*Electronic Journal of Differential Equations*, Vol. 2006(2006), No. 72, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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# STABILITY OF SOLITARY WAVE SOLUTIONS FOR EQUATIONS OF SHORT AND LONG DISPERSIVE WAVES

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Abstract. In this paper, we consider the existence and stability of a novel set of solitary-wave solutions for two models of short and long dispersive waves in a two layer fluid. We prove the existence of solitary waves via the Concentration Compactness Method. We then introduce the sets of solitary waves obtained through our analysis for each model and we show that them are stable provided the associated action is strictly convex. We also establish the existence of intervals of convexity for each associated action. Our analysis does not depend of spectral conditions.

## 1. Introduction

We study the existence and stability of solitary-wave solutions through of an analysis of type variational for two models of interaction between long waves and short waves under a weakly coupled nonlinearity in a two layer fluid and under the setting of deep and shallow flows. When the fluid depth of the lower layer is sufficiently large, in comparison with the wavelength of the internal wave, and the fluids have different densities, we have the following nonlinear coupled system (see Funakoshi and Oikawa [7])

$$\begin{gathered} iu_t + u_{xx} = \alpha vu, \\ v_t + \gamma D v_x = \beta(|u|^2)_x, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{gathered} \tag{1.1}$$

where $u = u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ denotes the short wave term and $v = v(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the long wave term. Here, $\alpha, \beta$ are positive constants, $\gamma \in \mathbb{R}$, and $D = \mathcal{H}\partial_x$ is a linear differential operator representing the dispersion of the internal wave, where $\mathcal{H}$ denotes the Hilbert transform defined by

$$\mathcal{H}f(x) = p.v.\,\frac{1}{\pi}\int \frac{f(y)}{x-y}\,dy.$$

2000 *Mathematics Subject Classification.* 35Q35, 35Q53, 35Q55, 35B35, 58E30, 76B15.
*Key words and phrases.* Dispersive wave equations; variational methods; stability; solitary wave solutions.
©2006 Texas State University - San Marcos.
Submitted May 30, 2005. Published July 10, 2006.
Supported by grant 300654/96-0 from CNPq Brazil.

When the fluid depth is sufficiently small in comparison with the wavelength of the internal wave, the model describing the interaction takes the form (see [7, 8]

$$\begin{gathered} iu_t + u_{xx} = \alpha v u \\ v_t + \gamma v_x + \eta v_{xxx} + \mu v v_x = \beta(|u|^2)_x, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{gathered} \tag{1.2}$$

where $\alpha, \gamma, \eta, \mu, \beta \in \mathbb{R}$. Equation (1.2) is sometimes called *the coupled Schrödinger - Korteweg - de Vries equation* (Schrödinger-KdV equation henceforth).

One of the interesting features of wave equations of the form (1.1) or (1.2) is that due to nonlinearity and dispersion they often possess solitary-wave solutions. Solitary waves for (1.1) or (1.2) are travelling-wave solutions of the form

$$\begin{gathered} u(x,t) = e^{i\omega t} e^{ic(x-ct)/2} \phi(x-ct), \\ v(x,t) = \psi(x-ct), \end{gathered} \tag{1.3}$$

where $\omega, c \in \mathbb{R}$ and $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are typically smooth functions such that for each $n \in \mathbb{N}$, $\phi^{(n)}(\xi) \to 0$ and $\psi^{(n)}(\xi) \to 0$, as $|\xi| \to \infty$. We will see more later that the existence of solitary-wave solutions plays a distinguished role in the long-time evolution of solutions of (1.1) or (1.2). Substituting (1.3) in model (1.1) it follows immediately that $(\phi, \psi)$ satisfies the pseudo-differential system

$$\begin{gathered} \phi'' - \sigma\phi = \alpha\, \psi\phi \\ \gamma \mathcal{H} \psi' - c\psi = \beta \phi^2. \end{gathered} \tag{1.4}$$

Similarly for model (1.2) we have that $(\phi, \psi)$ satisfies the differential system

$$\begin{gathered} \phi'' - \sigma\phi = \alpha\, \psi\phi \\ \eta\psi'' - (c-\gamma)\psi + \frac{\mu}{2}\psi^2 = \beta\phi^2, \end{gathered} \tag{1.5}$$

where in (1.4) and (1.5), "$'$" $= \frac{d}{d\xi}$, with $\xi = x - ct$ and $\sigma = \omega - \frac{c^2}{4}$.

Next we establish some results known about the models (1.1) and (1.2). System (1.1) has been considered under various settings. For example, Funakoshi and Oikawa [7]) have computed numerically solitary-wave solutions for (1.4). Recently, Angulo and Montenegro [2] have proved the existence of even solitary-wave solutions using the Concentration Compactness Method and the theory of symmetric decreasing rearrangements. We recall that explicitly solutions for (1.4) are not know for $\gamma \neq 0$. With regard to the initial value problem, Bekiranov, Ogawa and Ponce [3] proved a well-posedness theory for (1.1) in $H^s_{\mathbb{C}}(\mathbb{R}) \times H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R})$. More precisely, if $|\gamma| < 1$ and $s \geq 0$, then for any $(u_0, v_0) \in H^s_{\mathbb{C}}(\mathbb{R}) \times H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R})$ there exists $T > 0$ such that the initial value problem (1.1) admits a unique solution $(u(t), v(t)) \in C([0,T]; H^s_{\mathbb{C}}(\mathbb{R})) \times C([0,T]; H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R}))$. Moreover, for $T > 0$ the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $H^s_{\mathbb{C}}(\mathbb{R}) \times H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R})$ to $C([0,T]; H^s_{\mathbb{C}}(\mathbb{R})) \times C([0,T]; H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R}))$. For the case $|\gamma| = 1$, we get the same results as above, but for $s > 0$. We note that as a consequence of the conservation laws (1.14) and (1.16) below, we can take $T = +\infty$ if $s \geq 1$ and $\gamma < 0$.

For the system (1.2) we have the following results of well-posedness. Tsutsumi [17] showed a global well-posedness theory in $H^{m+\frac{1}{2}}_{\mathbb{C}}(\mathbb{R}) \times H^m_{\mathbb{R}}(\mathbb{R})$ for $m = 1, 2, 3 \dots$, Bekiranov, Ogawa and Ponce [5] proved a local theory in $H^s_{\mathbb{C}}(\mathbb{R}) \times H^{s-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R})$ for

$s \geq 0$, and Fernandez and Linares [6] showed a local result in $L^2_{\mathbb{C}}(\mathbb{R}) \times H_{\mathbb{R}}^{-\frac{3}{4}+}(\mathbb{R})$ and a global result in $H^1_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R})$ with the parameters in (1.2) having the same sign.

With regard to the existence and stability of solitary-wave solutions of the form (1.3), we have the results by Lin [11] and by Albert and Angulo [1]. More precisely, in [11], the existence of solutions of the form

$$\phi(x) = \pm\sqrt{2\sigma((c-\gamma) - 8\sigma)}\,\mathrm{sech}(\sqrt{\sigma}x)$$
$$\psi(x) = 2\sigma\,\mathrm{sech}^2(\sqrt{\sigma}x)$$

for (1.5) with $\alpha = \beta = -1$, $\eta = 2$, $\mu = 12$, $c > \gamma$ and $\sigma \in (0, (c-\gamma)/8)$, was found. Then, using stability theory of [8], he went on to show that this solution is orbitally stable provided $c - \gamma \leq 1$ and $\sigma \in (0, (c-\gamma)/12)$. In [1], for $\alpha = \beta = -1$, $\eta = 2$, $\gamma = 0$ and $\mu = 6q$ it was proved for a certain range of values of $q$, equation (1.5) has a non-empty set of ground-state solutions which is stable.

The result in the present paper are complementary to those in [1, 2, 11], where different techniques were used. The main purpose here is to show the existence and stability of a novel set of solitary waves solutions for equations (1.1) and (1.2). Our approach is based essentially in variational methods and techniques of convexity type.

Next we describe briefly our results. Our theory of existence of smooth real solutions for (1.4) follows from the work of Angulo and Montenegro [2] (a sketch of the proof is given in Theorem 2.1 below), where by using the Concentration Compactness Method (Lions [12, 13]) and the conditions $\alpha, \beta, \sigma, c > 0$ and $\gamma < 0$, it is obtained solutions for (1.4) as minimimizer of the variational problem

$$I_\lambda = \inf\{V(f,g) | (f,g) \in H^1_{\mathbb{R}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R}) \text{ and } F(f,g) = \lambda\}, \tag{1.6}$$

where $\lambda > 0$,

$$V(f,g) = \int_{\mathbb{R}} [(f'(x))^2 - \gamma(D^{1/2}g(x))^2 + \sigma f^2(x) + cg^2(x)]dx, \tag{1.7}$$

$$F(f,g) = \int_{\mathbb{R}} f^2(x)g(x)dx. \tag{1.8}$$

So, if we denote the set of minimizers associated to $I_\lambda$ by $G_\lambda$, namely,

$$G_\lambda = \{(f,g) \in H^1_{\mathbb{R}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R}) |\ V(f,g) = I_\lambda \text{ and } F(f,g) = \lambda\} \tag{1.9}$$

then $G_\lambda \neq \emptyset$ and each element of $G_\lambda$ yields a solution of (1.4) via a scaling argument.

With regard to solutions for system (1.5), we shall establish here a theory of existence with the conditions $\alpha\beta > 0$, $\eta, \sigma > 0$, $c > \gamma$ and $\mu = -3\alpha$. Our argument will be again via an compactness argument, but in this case the proof is more easy compared with that given for the existence of solution of (1.4) because we do not have the nonlocal term $D = \mathcal{H}\partial_x$. Here, we will use some results from Lopes [14, 15] for finding solutions of the minimization problem

$$J_\lambda = \inf\{Z(f,g) | (f,g) \in H^1_{\mathbb{R}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R}) \text{ and } N(f,g) = \lambda\}, \tag{1.10}$$

where $\lambda > 0$,

$$Z(f,g) = \int_{\mathbb{R}} [(f'(x))^2 + \eta(g'(x))^2 + \sigma f^2(x) + (c-\gamma)g^2(x)]dx\,, \tag{1.11}$$

$$N(f,g) = \int_{\mathbb{R}} g(x)[f^2(x) + g^2(x)]\,dx\,. \tag{1.12}$$

We note that once established the existence of solutions for (1.5) with the condition $\mu = -3\alpha$, we can obtain a existence result for solitary wave for (1.5) under the condition that $\mu$ and $\alpha$ have opposite sign. In fact, under this constraint for $\mu$ and $\alpha$, one can multiply the second equation in (1.5) by a positive constant and to obtain a equivalent system in which the condition $\mu = -3\alpha$ holds.

The following question arising in this point is whether the following set of solitary-wave solutions for (1.1),

$$\begin{aligned} \mathcal{S}_{c,\omega} = \{(e^{i\theta}e^{icx/2}\phi_{c,\omega}\psi_{c,\omega}) : (\phi_{c,\omega},\psi_{c,\omega}) \in H^1_{\mathbb{R}}(\mathbb{R}) \times H^{\frac{1}{2}}_{\mathbb{R}}(\mathbb{R}),\\ F(\phi_{c,\omega},\psi_{c,\omega}) = -\frac{1}{3\beta}V(\phi_{c,\omega},\psi_{c,\omega}) = -\frac{1}{27\beta^3}I_1^3\}, \end{aligned} \tag{1.13}$$

with $\alpha = 2\beta > 0$, $\omega > \frac{c^2}{4}$ and $c > 0$, is a stable set with respect to equation (1.1), in the sense that if $(h,g) \in S_{c,\omega}$ and a slight perturbation of $(h,g)$ is taken as initial data for (1.1), then the resulting solution of (1.1) can be said to have a profile which remains close to $S_{c,\omega}$ for all time. It is well known from Cazenave and Lions [5] that we can obtain a result of stability of this type if the functionals involved in the problem of minimization (1.6) are conserved quantities for equation (1.1), however, we do not have this ideal situation with the functionals $V$ and $F$. So, for overcome this problem, we consider the following functionals

$$H(u) = \int_{\mathbb{R}} |u(x)|^2\,dx, \tag{1.14}$$

$$G_1(u,v) \equiv \operatorname{Im}\int_{\mathbb{R}} u(x)\overline{u_x(x)}\,dx + \frac{\alpha}{2\beta}\int_{\mathbb{R}} v^2(x)\,dx, \tag{1.15}$$

$$E_1(u,v) \equiv \int_{\mathbb{R}} |u_x(x)|^2\,dx + \alpha v(x)|u(x)|^2 - \frac{\alpha\gamma}{2\beta}v(x)Dv(x)\,dx, \tag{1.16}$$

which are conserved quantities or invariants of motion for (1.1), *i.e.*, for $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ initial smooth functions, the solution of (1.1) emanating from $(u_0,v_0)$ has the property that $H(u(t)) = H(u_0)$, $G_1(u(t),v(t)) = G_1(u_0,v_0)$ and $E_1(u(t),v(t)) = E_1(u_0,v_0)$ for all $t$ for which the solution exists. Next, we define the following functional

$$d(c,\omega) = E_1(\Phi_{c,\omega},\Psi_{c,\omega}) + \omega H(\Phi_{c,\omega}) + cG_1(\Phi_{c,\omega},\Psi_{c,\omega}) \tag{1.17}$$

for every $(\Phi_{c,\omega},\Psi_{c,\omega}) \in \mathcal{S}_{c,\omega}$. So, by considering the following function of variable $\omega$,

$$d_c(\omega) \equiv d(c,\omega),$$

with $c > 0$ fixed and $\omega > c^2/4$, we obtain that the set of solitary waves solutions $\mathcal{S}_{c,\omega}$ will be stable with respect to equation (1.1) if $d_c(\omega)$ is a strictly convex function in $\omega$. In this paper we can prove the convexity of the function $d_c(\omega)$ with $\omega$ close to $c^2/4$.

A similar result of stability is also proved for the set of solitary-wave solutions of (1.2) obtained via the minimization problem (1.10). In this case we use the following conserved quantities for (1.2):

$$\begin{gathered} G_2(u,v) \equiv \operatorname{Im} \int_{\mathbb{R}} u(x)\overline{u_x(x)}\,dx + \frac{\alpha}{2\beta}\int_{\mathbb{R}} v^2(x)\,dx, \\ E_2(u,v) \equiv \int_{\mathbb{R}} |u_x(x)|^2 + \alpha v(x)|u(x)|^2 + \frac{\alpha\eta}{2\beta}v_x^2(x) - \frac{\alpha\gamma}{2\beta}v^2(x) - \frac{\alpha\mu}{6\beta}v^3(x)\,dx. \end{gathered} \tag{1.18}$$

We note that as we do not have explicit formulas for the solutions $(\phi_{c,\omega}, \psi_{c,\omega})$ in (1.4)-(1.5) and a argument of dilation is not available to obtain a explicit expression for the function $d$ in function of $c$ and $\omega$, we need to apply a Lemma of convexity of Shatah [16] (see Lemma 2.8 below).

This paper is organized as follows. In section 2, we give a sketch of the proof of existence of solutions for (1.4). These solutions are obtained via the Concentration Compactness Principle. We also prove that the set of solitary waves, $\mathcal{S}_{c,\omega}$, defined in (2.11) is stable in $H^1_{\mathbb{C}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R})$. In section 3, we give the corresponding theory of existence and stability of solitary waves solutions for system (1.2) following the same ideas established in section 2.

**Notation.** We shall denote by $\widehat{f}$ the Fourier transform of $f$, defined as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$. $|f|_{L^p}$ denotes the $L^p(\mathbb{R})$ norm of $f$, $1 \le p \le \infty$. In particular, $|\cdot|_{L^2} = \|\cdot\|$ and $|\cdot|_{L^\infty} = |\cdot|_\infty$. We denote by $H^s_{\mathbb{C}}(\mathbb{R})$ the Sobolev space of all $f$ (tempered distributions) for which the norm $\|f\|_s^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi$ is finite. For $s \ge 0$, $H^s_{\mathbb{R}}(\mathbb{R})$ denotes the space of all real-valued functions in $H^s_{\mathbb{C}}(\mathbb{R})$. The product norm in $H^s_{\mathbb{C}}(\mathbb{R}) \times H^r_{\mathbb{C}}(\mathbb{R})$ is denoted by $\|\cdot\|_{s\times r}$. We denote by $X_{\mathbb{R}}$ the product $H^1_{\mathbb{R}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R})$, $Y_{\mathbb{R}}$ the product $H^1_{\mathbb{R}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R})$, $X_{\mathbb{C}}$ the product $H^1_{\mathbb{C}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R})$, and $Y_{\mathbb{C}}$ the product $H^1_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R})$. $J^s = (1-\partial_x^2)^{s/2}$ and $D^s = (-\partial_x^2)^{s/2}$ are the Bessel and Riesz potentials of order $-s$, respectively, defined by $\widehat{J^s f}(\xi) = (1+\xi^2)^{s/2}\widehat{f}(\xi)$ and $\widehat{D^s f}(\xi) = |\xi|^s\widehat{f}(\xi)$.

## 2. Existence and stability of solitary waves solutions for equation (1.1)

In this section we give a theory of existence and stability of solitary waves solutions for equation (1.1). We begin with the problem of existence. Initially, we give a sketch of the proof of that $G_\lambda$ defined in (1.9) is not empty. In fact, we call $\{(f_n, g_n)\}_{n\ge1}$ in $X_{\mathbb{R}} = H^1_{\mathbb{R}}(\mathbb{R}) \times H^{1/2}_{\mathbb{R}}(\mathbb{R})$ a minimizing sequence for $I_\lambda$ if it satisfies

$$\begin{gathered} F(f_n, g_n) = \lambda, \quad \text{for all } n, \\ \lim_{n\to\infty} V(f_n, g_n) = I_\lambda. \end{gathered}$$

So we have the following Theorem of existence established in Angulo and Montenegro [2],

**Theorem 2.1.** *Let $\alpha, \beta, \sigma, c > 0$, $\gamma < 0$, and let $\lambda$ be any positive number. Then any minimizing sequence $\{(f_n, g_n)\}$ for $I_\lambda$ is relatively compact in $X_{\mathbb{R}}$ up to translation, i.e., there are subsequences $\{(f_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(f_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $X_{\mathbb{R}}$ to some $(f, g)$, which is a minimum of $I_\lambda$. Therefore, $G_\lambda \ne \emptyset$ and there are non-trivial solitary waves solutions for equation* (1.1).

*Sketch of the proof.* From $\lambda = \int_{\mathbb{R}} f^2(x)g(x)dx \leq \|f\|_1^2\|g\|_{\frac{1}{2}} \leq \|(f,g)\|^3_{1\times\frac{1}{2}}$ and $V(f,g) \geq C\|(f,g)\|^2_{1\times\frac{1}{2}}$, we obtain $0 < I_\lambda < \infty$ and each minimizing sequence is bounded in $X_{\mathbb{R}}$. Then there is a subsequence, still denoted by $(f_n, g_n)$, such that $\|(f_n,g_n)\|^2_{1\times\frac{1}{2}} \to \mu > 0$. We then apply the Concentration Compactness Lemma ([12, Lemma I.1]) with

$$\rho_n(x) = (f_n'(x))^2 + (f_n(x))^2 + (J^{1/2}g_n(x))^2.$$

The *Vanishing* case does not occur because for any $R > 0$

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-R}^{y+R}(f_n(x))^2dx = 0.$$

Thus $F(f_n, g_n)$ tends to zero as $n$ goes to infinity. But this contradicts the fact that $F(f_n, g_n) = \lambda > 0$.

In the *Dichotomy* case, one can show as in [2] that for some $\theta$ with $0 < \theta < \lambda$ and for all $\epsilon > 0$ there exist $\eta(\epsilon)$ (with $\eta(\epsilon) \to 0$ as $\epsilon \to 0$), two sequences $\mathbf{h}_n^{(1)} = \zeta_n\mathbf{h}_n$ and $\mathbf{h}_n^{(2)} = \varphi_n\mathbf{h}_n$ in $X_{\mathbb{R}}$, with $\varphi_n,\ \zeta_n \in C^\infty(\mathbb{R};\mathbb{R})$, $0 \leq \varphi_n,\ \zeta_n \leq 1$, and an integer $k$ such that for $n \geq k$ and $\mathbf{h}_n = (f_n, g_n)$,

$$\|\mathbf{h}_n^{(1)} + \mathbf{h}_n^{(2)} - \mathbf{h}_n\|_{1\times\frac{1}{2}} \leq \eta(\epsilon),$$

$$\Big|\int_{\mathbb{R}}\zeta_n^3(f_n)^2g_n\,dx - \theta\Big| \leq \eta(\epsilon),$$

$$\Big|\int_{\mathbb{R}}\varphi_n^3(f_n)^2g_n\,dx - (\lambda-\theta)\Big| \leq \eta(\epsilon).$$

Hence, these relations will imply that $I_\lambda \geq I_\theta + I_{\lambda-\theta}$. But this is a contradiction, since for $\tau > 0$ we have $I_{\tau\lambda} = \tau^{2/3}I_\lambda$, and therefore

$$I_\lambda \geq I_{\tau\lambda} + I_{(1-\tau)\lambda} = (\tau^{2/3} + (1-\tau)^{2/3})I_\lambda > I_\lambda,$$

where we have used that $\tau^{2/3} + (1-\tau)^{2/3} > 1$ for $\tau \in (0,1)$ and $I_\lambda > 0$.

Since the Vanishing and Dichotomy cases have been ruled out, it follows that there is a sequence $\{y_n\}_{n\geq1} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there is $R > 0$ large and $n_0 > 0$ such that for $n \geq n_0$,

$$\int_{|x-y_n|\leq R}\rho_n(x)\,dx \geq \mu - \epsilon, \quad \int_{|x-y_n|\geq R}\rho_n(x)\,dx \leq \epsilon.$$

Therefore

$$\Big|\int_{|x-y_n|\geq R}f_n^2g_n\,dx\Big| \leq C\|g_n\|_{\frac{1}{2}}|f_n|_\infty^{2/3}\Big(\int_{|x-y_n|\geq R}\rho_n(x)\,dx\Big)^{2/3} = O(\epsilon).$$

Hence

$$\Big|\int_{|x-y_n|\leq R}f_n^2g_n\,dx - \lambda\Big| \leq \epsilon.$$

Letting $\mathbf{h}_n^*(x) = (f_n^*(x), g_n^*(x)) \equiv (f_n(x-y_n), g_n(x-y_n))$, we have that $\{\mathbf{h}_n^*\}_{n\geq1}$ converges weakly in $X_{\mathbb{R}}$ to a vector-function $\mathbf{h}^* = (f_0, g_0)$. Then for $n \geq n_0$,

$$\lambda \geq \int_{-R}^{R}(f_n^*(x))^2g_n^*(x)\,dx \geq \lambda - \epsilon.$$

Since $H^1((-R,R))$ and $H^{1/2}((-R,R))$ are compactly embedded in $L^2((-R,R))$, we have from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\Big|\int_{-R}^{R}(f_n^*(x))^2g_n^*(x)\,dx-\int_{-R}^{R}(f_0(x))^2g_0(x)\,dx\Big|\\&\le C\Big(\|f_n^*-f_0\|_{L^2(-R,R)}+\|g_n^*-g_0\|_{L^2(-R,R)}\Big)\to 0,\quad \text{as } n\to\infty\,.\end{aligned}$$

Therefore,

$$\lambda\ge\int_{-R}^{R}f_0^2(x)g_0(x)\,dx\ge\lambda-\epsilon.$$

Thus for $\epsilon=1/j$, $j\in\mathbb{N}$, there exists $R_j>j$ such that

$$\lambda\ge\int_{-R_j}^{R_j}f_0^2(x)g_0(x)\,dx\ge\lambda-\frac{1}{j}.$$

So, for $j\to\infty$, we finally have that $F(f_0,g_0)=\lambda$. Furthermore, from the weak lower semicontinuity of $V$ and the invariance of $V$ by translations, we have

$$I_\lambda=\liminf_{n\to\infty}V(f_n^*,g_n^*)\ge V(f_0,g_0)\ge I_\lambda.$$

Thus the vector-function $\mathbf{h}^*=(f_0,g_0)\in G_\lambda$. Moreover, since

$$\|(f_n^*,g_n^*)\|_{1\times\frac12}\to\|(f_0,g_0)\|_{1\times\frac12},$$

we have that $(f_n^*,g_n^*)\to(f_0,g_0)$ strongly in $X_{\mathbb{R}}$. Thus the Theorem is proved. $\square$

**Remark 2.2.** Note that from [2, Theorem 3.5] we have that each component of $(f,g)\in G_\lambda$ is even (up translations) and strictly decreasing positive function on $(0,+\infty)$. More precisely, $f(x)=f^*(x+r)$, $g(x)=g^*(x+r)$ for some $r\in\mathbb{R}$, where $f^*$ and $g^*$ are the symmetric decreasing rearrangements of $f$ and $g$ respectively.

Our theory of stability has another variational characterization of solitary waves solutions for (1.1). We consider the following minimization problem in $X_{\mathbb{C}}=H^1_{\mathbb{C}}(\mathbb{R})\times H^{1/2}_{\mathbb{R}}(\mathbb{R})$ for $\lambda>0$,

$$\mathcal{M}_\lambda=\inf\{\mathcal{W}_{c,\omega}(h,g)|(h,g)\in X_{\mathbb{C}}\text{ and }\mathcal{F}(h,g)=\lambda\},$$

where

$$\mathcal{W}_{c,\omega}(h,g)=\int_{\mathbb{R}}|h'(x)|^2-\gamma(D^{1/2}g(x))^2+\omega|h(x)|^2+cg^2(x)\,dx+c\,\mathrm{Im}\int_{\mathbb{R}}h(x)\overline{h'(x)}\,dx,\tag{2.1}$$

$\gamma<0$, $\omega>c^2/4$, $c>0$, and

$$\mathcal{F}(h,g)=\int_{\mathbb{R}}|h(x)|^2g(x)\,dx.\tag{2.2}$$

Also, we denote the set of minimizers for $\mathcal{M}_\lambda$ by $\mathcal{G}_\lambda$, namely,

$$\mathcal{G}_\lambda=\{(h,g)\in X_{\mathbb{C}}:\mathcal{W}_{c,\omega}(h,g)=\mathcal{M}_\lambda\text{ and }\mathcal{F}(h,g)=\lambda\}.\tag{2.3}$$

Next show that every minimizing sequence for $\mathcal{M}_\lambda$ converges strongly in $X_{\mathbb{C}}$, up to rotations and translations, to some element of $\mathcal{G}_\lambda$. Initially, we establish a similar result as in Theorem 2.1 but considering complex-valued functions. More precisely, we consider the following minimization problem

$$I_\lambda^{\mathbb{C}}=\inf\{V_{\mathbb{C}}(h,g)|(h,g)\in X_{\mathbb{C}}\text{ and }\mathcal{F}(h,g)=\lambda\},\tag{2.4}$$

where $\lambda > 0$,

$$V_{\mathbb{C}}(h,g) = \int_{\mathbb{R}} [|h'(x)|^2 - \gamma(D^{1/2}g(x))^2 + \sigma|h(x)|^2 + cg^2(x)]dx$$

and $\mathcal{F}$ is defined as in (2.2). So we have the following Theorem.

**Theorem 2.3.** *Let $\sigma, c > 0$, $\gamma < 0$, and let $\lambda$ be any positive number. Then any minimizing sequence $\{(h_n, g_n)\}$ for $I_\lambda^{\mathbb{C}}$ is relatively compact in $X_{\mathbb{C}}$ up to translation, i.e., there are subsequences $\{(h_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(h_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $X_{\mathbb{C}}$ to some $(h,g)$ which is a minimum of $I_\lambda^{\mathbb{C}}$. Moreover, $(h,g) = (e^{i\theta}f, g)$, where $\theta \in \mathbb{R}$ and $(f,g) \in G_\lambda$.*

*Proof.* The existence of a minimum is proved as in Theorem 2.1 (we apply the Concentration Compactness Lemma to $\rho_n(x) = |f_n'(x)|^2 + |f_n(x)|^2 + |J^{1/2}g_n(x)|^2$). Now, let $(h,g)$ be a minimizer of the problem (2.4) and consider $h = h_1 + ih_2$, then $h_0 = |h_1| + i|h_2|$ is a minimizer of problem (2.4). In fact, from the inequality

$$\int_{\mathbb{R}} |h_i'(x)|^2 dx \geq \int_{\mathbb{R}} |\,|h_i|'(x)|^2 dx$$

and the condition $\mathcal{F}(h_0, g) = \mathcal{F}(h,g) = \lambda$, it follows that

$$I_\lambda^{\mathbb{C}} = V_{\mathbb{C}}(h,g) \geq V_{\mathbb{C}}(h_0,g) \geq I_\lambda^{\mathbb{C}}.$$

Therefore, there exists $K > 0$ (Lagrange multiplier) such that

$$\begin{gathered} -h_i'' + \sigma h_i = K h_i g \\ -|h_i|'' + \sigma|h_i| = K|h_i|g, \quad \text{for } i = 1,2. \end{gathered} \tag{2.5}$$

Since $|h_i| > 0$ it follows from the Sturm-Liouville Theory that $-\sigma$ is the smallest eigenvalue of operator $-\frac{d^2}{dx^2} - Kg$, and therefore is simple. Hence, from (2.5) there are $\mu_i \in \mathbb{R} - \{0\}$ such that $h_i = \mu_i h_0^*$, where $h_0^*$ is a positive function. Therefore, there exists a positive function $f$ and $\theta \in \mathbb{R}$ such that $h = e^{i\theta}f$. Moreover, from the relations $F(f,g) = \mathcal{F}(h,g) = \lambda$, $I_\lambda^{\mathbb{C}} = V_{\mathbb{C}}(h,g) = V(f,g) \geq I_\lambda$, and $I_\lambda \geq I_\lambda^{\mathbb{C}}$, we have that $(f,g) \in G_\lambda$. This finishes the proof. □

The following Theorem proves the existence a minimum for $\mathcal{M}_\lambda$.

**Theorem 2.4.** *Let $\gamma < 0$, $c > 0$, $\omega > \frac{c^2}{4}$, and $\lambda > 0$. Then, any minimizing sequence $\{(h_n, g_n)\}$ for $\mathcal{M}_\lambda$ is relatively compact in $X_{\mathbb{C}}$ up to rotations and translation, i.e., there are subsequences $\{(h_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(e^{icy_k/2}h_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $X_{\mathbb{C}}$ to some $(h,g)$ which is a minimum of $\mathcal{M}_\lambda$. Moreover, $(h,g) = (e^{i\theta}e^{icx/2}f, g)$ where $(f,g) \in G_\lambda$.*

*Proof.* Let $\{(h_n, g_n)\}$ be a minimizing sequence for $\mathcal{M}_\lambda$. Then we have that $\lim_{n\to\infty} \mathcal{W}_{c,\omega}(h_n, g_n) = \mathcal{M}_\lambda$ and $\mathcal{F}(h_n, g_n) = \lambda$. If $f_n \equiv e^{-icx/2}h_n$, then we have $\mathcal{F}(f_n, g_n) = \lambda$ and

$$\mathcal{W}_{c,\omega}(h_n, g_n) = \mathcal{W}_{c,\omega}(e^{icx/2}f_n, g_n) = V_{\mathbb{C}}(f_n, g_n) \geq I_\lambda^{\mathbb{C}}. \tag{2.6}$$

Since $I_\lambda^{\mathbb{C}} \geq \mathcal{M}_\lambda$, it follows from (2.6) that $\{(f_n, g_n)\}$ is a minimizing sequence for $I_\lambda^{\mathbb{C}}$. Therefore, from Theorem 2.3 there are subsequences $\{(f_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(f_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $X_{\mathbb{C}}$ to some $(h_0, g)$ which is a minimum of $I_\lambda^{\mathbb{C}}$. Then $(h_0, g) = (e^{i\theta}f, g)$ where $\theta \in \mathbb{R}$ and $(f,g) \in G_\lambda$. Hence, from the definition of $f_n$ we have that

$$(e^{icy_k/2}h_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k})) \to (e^{i\theta}e^{icx/2}f, g) \quad \text{in } X_{\mathbb{C}}.$$

So, $(h, g) = (e^{i\theta}e^{icx/2}f, g) \in \mathcal{G}_\lambda$ and this proves the Theorem. $\square$

**Corollary 2.5.** *Let $\gamma < 0$, $c > 0$, $\omega > \frac{c^2}{4}$, and $\lambda > 0$. Then the set $\mathcal{G}_\lambda$ is nonempty. Moreover, if $\{(h_n, g_n)\}$ is any minimizing sequence for $\mathcal{M}_\lambda$, then*

- (i) *There exist sequences $\{y_n\}$, $\{\theta_n\}$ and an element $(h, g) \in \mathcal{G}_\lambda$ such that $\{(e^{i\theta_n}h_n(\cdot + y_n), g_n(\cdot + y_n))\}$ has a subsequence converging strongly in $X_\mathbb{C}$ to $(h, g)$.*
- (ii) $\lim_{n\to\infty} \inf_{\theta,y\in\mathbb{R};\vec{\psi}\in\mathcal{G}_\lambda} \|(e^{i\theta}h_n(\cdot + y), g_n(\cdot + y)) - \vec{\psi}\|_{1\times\frac{1}{2}} = 0.$
- (iii) $\lim_{n\to\infty} \inf_{\vec{\psi}\in\mathcal{G}_\lambda} \|(h_n, g_n) - \vec{\psi}\|_{1\times\frac{1}{2}} = 0.$

*Proof.* By Theorem 2.4 we have that $\mathcal{G}_\lambda$ is nonempty and the item $(i)$ holds.

Now, suppose that the item (ii) does not hold; then there exist a subsequence $\{(h_{n_k}, g_{n_k})\}$ of $\{(h_n, g_n)\}$ and a number $\epsilon > 0$, such that

$$\inf_{\theta,y\in\mathbb{R};\vec{\psi}\in\mathcal{G}_\lambda} \|(e^{i\theta}h_{n_k}(\cdot + y), g_{n_k}(\cdot + y)) - \vec{\psi}\|_{1\times\frac{1}{2}} \geq \epsilon$$

for all $k \in \mathbb{N}$. But, since $\{(h_{n_k}, g_{n_k})\}$ itself is a minimizing sequence for $\mathcal{M}_\lambda$, from statement (i), it follows that there exist sequences $\{y_{n_k}\}$, $\{\theta_{n_k}\}$ and $\vec{\psi} \in \mathcal{G}_\lambda$ such that

$$\liminf_{n\to\infty} \inf_{\theta,y\in\mathbb{R};\vec{\psi}\in\mathcal{G}_\lambda} \|(e^{i\theta_{n_k}}h_{n_k}(\cdot + y_{n_k}), g_{n_k}(\cdot + y_{n_k})) - \vec{\psi}\|_{1\times\frac{1}{2}} = 0.$$

This contradiction proves statement (ii).

Finally, since the functionals $\mathcal{W}_{c,\omega}$ and $\mathcal{F}$ are invariants under rotations and translations, $\mathcal{G}_\lambda$ contains any rotations and translation of $\vec{\psi}$, if it contains $\vec{\psi}$, and hence statement (iii) follows immediately from statement (ii). This completes the Corollary. $\square$

In the following we establish some remarks and some sets which will be used for the problem of stability. If we define the minimization problem

$$M_c(\omega) = \inf_{(h,g)\in X_\mathbb{C}} \frac{\mathcal{W}_{c,\omega}(h, g)}{[\mathcal{F}(h, g)]^{2/3}}, \tag{2.7}$$

it is easy to see that for $a, b \in \mathbb{R} - \{0\}$ and $a^2 = b^2$, we have

$$\frac{\mathcal{W}_{c,\omega}(ah, bg)}{[\mathcal{F}(ah, bg)]^{2/3}} = \frac{\mathcal{W}_{c,\omega}(h, g)}{[\mathcal{F}(h, g)]^{2/3}}. \tag{2.8}$$

Moreover,

$$M_c(\omega) = \inf_{(h,g)\in X_\mathbb{C}} \mathcal{W}_{c,\omega}(h, g) : \mathcal{F}(h, g) = 1\}. \tag{2.9}$$

so, if $(h, g) \in X_\mathbb{C}$ and satisfies $\mathcal{W}_{c,\omega}(h, g) = M_c(\omega)$ and $\mathcal{F}(h, g) = 1$, then from Theorems 2.3 and 2.4 we obtain that there are $\theta \in \mathbb{R}$, a positive function $f$ and $K > 0$, such that $h = e^{i\theta}e^{icx/2}f$ and $(\phi, \psi) = (\pm \frac{K}{\sqrt{2\alpha\beta}}f, -\frac{K}{\alpha}g)$ is a solution of (1.4). Hence, $M_c(\omega) = \mathcal{W}_{c,\omega}(h, g) = V(f, g)$ and $F(f, g) = 1$, and so (2.9) can be written as

$$M_c(\omega) = \inf_{(f,g)\in X_\mathbb{R}} \{V(f, g) : F(f, g) = 1\} = I_1. \tag{2.10}$$

Next, for $\alpha = 2\beta$, $\omega > c^2/4$ and $c > 0$, we define our main set in the study of stability,

$$\mathcal{S}_{c,\omega} = \{(e^{i\theta}e^{icx/2}\phi, \psi) : (\phi, \psi) \in X_\mathbb{R}, F(\phi, \psi) = -\frac{1}{3\beta}V(\phi, \psi) = -\frac{1}{27\beta^3}[M_c(\omega)]^3\}. \tag{2.11}$$

Hence, for $(e^{i\theta}e^{icx/2}\phi,\psi)\in\mathcal{S}_{c,\omega}$ we have that $(\phi,\psi)$ satisfies (1.4) with $\alpha=2\beta$. In fact, let $F(\phi,\psi)=\lambda$, then since $\mathcal{W}_{c,\omega}(e^{i\theta}e^{icx/2}\phi,\psi)=V(\phi,\psi)$ it follows from (2.8) that

$$\mathcal{W}_{c,\omega}(e^{i\theta}e^{icx/2}\frac{1}{\lambda^{1/3}}\phi,\frac{1}{\lambda^{1/3}}\psi)=\frac{\mathcal{W}_{c,\omega}(e^{icx/2}\phi,\psi)}{[\mathcal{F}(\phi,\psi)]^{2/3}}=\frac{V(\phi,\psi)}{[F(\phi,\psi)]^{2/3}}=M_c(\omega),$$

thus $(e^{icx/2}\frac{1}{\lambda^{1/3}}\phi,\frac{1}{\lambda^{1/3}}\psi)\in\mathcal{G}_1$ and therefore there is $K_0\in\mathbb{R}$ such that

$$\begin{gathered}-\phi''+(\omega-\frac{c^2}{4})\phi=\frac{K_0}{\lambda^{1/3}}\psi\phi\\ -\gamma\mathcal{H}\psi'+c\psi=\frac{K_0}{2\lambda^{1/3}}\phi^2.\end{gathered}$$

Hence $V(\phi,\psi)=\frac{3K_0}{2\lambda^{1/3}}F(\phi,\psi)$ and so it follows that $-\beta=\frac{K_0}{2\lambda^{1/3}}$. This shows the claim.

Now we are going to give our definition of stability used here.

**Definition 2.6.** Let $(X,\|\cdot\|_X)$ be a Hilbert space and $Y$ a subspace of $X$. A set $\mathcal{S}\subset X$ is $X$-stable with respect to (1.1) (or to (1.2)) if for all $\epsilon>0$, there is $\delta>0$ such that for all $(u_0,v_0)\in Y$ with

$$\inf_{(\Phi,\Psi)\in\mathcal{S}}\|(u_0,v_0)-(\Phi,\Psi)\|_X<\delta$$

the solution $(u(t),v(t))$ of (1.1) (or (1.2)) with $(u(0),v(0))=(u_0,v_0)$ can be extended to a global solution in $C([0,\infty);Y)$ and

$$\sup_{0\le t<\infty}\inf_{(\Phi,\Psi)\in\mathcal{S}}\|(u(t),v(t))-(\Phi,\Psi)\|_X<\epsilon.$$

Otherwise $\mathcal{S}$ is called $X$-unstable.

We shall show here that the set $\mathcal{S}_{c,\omega}$ defined in (2.11) is $X_{\mathbb{C}}$-stable (Theorem 2.12 below). In order to prove it we need several lemmas. Initially, for $(\Phi_{c,\omega}(\xi),\Psi_{c,\omega}(\xi))=(e^{ic\xi/2}\phi_{c,\omega}(\xi),\psi_{c,\omega}(\xi))\in\mathcal{S}_{c,\omega}$ we define the following functional

$$d(c,\omega)=E_1(\Phi_{c,\omega},\Psi_{c,\omega})+\omega H(\Phi_{c,\omega})+c\,G_1(\Phi_{c,\omega},\Psi_{c,\omega}),\tag{2.12}$$

where $E_1$, $H$, and $G_1$ are defined in (1.14) and (1.16). Also, we define the following function a one parameter $\omega$,

$$d_c(\omega)\equiv d(c,\omega)\tag{2.13}$$

where $c>0$ is fixed and $\omega\in(\frac{c^2}{4},\infty)$. So, we have the following two basic features of the function $d_c$, namely, $d_c(\cdot)$ is constant on $S_{c,\omega}$ and $d_c(\cdot)$ is strictly increasing. In fact, from (2.1), (2.2), (2.11) and (2.12) we get that for any $(\Phi_{c,\omega},\Psi_{c,\omega})\in S_{c,\omega}$

$$\begin{aligned}d_c(\omega)&=\mathcal{W}_{c,\omega}(e^{ic\xi/2}\phi_{c,\omega},\psi_{c,\omega})+\alpha\mathcal{F}(e^{ic\xi/2}\phi_{c,\omega},\psi_{c,\omega})\\&=V(\phi_{c,\omega},\psi_{c,\omega})+\alpha F(\phi_{c,\omega},\psi_{c,\omega})=\frac{1}{3}V(\phi_{c,\omega},\psi_{c,\omega})\\&=-\beta F(\phi_{c,\omega},\psi_{c,\omega})=-\beta\mathcal{F}(\Phi_{c,\omega},\Psi_{c,\omega})\\&=\frac{1}{27\beta^2}[M_c(\omega)]^3.\end{aligned}\tag{2.14}$$

Now, let $\omega<\omega_1$ and let $(h,g)$ be a minimizer for $M_c(\omega_1)$, then it follows that

$$M_c(\omega)\le\frac{\mathcal{W}_{c,\omega}(h,g)}{[\mathcal{F}(h,g)]^{2/3}}=M_c(\omega_1)+(\omega-\omega_1)\frac{\int_{\mathbb{R}}|h|^2dx}{[\mathcal{F}(h,g)]^{2/3}}<M_c(\omega_1),\tag{2.15}$$

and so from (2.14) we obtain that $d_c(\omega)$ is strictly increasing.

**Remark 2.7.** (i) For a fixed $c>0$ , it is easy to show that $M_c(\omega)$ is a continuous function on $(\frac{c^2}{4},\infty)$. In fact, from the relations

$$0 \le M_c(\omega_1) - M_c(\omega) \le \frac{\omega_1-\omega}{\omega-(c^2/4)} M_c(\omega) \quad \text{for } \omega_1>\omega$$
$$0 \le M_c(\omega) - M_c(\omega_1) \le \frac{\omega-\omega_1}{\omega_1-(c^2/4)} M_c(\omega) \quad \text{for } \omega_1<\omega,$$

it follows the continuity.

(ii) If we consider

$$\alpha_c(\omega)=\inf\Big\{\int_{\mathbb{R}} |\Phi_{c,\omega}(x)|^2dx : (\Phi_{c,\omega},\Psi_{c,\omega})\in S_{c,\omega}\Big\}$$
$$\beta_c(\omega)=\sup\Big\{\int_{\mathbb{R}} |\Phi_{c,\omega}(x)|^2dx : (\Phi_{c,\omega},\Psi_{c,\omega})\in S_{c,\omega}\Big\},$$

then we get from (2.15) and (2.11) that for $\omega<\omega_1$,

$$\frac{9\beta^2\alpha_c(\omega_1)}{[M_c(\omega_1)]^2} \le \frac{M_c(\omega_1)-M_c(\omega)}{\omega_1-\omega} \le \frac{9\beta^2\beta_c(\omega)}{[M_c(\omega)]^2}. \tag{2.16}$$

Hence from (2.16) it is possible to show that $M_c$ is differentiable at $\omega_1$ if and only if $\alpha_c(\omega_1)=\beta_c(\omega_1)$ (see [10, Lemma 4.3]). Therefore from the last affirmation and from (2.14) we can conclude that $d_c(\cdot)$ is differentiable at all but countably many points of $(\frac{c^2}{4},\infty)$.

From item (ii) we can assume, without losing of generality, that $M_c$ is differentiable. We now state without proof a lemma due to Shatah [16] related to strictly convex functions.

**Lemma 2.8.** *Let $h$ be any function which is strictly convex in an interval $I$ about $\omega$. Then given $\epsilon>0$, there exists $N(\epsilon)>0$ such that for $\omega_1\in I$ and $|\omega_1-\omega|\ge\epsilon$ we have*

(1) *For $\omega_1<\omega<\omega_0$, $|\omega_0-\omega|<\epsilon/2$, $\omega_0\in I$, then*
$$\frac{h(\omega_1)-h(\omega_0)}{\omega_1-\omega_0} \le \frac{h(\omega)-h(\omega_0)}{\omega-\omega_0} - \frac{1}{N(\epsilon)}.$$
(2) *For $\omega_0<\omega<\omega_1$, $|\omega_0-\omega|<\epsilon/2$, $\omega_0\in I$, then*
$$\frac{h(\omega_1)-h(\omega_0)}{\omega_1-\omega_0} \ge \frac{h(\omega)-h(\omega_0)}{\omega-\omega_0} + \frac{1}{N(\epsilon)}.$$

It follows from Lemma 2.8 and from the inequalities in (2.16) the following result for the function $d_c(\cdot)$.

**Lemma 2.9.** *Suppose that $d_c(\cdot)$ is strictly convex in an interval $I$ around $\omega$. Then given $\epsilon>0$, there exists $N(\epsilon)>0$ such that for $\omega_1\in I$ and $|\omega_1-\omega|\ge\epsilon$ we have*

$$d_c(\omega_1)\ge d_c(\omega)+\beta_c(\omega)(\omega_1-\omega)+\frac{1}{N(\epsilon)}(\omega-\omega_1) \quad \textit{for } \omega_1<\omega,$$
$$d_c(\omega_1)\ge d_c(\omega)+\alpha_c(\omega)(\omega_1-\omega)+\frac{1}{N(\epsilon)}(\omega_1-\omega) \quad \textit{for } \omega_1>\omega.$$

For $\epsilon > 0$ define the following $\epsilon$-neighborhood of set $\mathcal{S}_{c,\omega}$,

$$U_{c,\omega,\epsilon} = \{(u,v) \in X_{\mathbb{C}} : \inf_{(\Phi,\Psi)\in\mathcal{S}_{c,\omega}} \|(u,v) - (\Phi,\Psi)\|_{1\times\frac{1}{2}} < \epsilon\},$$

then we have the following Lemma.

**Lemma 2.10.** *Let $\alpha = 2\beta$ and a fixed $c > 0$. We consider for $(\Phi_{c,\omega}, \Psi_{c,\omega}) \in \mathcal{S}_{c,\omega}$ the function*

$$d_c(\omega) \equiv -\beta\mathcal{F}(\Phi_{c,\omega}, \Psi_{c,\omega})$$

*with $\omega \in (c^2/4, \infty)$. Then, there is a small $\epsilon$ and a $C^1$-map $\rho : U_{c,\omega,\epsilon} \to (c^2/4, \infty)$, defined by*

$$\rho(u,v) = d_c^{-1}(-\beta\mathcal{F}(u,v)), \tag{2.17}$$

*such that $\rho(\Phi_{c,\omega}, \Psi_{c,\omega}) = \omega$ for any $(\Phi_{c,\omega}, \Psi_{c,\omega}) \in \mathcal{S}_{c,\omega}$.*

*Proof.* Since $d_c(\cdot)$ is a strictly increasing continuous mapping, $\mathcal{S}_{c,\omega}$ is a bounded set in $X_{\mathbb{C}}$ and the function $(h,g) \to \mathcal{F}(h,g)$ is uniformly continuous on bounded set, it follows immediately the Lemma. $\square$

**Lemma 2.11.** *Let $\alpha = 2\beta$ and a fixed $c > 0$. Suppose that $d_c$ is strictly convex in an interval $I$ around $\omega$. Then there exists $\epsilon > 0$ such that for all $\vec{u} = (u,v) \in U_{c,\omega,\epsilon}$ and any $\vec{\Phi} = (\Phi_{c,\omega}, \Psi_{c,\omega}) \in \mathcal{S}_{c,\omega}$,*

$$E_1(\vec{u}) - E_1(\vec{\Phi}) + \rho(\vec{u})(H(\vec{u}) - H(\vec{\Phi})) + c(G_1(\vec{u}) - G_1(\vec{\Phi})) \geq \frac{1}{N(\epsilon)}|\rho(\vec{u}) - \omega|, \tag{2.18}$$

*where $\rho(\vec{u})$ is defined in (2.17) and $N(\epsilon)$ is given by Lemma 2.9.*

*Proof.* Let $\epsilon$ be small enough such that $\rho(U_{c,\omega,\epsilon}) \subset (\omega - \eta, \infty) \subset (\frac{c^2}{4}, \infty)$ for $\eta > 0$ small. Then, since

$$E_1(\vec{u}) + \rho(\vec{u})H(\vec{u}) + cG_1(\vec{u}) = \mathcal{W}_{c,\rho(\vec{u})}(\vec{u}) + \alpha\mathcal{F}(\vec{u}), \tag{2.19}$$

$d_c(\rho(\vec{u})) = -\beta\mathcal{F}(\vec{u})$ and $d_c(\rho(\vec{u})) = -\beta\mathcal{F}(\Phi_{c,\rho(\vec{u})}, \Psi_{c,\rho(\vec{u})})$ (see (2.14)), we get that $\mathcal{F}(\vec{u}) = \mathcal{F}(\Phi_{c,\rho(\vec{u})}, \Psi_{c,\rho(\vec{u})})$. Therefore

$$\mathcal{W}_{c,\rho(\vec{u})}(\vec{u}) \geq \mathcal{W}_{c,\rho(\vec{u})}(\Phi_{c,\rho(\vec{u})}, \Psi_{c,\rho(\vec{u})}). \tag{2.20}$$

Then from (2.19), (2.20), the first equality in (2.14), Remark 2.7 and Lemma 2.9 it follows

$$\begin{aligned} E_1(\vec{u}) + \rho(\vec{u})H(\vec{u}) + cG_1(\vec{u}) &\geq \mathcal{W}_{c,\rho(\vec{u})}(\Phi_{c,\rho(\vec{u})}, \Psi_{c,\rho(\vec{u})}) + \alpha\mathcal{F}(\Phi_{c,\rho(\vec{u})}, \Psi_{c,\rho(\vec{u})}) \\ &= d_c(\rho(\vec{u})) \\ &\geq d_c(\omega) + H(\vec{\Phi})(\rho(\vec{u}) - \omega) + \frac{1}{N(\epsilon)}|\rho(\vec{u}) - \omega| \\ &= E_1(\vec{\Phi}) + cG_1(\vec{\Phi}) + \rho(\vec{u})H(\vec{\Phi}) + \frac{1}{N(\epsilon)}|\rho(\vec{u}) - \omega|. \end{aligned}$$

This proves the Lemma. $\square$

Now we are ready to prove our theorem of stability of the set of travelling waves $\mathcal{S}_{c,\omega}$ in $X_{\mathbb{C}}$.

**Theorem 2.12.** *Let $\alpha = 2\beta$ and a fixed $c > 0$. Suppose that $d_c$ is strictly convex in an interval $I$ around $\omega$ then the set $\mathcal{S}_{c,\omega}$ is $X_{\mathbb{C}}$-stable with respect to equation (1.1).*

*Proof.* Assume that $\mathcal{S}_{c,\omega}$ is $X_{\mathbb{C}}$-unstable and choose initial data $\vec{u}_k(0) \in U_{c,\omega,1/k}$, such that

$$\sup_{0\le t<\infty} \inf_{\vec{\Phi}\in\mathcal{S}_{c,\omega}} ; \|\vec{u}_k(t) - \vec{\Phi}\|_{1\times\frac{1}{2}} \ge \delta,$$

where $\vec{u}_k(t) = (u_k(t), v_k(t))$ is the solution of (1.1) with initial data $\vec{u}_k(0)$. Then, by continuity in $t$, we can find $t_k$ such that

$$\inf_{\vec{\Phi}\in\mathcal{S}_{c,\omega}} \|\vec{u}_k(t_k) - \vec{\Phi}\|_{1\times\frac{1}{2}} = \delta. \tag{2.21}$$

By definition of $U_{c,\omega,1/k}$, and since $E_1, H$, and $G_1$ are invariants of the equation (1.1), we can find $\vec{\Phi}_k \in \mathcal{S}_{c,\omega}$ such that

$$\begin{gathered}
|E_1(\vec{u}_k(t_k)) - E_1(\vec{\Phi}_k)| = |E_1(\vec{u}_k(0)) - E_1(\vec{\Phi}_k)| \to 0\\
|H(\vec{u}_k(t_k)) - H(\vec{\Phi}_k)| = |H(\vec{u}_k(0)) - H(\vec{\Phi}_k)| \to 0\\
|G_1(\vec{u}_k(t_k)) - G_1(\vec{\Phi}_k)| = |G_1(\vec{u}_k(0)) - G_1(\vec{\Phi}_k)| \to 0
\end{gathered}$$

as $k \to \infty$. Moreover, by choosing $\delta$ small enough in Lemma 2.11 it follows that

$$\begin{aligned}
&E_1(\vec{u}_k(t_k)) - E_1(\vec{\Phi}_k) + \rho(\vec{u}_k(t_k))(H(\vec{u}_k(t_k)) - H(\vec{\Phi}_k)) + c(G_1(\vec{u}_k(t_k)) - G_1(\vec{\Phi}_k))\\
&\ge \frac{1}{N(\epsilon)}|\rho(\vec{u}_k(t_k)) - \omega|.
\end{aligned}$$

Since $\vec{u}_k(t_k)$ is uniformly bounded for $k$, it follows from the last inequality that $\rho(\vec{u}_k(t_k)) \to \omega$, as $k \to \infty$. Hence, by (2.17) and the continuity of $d_c$ we have

$$\lim_{k\to\infty} \beta\mathcal{F}(\vec{u}_k(t_k)) = -d_c(\omega). \tag{2.22}$$

On the other hand, for (2.1) and (2.14) ($d_c(\cdot)$ is constant on $S_{c,\omega}$) we have

$$\begin{aligned}
\mathcal{W}_{c,\omega}(\vec{u}_k(t_k)) &= E_1(\vec{u}_k(t_k)) + \omega H(\vec{u}_k(t_k)) + cG_1(\vec{u}_k(t_k)) - \alpha\mathcal{F}(\vec{u}_k(t_k))\\
&= d_c(\omega) + E_1(\vec{u}_k(t_k)) - E_1(\vec{\Phi}_k) + c(G_1(\vec{u}_k(t_k)) - G(\vec{\Phi}_k))\\
&\quad + \omega(H(\vec{u}_k(t_k)) - H(\vec{\Phi}_k)) - \alpha\mathcal{F}(\vec{u}_k(t_k)),
\end{aligned}$$

then by (2.22) and (2.14)

$$\lim_{k\to\infty} \mathcal{W}_{c,\omega}(\vec{u}_k(t_k)) = d_c(\omega) + 2d_c(\omega) = 3d_c(\omega) = \frac{1}{9\beta^2}[M_c(\omega)]^3.$$

Let $\vec{w}_k(t_k) = [\mathcal{F}(\vec{u}_k(t_k))]^{-1/3}\vec{u}_k(t_k)$, then $\mathcal{F}(\vec{w}_k(t_k)) = 1$, and so from (2.14) and (2.22) we conclude that

$$\begin{aligned}
\lim_{k\to\infty} \mathcal{W}_{c,\omega}(\vec{w}_k(t_k)) &= \lim_{k\to\infty} [\mathcal{F}(\vec{u}_k(t_k))]^{-2/3}\mathcal{W}_{c,\omega}(\vec{u}_k(t_k))\\
&= \Big(\frac{\beta}{d_c(\omega)}\Big)^{2/3}\frac{1}{9\beta^2}[M_c(\omega)]^3 \qquad = M_c(\omega).
\end{aligned}$$

Therefore, $\vec{w}_k(t_k)$ is a minimizing sequence for $\mathcal{M}_1$ and by Theorem 2.4 and Corollary 2.5 there exists $\vec{\psi}_k \in \mathcal{G}_1$ such that

$$\lim_{k\to\infty} \|\vec{w}_k(t_k) - \vec{\psi}_k\|_{1\times\frac{1}{2}} = 0. \tag{2.23}$$

Now from Theorem 2.4, $\vec{\psi}_k = (e^{icx/2}f_k, g_k)$ for $(f_k, g_k) \in G_1$, hence there exists $K > 0$ such that $(\phi_k, \psi_k) = (-\frac{K}{\alpha}f_k, -\frac{K}{\alpha}g_k)$ is a solution of (1.4). Then, since

$K = \frac{2}{3}M_c(\omega)$, it follows that $\vec{\Psi}_k = (e^{icx/2}\phi_k, \psi_k) \in \mathcal{S}_{c,\omega}$ and so from (2.23)

$$\lim_{k\to\infty} \|\vec{w}_k(t_k) - 3\beta[M_c(\omega)]^{-1}\vec{\Psi}_k\|_{1\times\frac{1}{2}} = 0. \tag{2.24}$$

Therefore, from (2.24) and being $\mathcal{S}_{c,\omega}$ a bounded set in $X_{\mathbb{C}}$ we have

$$\begin{aligned} \|\vec{u}_k(t_k) - \vec{\Psi}_k\|_{1\times\frac{1}{2}} &= |\mathcal{F}(\vec{u}_k(t_k))|^{1/3}\|[\mathcal{F}(\vec{u}_k(t_k))]^{-1/3}(\vec{u}_k(t_k) - \vec{\Psi}_k)\|_{1\times\frac{1}{2}} \\ &\le |\mathcal{F}(\vec{u}_k(t_k))|^{1/3}\Big[\|\vec{w}_k(t_k) - 3\beta[M_c(\omega)]^{-1}\vec{\Psi}_k\|_{1\times\frac{1}{2}} \\ &\quad + A|[\mathcal{F}(\vec{u}_k(t_k))]^{-1/3} + 3\beta[M_c(\omega)]^{-1}| \end{aligned}$$

and therefore we have that $\|\vec{u}_k(t_k) - \vec{\Psi}_k\|_{1\times\frac{1}{2}} \to 0$ as $k \to \infty$. But by (2.21) we get a contradiction. This shows the Theorem. $\square$

Next, we show the existence of intervals close to $c^2/4$ where the function $\omega \to d_c(\omega)$ is convex. In fact, for $\sigma = \omega - \frac{c^2}{4} > 0$ define

$$f(x) = \frac{\sigma^2}{1 + (\sqrt{\sigma}x)^2}, \quad g(x) = \frac{\sigma^{5/2}}{1 + (\sqrt{\sigma}x)^2},$$

functions in $H^1_{\mathbb{R}}(\mathbb{R})$ and $H^{1/2}_{\mathbb{R}}(\mathbb{R})$ respectively. Then, from $\int_{-\infty}^{\infty}[f'(x)]^2\,dx = k_0\sigma^{9/2}$, $\int_{-\infty}^{\infty}[D^{1/2}g(x)]^2\,dx = k_1\sigma^5$, $\int_{-\infty}^{\infty}[f(x)]^2\,dx = k_2\sigma^{7/2}$, $\int_{-\infty}^{\infty}[g(x)]^2\,dx = k_3\sigma^{9/2}$, and $\int_{-\infty}^{\infty} g(x)f^2(x)\,dx = k_4\sigma^6$, with $k_i > 0$, we have that $V(f,g) = k_0\sigma^{9/2} + (-\gamma k_1)\sigma^5 + k_2\sigma^{9/2} + ck_3\sigma^{9/2}$, where $\gamma < 0$ and $c > 0$. Therefore, from (2.7) and (2.10) it follows

$$M_c(\omega) \le \frac{V(f,g)}{[N(f,g)]^{2/3}} \le k_5(\sigma^{1/2} + \sigma).$$

Hence, (2.14) implies the inequality

$$0 < d_c(\omega) \le k_6\Big((\omega - \frac{c^2}{4})^{3/2} + (\omega - \frac{c^2}{4})^3\Big) \equiv j_{c^2/4}(\omega). \tag{2.25}$$

Therefore, since the function $\omega \in [c^2/4, \infty) \to j_{c^2/4}(\omega)$ vanishes to first order at $\omega = c^2/4$ and is convex, we obtain from (2.25) and from the positivity and monotonicity of $d_c$ (as function of $\omega$), the existence of intervals of convexity close to $c^2/4$.

## 3. Existence and stability of solitary waves for the Schrödinger-KdV equation

In this section, we give a theory of existence and stability of solitary waves solutions for equation (1.2) based on the same ideas exposed in the second section. Initially, we have the results of existence, which are based essentially on the works of Lopes [14, 15] on the Concentration Compactness Principle. In fact, if we denote the set of minimizers in $Y_{\mathbb{R}} = H^1_{\mathbb{R}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R})$ for $J_\lambda$ (defined in (1.10)) by

$$P_\lambda = \{(f,g) \in Y_{\mathbb{R}} : Z(f,g) = J_\lambda \text{ and } N(f,g) = \lambda\}, . \tag{3.1}$$

We have the following existence theroem.

**Theorem 3.1.** *Let $\alpha\beta > 0$, $\mu = -3\alpha$, $\sigma, \eta > 0$, $c > \gamma$, and let $\lambda$ be any positive number. Then, any minimizing sequence $\{(f_n, g_n)\}$ for $J_\lambda$ is relatively compact in $Y_{\mathbb{R}}$ up to translation, i.e., there are subsequences $\{(f_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(f_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $Y_{\mathbb{R}}$ to some $(f,g)$ which* *is a minimum of $J_\lambda$. Therefore, $P_\lambda \neq \emptyset$ and there are non-trivial solitary waves solutions $(\phi,\psi) = (\pm\frac{K}{\sqrt{2\alpha\beta}}f, -\frac{K}{\alpha}g)$ for equation* (1.5) *with $K > 0$.*

The proof of the above theorem is an immediate application of the results in [14, 15].

**Remark 3.2.** ¿From the equality

$$\psi(x) = -\frac{1}{2\sqrt{\eta(c-\gamma)}}\,\mathcal{K} * (\frac{3\alpha}{2}\psi^2 + \beta\phi^2)(x), \tag{3.2}$$

where $\mathcal{K}(x) = \exp(-\frac{\sqrt{c-\gamma}}{\sqrt{\eta}}|x|)$, it follows that for $\alpha > 0$, we have $\psi < 0$, and for $\alpha < 0$, we have $\psi > 0$.

**Remark 3.3.** Following the same techniques used in [2], we can show that if $(f,g) \in P_\lambda$ then $(|f|,g) \in P_\lambda$, and therefore $\phi(x) > 0$ for all $x$, or, $\phi(x) < 0$ for all $x$. Moreover, we can show that each component of $(f,g)$ is even and is a strictly decreasing positive functions on $(0,+\infty)$, up to translations.

Now, in the same spirit of section 2 we consider the following minimization problem in $Y_{\mathbb{C}} = H^1_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{R}}(\mathbb{R})$ for $\lambda > 0$

$$\mathcal{J}_\lambda = \inf\{\mathcal{Q}_{c,\omega}(h,g) : (h,g) \in Y_{\mathbb{C}} \text{ and } \mathcal{N}(h,g) = \lambda\},$$

where

$$\mathcal{Q}_{c,\omega}(h,g) = \int_{\mathbb{R}} |h'(x)|^2 + \eta(g'(x))^2 + \omega|h(x)|^2 + (c-\gamma)g^2(x)\,dx + c\,\mathrm{Im}\int_{\mathbb{R}} h(x)\overline{h'(x)}\,dx,$$

$\eta > 0$, $\omega > \frac{c^2}{4}$, $c > \gamma$, and

$$\mathcal{N}(h,g) = \int_{\mathbb{R}} g(x)[|h(x)|^2 + g^2(x)]\,dx. \tag{3.3}$$

Also, we denote the set of minimizers for $\mathcal{J}_\lambda$ by $\mathcal{P}_\lambda$, namely,

$$\mathcal{P}_\lambda = \{(h,g) \in Y_{\mathbb{C}} : \mathcal{Q}_{c,\omega}(h,g) = \mathcal{J}_\lambda \text{ and } \mathcal{N}(h,g) = \lambda\}. \tag{3.4}$$

Next, we shall show that every minimizing sequence for $\mathcal{J}_\lambda$ converges strongly in $Y_{\mathbb{C}}$, up to rotations and translations, to some element of $\mathcal{P}_\lambda$. Initially, we have a similar result as in Theorem 2.3. Let the following minimization problem be

$$J^{\mathbb{C}}_\lambda = \inf\{Z_{\mathbb{C}}(h,g) : (h,g) \in Y_{\mathbb{C}} \text{ and } \mathcal{N}(h,g) = \lambda\},$$

where $\lambda > 0$,

$$Z_{\mathbb{C}}(h,g) = \int_{\mathbb{R}}[|h'(x)|^2 + \eta(g'(x))^2 + \sigma|h(x)|^2 + (c-\gamma)g^2(x)]dx,$$

and $\mathcal{N}$ as in (3.3), then we have the following results.

**Lemma 3.4.** *Let $\alpha\beta > 0$, $\sigma,\eta > 0$, $\mu = -3\alpha$, $c > \gamma$, and let $\lambda$ be any positive number. Then, any minimizing sequence $\{(h_n,g_n)\}$ for $J^{\mathbb{C}}_\lambda$ is relatively compact in $Y_{\mathbb{C}}$ up to translation, i.e., there are subsequences $\{(h_{n_k},g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(h_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $Y_{\mathbb{C}}$ to some $(h,g)$ which is a minimum of $J^{\mathbb{C}}_\lambda$. Moreover, $(h,g) = (e^{i\theta}f,g)$ where $\theta \in \mathbb{R}$ and $(f,g) \in P_\lambda$.*

The proof of the lemma above is similar to the proof of Theorem 2.3.

**Theorem 3.5.** *Let $\alpha\beta > 0$, $\mu = -3\alpha$, $\eta > 0$, $c > \gamma$, $\omega > \frac{c^2}{4}$ and $\lambda > 0$. Then, any minimizing sequence $\{(h_n, g_n)\}$ for $\mathcal{J}_\lambda$ is relatively compact in $Y_{\mathbb{C}}$ up to rotations and translation, i.e., there are subsequences $\{(h_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that $(e^{icy_k/2}h_{n_k}(\cdot - y_{n_k}), g_{n_k}(\cdot - y_{n_k}))$ converges strongly in $Y_{\mathbb{C}}$ to some $(h, g)$ which is a minimum of $\mathcal{J}_\lambda$. Moreover, $(h, g) = (e^{i\theta}e^{icx/2}f, g)$ where $(f, g) \in P_\lambda$.*

The proof of the above theorem is similar to the proof in Theorem 2.4.

**Corollary 3.6.** *Let $\alpha\beta > 0$, $\mu = -3\alpha$, $\eta > 0$, $c > \gamma$, $\omega > \frac{c^2}{4}$ and $\lambda > 0$. Then, the set $\mathcal{P}_\lambda$ is nonempty. Moreover, if $\{(h_n, g_n)\}$ is any minimizing sequence for $\mathcal{J}_\lambda$ then*

(i) *There exist sequences $\{y_n\}$, $\{\theta_n\}$ and an element $(h, g) \in \mathcal{P}_\lambda$ such that $\{(e^{i\theta_n}h_n(\cdot + y_n), g_n(\cdot + y_n))\}$ has a subsequence converging strongly in $Y_{\mathbb{C}}$ to $(h, g)$.*
(ii) $\lim_{n\to\infty} \inf_{\theta, y\in\mathbb{R}; \vec{\psi}\in\mathcal{J}_\lambda} \|(e^{i\theta}h_n(\cdot + y), g_n(\cdot + y)) - \vec{\psi}\|_{1\times 1} = 0$.
(iii) $\lim_{n\to\infty} \inf_{\vec{\psi}\in\mathcal{J}_\lambda} \|(h_n, g_n) - \vec{\psi}\|_{1\times 1} = 0$.

The proof of the above corollary is similar to the proof of Corollary 2.5.

Now, defining the following minimization problem

$$T_c(\omega) = \inf_{(h,g)\in Y_{\mathbb{C}}} \frac{\mathcal{Q}_{c,\omega}(h, g)}{[\mathcal{N}(h, g)]^{2/3}}, \tag{3.5}$$

we have that

$$T_c(\omega) = \inf_{(h,g)\in Y_{\mathbb{C}}} \{\mathcal{Q}_{c,\omega}(h, g) : \mathcal{N}(h, g) = 1\}, \tag{3.6}$$

and therefore from Theorem 2.3 it follows that (3.6) can be written as

$$T_c(\omega) = \underset{(f,g)\in Y_{\mathbb{R}}}{\rightarrow} Inf\ \{\ Z(f, g) : N(f, g) = 1\}.$$

For $\alpha = 2\beta$, $\mu = -3\alpha$, $\omega > \frac{c^2}{4}$ and $c > \gamma$, we define the set

$$\mathcal{B}_{c,\omega} = \{(e^{i\theta}e^{icx/2}\phi, \psi)|\ (\phi, \psi) \in Y_{\mathbb{R}}, N(\phi, \psi) = -\frac{1}{3\beta}Z(\phi, \psi) = -\frac{1}{27\beta^3}[T_c(\omega)]^3\}. \tag{3.7}$$

Therefore, if $(e^{i\theta}e^{icx/2}\phi, \psi) \in \mathcal{B}_{c,\omega}$ then $(\phi, \psi)$ satisfies (1.5) with $\alpha = 2\beta$ and $\mu = -3\alpha$.

To establish stability for (1.2), for $(\Pi_{c,\omega}(\xi), \Theta_{c,\omega}(\xi)) = (e^{ic\xi/2}\phi_{c,\omega}(\xi), \psi_{c,\omega}(\xi))$ in $\mathcal{B}_{c,\omega}$, we define the following function with a parameter $\omega$,

$$d_c^{(2)}(\omega) \equiv d^{(2)}(c, \omega) \tag{3.8}$$

where

$$d^{(2)}(c, \omega) = E_2(\Pi_{c,\omega}, \Theta_{c,\omega}) + \omega\ H(\Pi_{c,\omega}) + cG_2(\Pi_{c,\omega}, \Theta_{c,\omega}),$$

with $E_2$, $H$, $G_2$ defined in (1.14) and (1.18), $c > \gamma$ and $\omega \in (\frac{c^2}{4}, \infty)$. Therefore, as in section 2, we get that $d_c^{(2)}$ is constant on $\mathcal{B}_{c,\omega}$ and is strictly increasing as a function of $\omega$. Moreover, for any $(\Pi_{c,\omega}, \Theta_{c,\omega}) \in \mathcal{B}_{c,\omega}$ we have

$$d_c^{(2)}(\omega) = -\beta\mathcal{N}(\Pi_{c,\omega}, \Theta_{c,\omega}) = \frac{1}{27\beta^2}[T_c(\omega)]^3. \tag{3.9}$$

So, we have the following result of nonlinear stability of the set $\mathcal{B}_{c,\omega}$. Its proof follows the same lines of the proof of Theorem 2.12.

**Theorem 3.7.** *Let $\alpha = 2\beta$, $\mu = -3\alpha$, $\eta > 0$, $\omega > \frac{c^2}{4}$ and $c > \gamma$. Suppose that for $c$ fixed, $d_c^2$ is strictly convex in an interval $I$ around $\omega$, then the set $\mathcal{B}_{c,\omega}$ is $Y_{\mathbb{C}}$-stable with respect to* (1.2).

Finally, we note that is possible to show the existence of intervals around of $c^2/4$ where the function $\omega \to d_c^{(2)}(\omega)$ is convex. In fact, for $\sigma = \omega - \frac{c^2}{4} > 0$ define

$$
\begin{gathered}
f(x) = e^{-\sqrt{\sigma}|x|} \sin \sigma^{3/2}|x|, \\
g(x) = e^{-\sqrt{\sigma}|x|} \sin \sigma^{2}|x|
\end{gathered}
$$

functions in $H^1_{\mathbb{R}}(\mathbb{R})$. Then from the relations

$$
\begin{gathered}
\int_{-\infty}^{\infty} [f'(x)]^2\,dx = \frac{\sigma^{5/2}}{2}, \quad \int_{-\infty}^{\infty} [g'(x)]^2\,dx = \frac{\sigma^{7/2}}{2}, \\
\int_{-\infty}^{\infty} [f(x)]^2\,dx = \frac{\sigma^{3/2}}{2(1+\sigma^2)}, \quad \int_{-\infty}^{\infty} [g(x)]^2\,dx = \frac{\sigma^{5/2}}{2(1+\sigma^3)}, \\
\int_{-\infty}^{\infty} g(x) f^2(x)\,dx = -\frac{4\sigma^3(\sigma^3 - 27 - 4\sigma^2)}{\sigma^3(\sigma^3 - 27 - 4\sigma^2)^2 + (9\sigma^3 - 27 - 12\sigma^2)^2}, \\
\int_{-\infty}^{\infty} g^3(x)\,dx = \frac{4\sigma^4}{3(1+\sigma^3)(9+\sigma^3)},
\end{gathered}
$$

we have

$$
\begin{gathered}
Z(f,g) \le [1 + (c-\gamma)]\sigma^{5/2} + \sigma^{7/2} \\
N(f,g) \ge \frac{4\sigma^3}{\sigma^3(\sigma^3 - 27 - 4\sigma^2)^2 + (9\sigma^3 - 27 - 12\sigma^2)^2},
\end{gathered}
$$

where in the last inequality we choose $\sigma$ such that $27 + 4\sigma^2 - \sigma^3 \ge 1$, for example $\sigma < 1$. Therefore, from (3.5),

$$
\begin{aligned}
T_c(\omega) &\le \frac{k_0}{\sigma^2}([1 + (c-\gamma)]\sigma^{5/2} + \sigma^{7/2})(\sigma^2 + 1) \\
&\le k_0([1 + (c-\gamma)]\sigma^{5/2} + \sigma^{7/2} + \sigma^{3/2} + [1 + (c-\gamma)]\sigma^{1/2}).
\end{aligned}
$$

Hence, (3.9) implies

$$
\begin{aligned}
d_c^{(2)}(\omega) &\le k_1\Big([1 + (c-\gamma)]^3[(\omega - \frac{c^2}{4})^{15/2} + (\omega - \frac{c^2}{4})^{3/2}] \\
&\quad + (\omega - \frac{c^2}{4})^{21/2} + (\omega - \frac{c^2}{4})^{9/2}\Big) \\
&\equiv h_{c^2/4}(\omega).
\end{aligned} \tag{3.10}
$$

Therefore, since the function $\omega \in [c^2/4, \infty) \mapsto h_{c^2/4}(\omega)$ vanishes to first order at $\omega = c^2/4$ and is convex, we obtain from (3.10) and from the positivity and monotonicity of $d_c^{(2)}$ (as function of $\omega$), the existence of intervals of convexity close to $c^2/4$.

**Acknowledgments.** The author is grateful to the referee for his/her helpful comments which led to the improvement of this paper.

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