(h)\sum_{i=1}^l q_{ij}^M L_i^{\hat{g}} |x_i(n-\sigma_{ij}(n))-x_i^*(n-\sigma_{ij}(n))|,
\end{aligned}
\tag{3.19}
$$

where $i=1,2,\dots,l$, $j=1,2,\dots,m$. Now we consider functions $\Gamma_i(\cdot,\cdot)$ and $\Theta_j(\cdot,\cdot)$, $i=1,2,\dots,l$, $j=1,2,\dots,m$, defined by

$$
\Gamma_i(\mu_i,n)=1-\mu_i e^{-a_i(n)h}-\mu_i\alpha_i(h)\sum_{j=1}^m b_{ij}^M L_i^{\hat{f}}-\sum_{j=1}^m q_{ij}^M L_j^{\hat{g}}\theta_i(h)\mu_i^{\sigma_{ij}^M+1}
$$

$$
\Theta_j(\nu_j,n)=1-\nu_j e^{-b_j(n)h}-\nu_j\alpha_j(h)\sum_{i=1}^l a_{ji}^M L_j^f-\sum_{i=1}^l p_{ji}^M L_j^g\theta_j(h)\nu_j^{\tau_{ji}^M+1},
$$

where $\mu_i,\nu_j\in[1,\infty)$, $n\in[0,\omega-1]_{\mathbb{Z}}$, $i=1,2,\dots,l$, $j=1,2,\dots,m$. Since

$$
\begin{aligned}
\Gamma_i(1,n)&=1-e^{-a_i(n)h}-\alpha_i(h)\sum_{j=1}^m b_{ij}^M L_i^{\hat{f}}-\alpha_i(h)\sum_{j=1}^m q_{ij}^M L_j^{\hat{g}} \\
&=\alpha_i(h)\Big[a_i(n)-\sum_{j=1}^m b_{ij}^M L_i^{\hat{f}}-\sum_{j=1}^m q_{ij}^M L_j^{\hat{g}}\Big] \\
&\ge \alpha_i(h)\Big[a_i^m-\sum_{j=1}^m b_{ij}^M L_i^{\hat{f}}-\sum_{j=1}^m q_{ij}^M L_j^{\hat{g}}\Big]>0,
\end{aligned}
$$

for $n\in[0,\omega-1]_{\mathbb{Z}}$, $i=1,2,\dots,l$, and

$$
\begin{aligned}
\Theta_j(1,n)&=1-e^{-b_j(n)h}-\beta_j(h)\sum_{i=1}^l a_{ji}^M L_j^f-\beta_j(h)\sum_{i=1}^l p_{ji}^M L_i^g \\
&=\beta_j(h)\Big[b_j(n)-\sum_{i=1}^l a_{ji}^M L_j^f-\sum_{i=1}^l p_{ji}^M L_i^g\Big] \\
&\ge \beta_j(h)\Big[b_j^m-\sum_{i=1}^l a_{ji}^M L_j^f-\sum_{i=1}^l p_{ji}^M L_i^g\Big]>0,
\end{aligned}
$$

for $n\in[0,\omega-1]_{\mathbb{Z}}$, $j=1,2,\dots,m$. Using the continuity of $\Gamma_i(\mu_i,n)$ and $\Theta_j(\nu_j,n)$ on $[1,\infty)$ with respect to μ_i and ν_j, respectively, for every $n\in[0,\omega-1]_{\mathbb{Z}}$ and the fact that $\Gamma_i(\mu_i,n)\to-\infty$ as $\mu_i\to\infty$ and $\Theta_j(\nu_j,n)\to-\infty$ as $\nu_j\to\infty$ uniformly in $n\in[0,\omega-1]_{\mathbb{Z}}$, $i=1,2,\dots,l$, $j=1,2,\dots,m$, we see that there exist $\nu_i^*(n),\nu_j^*(n)\in(1,\infty)$ such that $\Gamma_i(\mu_i^*(n),n)=0$ and $\Theta_j(\nu_j^*(n),n)=0$ for $n\in$

$[0,\omega-1]_{\mathbb{Z}}$, $i=1,2,\dots,l$, $j=1,2,\dots,m$. By choosing $\lambda=\min\{\mu_i^*(n),\nu_j^*(n),n\in[0,\omega-1]_{\mathbb{Z}},i=1,2,\dots,l,j=1,2,\dots,m\}$, where $\lambda>1$, we obtain $\Gamma_i(\lambda,n)\ge 0$ and $\Theta_j(\lambda,n)\ge 0$ for all $n\in[0,\omega-1]_{\mathbb{Z}}$, $i=1,2,\dots,l$, $j=1,2,\dots,m$, that is,

$$\lambda e^{-a_i(n)h}+\lambda\alpha_i(h)\sum_{j=1}^m b_{ij}^M L_i^{\hat f}+\sum_{j=1}^m q_{ij}^M L_j^{\hat g}\theta_i(h)\lambda^{\sigma_{ij}^M+1}\le 1,\ n\in[0,\omega-1]_{\mathbb{Z}},$$
$$\lambda e^{-b_j(n)h}+\lambda\alpha_j(h)\sum_{i=1}^l a_{ji}^M L_j^f+\sum_{i=1}^l p_{ji}^M L_j^g\theta_j(h)\lambda^{\tau_{ji}^M+1}\le 1,\ n\in[0,\omega-1]_{\mathbb{Z}},$$

for $i=1,2,\dots,l$, $j=1,2,\dots,m$. Hence

$$\begin{gathered}\lambda e^{-a_i^m h}+\lambda\alpha_i(h)\sum_{j=1}^m b_{ij}^M L_i^{\hat f}+\sum_{j=1}^m q_{ij}^M L_j^{\hat g}\theta_i(h)\lambda^{\sigma_{ij}^M+1}\le 1,\\ \lambda e^{-b_j^m h}+\lambda\alpha_j(h)\sum_{i=1}^l a_{ji}^M L_j^f+\sum_{i=1}^l p_{ji}^M L_j^g\theta_j(h)\lambda^{\tau_{ji}^M+1}\le 1,\end{gathered}\tag{3.20}$$

for $i=1,2,\dots,l$, $j=1,2,\dots,m$. Now let us consider

$$\begin{aligned}u_i(n)&=\lambda^n\frac{|x_i(n)-x_i^*(n)|}{\alpha_i(h)},n\in[-\tau,\infty)_{\mathbb{Z}},\quad i=1,2,\dots,l,\\ v_j(n)&=\lambda^n\frac{|y_j(n)-y_j^*(n)|}{\beta_j(h)},n\in[-\tau,\infty)_{\mathbb{Z}},\quad j=1,2,\dots,m.\end{aligned}\tag{3.21}$$

Using (3.4) and (3.21), we derive that

$$\begin{aligned}\Delta u_i(n)&\le-(1-\lambda e^{-a_i^m h})u_i(n)+\lambda\sum_{j=1}^m\beta_j(h)a_{ji}^M L_j^f v_j(n)\\ &\quad+\sum_{j=1}^m\beta_j(h)p_{ji}^M L_j^g\lambda^{\tau_{ji}+1}v_j(n-\tau_{ji}),\quad i=1,2,\dots,l,\\ \Delta v_j(n)&\le-(1-\lambda e^{-b_j^m h})v_j(n)+\lambda\sum_{i=1}^l\alpha_i(h)b_{ij}^M L_i^{\hat f}u_i(n)\\ &\quad+\sum_{i=1}^l\alpha_i(h)q_{ij}^M L_i^{\hat g}\lambda^{\sigma_{ij}+1}u_i(n-\sigma_{ij}),\quad j=1,2,\dots,m.\end{aligned}\tag{3.22}$$

We consider the Lyapunov function

$$\begin{aligned}V(n)&=\sum_{i=1}^l\Big(u_i(n)+\sum_{j=1}^m\beta_j(h)p_{ji}^M L_j^g\lambda^{\tau_{ji}^M+1}\sum_{s=n-\tau_{ji}}^{n-1}v_j(s)\Big)\\ &\quad+\sum_{j=1}^m\Big(v_i(n)+\sum_{i=1}^l\alpha_i(h)q_{ij}^M L_i^{\hat g}\lambda^{\sigma_{ij}^M+1}\sum_{s=n-\sigma_{ij}}^{n-1}u_i(s)\Big).\end{aligned}\tag{3.23}$$

Calculating the difference $\Delta V(n)=V(n+1)-V(n)$ along (3.22), we obtain

$$\Delta V(n)\le-\sum_{i=1}^l\Big((1-\lambda e^{-a_i^m h})u_i(n)-\lambda\sum_{j=1}^m\beta_j(h)a_{ji}^M L_j^f v_j(n)$$

$$
\begin{aligned}
&- \sum_{j=1}^{m} \beta_j(h) p_{ji}^M L_j^g \lambda^{\tau_{ji}^M+1} v_j(n)\Big) - \sum_{j=1}^{m} \Big((1-\lambda e^{-b_j^m h}) v_j(n) \\
&- \lambda \sum_{i=1}^{l} \alpha_i(h) b_{ij}^M L_i^{\hat f} u_i(n) - \sum_{i=1}^{l} \alpha_i(h) q_{ij}^M L_j^{\hat g} \lambda^{\sigma_{ij}^M+1} u_i(n)\Big) \\
= &- \sum_{i=1}^{l} \Big(1 - \lambda e^{-a_i^m h} - \lambda \alpha_i(h) \sum_{j=1}^{m} b_{ij}^M L_i^{\hat f} - \alpha_i(h) \sum_{j=1}^{m} q_{ij}^M L_j^{\hat g} \lambda^{\sigma_{ij}^M+1}\Big) u_i(n) \\
&- \sum_{j=1}^{m} \Big(1 - \lambda e^{-b_j^m h} - \lambda \beta_j(h) \sum_{i=1}^{l} a_{ji}^M L_j^f - \beta_j(h) \sum_{i=1}^{l} p_{ji}^M L_j^g \lambda^{\tau_{ji}^M+1}\Big) v_j(n).
\end{aligned}
$$

Using (3.20) in the above we deduce that $\Delta V(n) \le 0$ for $n \in \mathbb{Z}_0^+$. From this result and (3.23) it follows that

$$
\sum_{i=1}^{l} u_i(n) + \sum_{j=1}^{m} v_j(n) \le V(n) \le V(0) \quad \text{for } n \in \mathbb{Z}^+. \tag{3.24}
$$

Thus

$$
\begin{aligned}
&\sum_{i=1}^{l} u_i(n) + \sum_{j=1}^{m} v_j(n) \\
&= \lambda^n \sum_{i=1}^{l} \frac{|x_i(n) - x_i^*(n)|}{\alpha_i(h)} + \lambda^n \sum_{j=1}^{m} \frac{|y_i(n) - y_i^*(n)|}{\beta_j(h)} \\
&\le \sum_{i=1}^{l} \Big(u_i(0) + \sum_{j=1}^{m} \beta_j(h) p_{ji}^M L_j^g \lambda^{\tau_{ji}^M+1} \sum_{s=-\tau_{ji}}^{-1} v_j(s)\Big) \\
&\quad + \sum_{j=1}^{m} \Big(v_i(0) + \sum_{i=1}^{l} \alpha_i(h) q_{ij}^M L_i^{\hat g} \lambda^{\sigma_{ij}^M+1} \sum_{s=-\sigma_{ij}}^{-1} u_i(s)\Big) \\
&\le \sum_{i=1}^{l} \Big(1 + \alpha_i(h) \sum_{j=1}^{m} q_{ij}^M L_i^{\hat g} \lambda^{\sigma_{ij}^M+1} \sigma_{ij}\Big) \frac{1}{\alpha_i(h)} \sup_{s\in[-\sigma,0]} \{|x_i(s) - x_i^*(s)|\} \\
&\quad + \sum_{j=1}^{m} \Big(1 + \beta_j(h) \sum_{i=1}^{l} p_{ji}^M L_j^g \lambda^{\tau_{ji}^M+1} \tau_{ji}\Big) \frac{1}{\beta_j(h)} \sup_{s\in[-\tau,0]} \{|y_j(s) - y_j^*(s)|\}.
\end{aligned}
$$

Therefore, we obtain the assertion (3.18), where

$$
\delta = \frac{\max\{\max_{1\le i\le l} \alpha_i(h), \max_{1\le j\le m} \beta_j(h)\}}{\min\{\min_{1\le i\le l} \alpha_i(h), \min_{1\le j\le m} \beta_j(h)\}} \sigma \ge 1
$$

and

$$
\sigma = \max_{1\le i\le l,\, 1\le j\le m} \Big\{1 + \alpha_i(h) \sum_{j=1}^{m} q_{ij}^M L_i^{\hat g} \lambda^{\sigma_{ij}^M+1} \sigma_{ij}, 1 + \beta_j(h) \sum_{i=1}^{l} p_{ji}^M L_j^g \lambda^{\tau_{ji}^M+1} \tau_{ji}\Big\} \ge 1.
$$

We conclude from (3.18) that the unique periodic solution of (3.4) is globally exponentially stable and this completes the proof. $\square$

4. An Example

Consider the following BAM neural networks system with discrete delays

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= -a_i(t)x_i(t) + \sum_{j=1}^{2} p_{ji}(t)f_j(y_j(t-\tau_{ji})) + I_i(t), \quad i = 1,2, \\ \frac{\mathrm{d}y_j}{\mathrm{d}t} &= -b_j(t)y_j(t) + \sum_{i=1}^{2} q_{ij}(t)g_i(x_i(t-\sigma_{ij})) + J_j(t), \quad j = 1,2, \end{aligned} \tag{4.1}$$

in which for $i,j = 1,2$, $f_j(u) = \arctan(u+j)$, $g_i(u) = \frac{u}{(i+u^2)}$, $a_i(t) = 5(\cos t + 2)$, $b_j(t) = \sin t + 4$, $p_{ji}(t) = -\sin(i+j)t$, $q_{ij}(t) = \frac{i}{j}\sin t$, τ_{ji}, σ_{ij} are positive constants, $I_i(t), J_j(t)$ are any continuous 2π-periodic functions. Then it is easy to show that (4.1) satisfies all the conditions of Theorem 2.3, hence (4.1) has a unique 2π-periodic solution which is globally exponential stable.

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