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CONDITIONS FOR THE LOCAL REGULARITY OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS NEAR THE BOUNDARY

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Abstract. In this paper we present conditions for the local regularity of weak solutions of the Navier-Stokes equations near the smooth boundary.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, let $T > 0$ and $Q_T = \Omega \times (0, T)$. We consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $u = (u_1, u_2, u_3)$ and the pressure ϕ in Q_T:

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla \phi = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

$$u|_{t=0} = u_0, \tag{1.4}$$

where $\nu > 0$ is the viscosity coefficient. The initial data u_0 satisfy the compatibility conditions $u_0|_{\partial\Omega} = 0$ and $\nabla \cdot u_0 = 0$ and for our purposes we can suppose without loss of generality that u_0 is sufficiently smooth. The existence of a weak solution $u \in L^2(0, T; W_0^{1,2}(\Omega)^3) \cap L^\infty(0, T; L^2_\sigma(\Omega))$ of (1.1)–(1.4) is well known (see [4] or [14]). The associated pressure ϕ is a scalar function such that u and ϕ satisfy the equation (1.1) in Q_T in the sense of distributions.

Let $q > 1$. Let us set

$$L^q_\sigma(\Omega) = \text{ closure of } \{\varphi \in (C_0^\infty(\Omega))^3; \nabla \cdot \varphi = 0 \text{ in } \Omega\} \text{ in } (L^q(\Omega))^3,$$
$$G^q(\Omega) = \{\nabla p; p \in W^{1,q}(\Omega)\}.$$

We then have the Helmholtz decomposition

$$(L^q(\Omega))^3 = L^q_\sigma(\Omega) \oplus G^q(\Omega) \quad \text{(direct sum)}. \tag{1.5}$$

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Let P_σ^q be the continuous projection from $(L^q(\Omega))^3$ onto $L^q_\sigma(\Omega)$ associated with Helmholtz decomposition. If Δ denotes the Laplace operator with zero boundary condition, then the Stokes operator is defined as $A_q = -P_\sigma^q\Delta$ with $D(A_q) = W^{2,q}(\Omega)^3 \cap W_0^{1,q}(\Omega)^3 \cap L^q_\sigma(\Omega)$.

In this paper we use both scalar and vector functions and for the sake of simplicity we denote by S any space S^3 of vector functions with the exception of the notation in Lemma 2.1. We use the standard notation for the Lebesgue spaces $L^p(\Omega)$ and their norms $\|\cdot\|_{p,\Omega}$. The Sobolev spaces are denoted by $W^{k,p}(\Omega)$. Sometimes we drop Ω and write only L^p, $\|\cdot\|_p$ and $W^{k,p}$. Further, if $A = B\times(t_1,t_2)$ then $L^{p,q}$ or $L^{p,q}(A)$ denote the space $L^q(t_1,t_2;L^p(B))$ with the norm $\|\cdot\|_{p,q,A}$ or simply $\|\cdot\|_{p,q}$. $L^{p,p}(A)$ is also denoted as $L^p(A)$ or L^p. $C^\beta(\overline{\Omega})$ is the space of Hölder continuous functions on Ω with the norm $\|f\|_{C^\beta(\overline{\Omega})} = \sup_{x\in\Omega}|f(x)| + \sup_{x,y\in\Omega,x\neq y}|f(x)-f(y)|/|x-y|^\beta$.

$L^p_w(\Omega)$ denote the weak Lebesgue space on Ω with the quasi-norm $\|\cdot\|_{p,w,\Omega}$ defined by $\|\phi\|_{p,w,\Omega} = \sup_{R>0} R\mu\{x\in\Omega; |\phi(x,t)| > R\}^{1/p}$, where μ is the Lebesgue measure. There exists an equivalent norm to $\|\cdot\|_{p,w,\Omega}$, so we may understand $L^p_w(\Omega)$ as a Banach space. Let us note that $L^p(\Omega)\subset L^p_w(\Omega)$ and $\|\phi\|_{p,w,\Omega}\le\|\phi\|_p$ for every $\phi\in L^p(\Omega)$. Sometimes we write L^p_w and $\|\phi\|_{p,w}$ instead of $L^p_w(\Omega)$ and $\|\phi\|_{p,w,\Omega}$, respectively.

For $(x_0,t_0)\in\overline{\Omega}\times(0,T)$ and $r>0$ we will denote $B_r = B_r(x_0)$ the open ball centered at x_0 with radius r, $D_r = D_r(x_0) = B_r(x_0)\cap\Omega$, $Q_r = Q_r(x_0,t_0) = D_r(x_0)\times(t_0-r^2,t_0+r^2)$. A point $(x_0,t_0)\in\overline{\Omega}\times(0,T)$ is called a regular point of a weak solution u if $u\in L^\infty(Q_r)$ for some $r>0$. Otherwise, (x_0,t_0) is called a singular point of u.

Let us now present some recent results concerning the regularity of weak solutions near the boundary. S.Takahashi showed in [12] and [13] that if $u\in L^{p,q}(Q_r)$, where $(x_0,t_0)\in\partial\Omega\times(0,T)$, $r>0$, $p,q\in(1,\infty)$ and $3/p+2/q\le 1$, then $u\in L^\infty(Q_{\tilde r})$ for any $\tilde r\in(0,r)$ provided that $B_r\cap\partial\Omega$ is a part of a plane.

Takahashi's result was improved in [11], where the following theorem was proved.

Theorem 1.1. *Let u be an arbitrary weak solution of* (1.1) - (1.4)*, $(x_0,t_0)\in\partial\Omega\times(0,T)$, $r>0$. We suppose that $u\in L^{p,q}(Q_r)$, where $2/q+3/p=1$ and $p,q\in(1,\infty)$. Then*

$$u\in L^\infty(t_0-\tilde r^2,t_0+\tilde r^2;C^\beta(\overline{D}_{\tilde r})) \tag{1.6}$$

for every $\beta\in(0,1)$ and $\tilde r\in(0,r)$.

Neustupa [9] proved a similar result. He supposed that $u\in L^q(t_1,t_2;L^p(U_r^*))$ for some $r>0$, $0<t_1<t_2<T$, $p,q\in(1,\infty)$ with $3/p+2/q=1$, where $U_r^* = \{x\in\Omega;\operatorname{dist}(x,\partial\Omega)<r\}$. He proved under this assumption that if u is a weak solution of (1.1)–(1.4) satisfying the strong energy inequality then $u\in L^\infty(t_1+\zeta,t_2-\zeta;W^{2+\delta,2}(U_\rho^*))$ and $\partial u/\partial t,\nabla\phi\in L^\infty(t_1+\zeta,t_2-\zeta;W^{\delta,2}(U_\rho^*))$ for each $\delta\in[0,1/2)$, $\rho\in(0,r)$ and such $\zeta>0$ that $t_1+\zeta<t_2-\zeta$.

The local boundary regularity of u was also studied in [2], [10] and [6]. It was proved in [2] that a suitable weak solution u is bounded locally near the boundary if $u\in L^{p,q}$, $3/p+2/q=1$, $p,q\in(1,\infty)$ and the pressure ϕ is bounded at the boundary. Moreover, better regularity of ϕ gives better local regularity of u. Seregin presented in [10] a condition for local Hölder continuity for suitable weak solutions near the plane boundary which has the form of the famous Caffarelli-Kohn-Nirenberg condition for boundedness of suitable weak solutions in a neighborhood of an interior point of Q_T. Also Kang [6] studied boundary regularity of weak solutions in the half-space. He proved that a weak solution u which is locally in the class $L^{p,q}$ with $3/p+2/q=1$ and $p,q\in(1,\infty)$ near the boundary is Hölder continuous up to the boundary. The main tool in the proof of this result is a pointwise estimate for the fundamental solution of the Stokes system.

In this paper we present some conditions ensuring the local regularity of weak solutions near the boundary. The following theorem is our main result.

Theorem 1.2. *Let $u=(u_1,u_2,u_3)$ be an arbitrary weak solution of (1.1)-(1.4). There exists $\varepsilon>0$ such that if $(x_0,t_0)\in\partial\Omega\times(0,T)$, $r>0$ and at least one of the following conditions is fulfilled:*

- (A1) $\|u\|_{L^\infty(t_0-r^2,t_0+r^2;L^3_\omega(D_r))}<\varepsilon$,
- (A2) $\nabla u\in L^{3,2}(Q_r)$,
- (A3) $\nabla u_1,\nabla u_2\in L^{p,q}(Q_r)$, $p\in(3/2,\infty]$, $q\in(2,\infty]$, $3/p+2/q\le 2$,
- (A4) $\nabla u_1,\nabla u_2\in L^{p,q}(Q_r)$, $p\in(3,\infty]$, $q=2$.

Then, for every $\beta\in(0,1)$ and $\tilde r\in(0,r)$,

$$u\in L^\infty(t_0-\tilde r^2,t_0+\tilde r^2;C^\beta(\overline{D}_{\tilde r}))\,. \tag{1.7}$$

Remark 1.3. We can compare conditions (A3) and (A4) with Theorem 1.2 from [1], where the regularity of u was proved under the assumption that $\nabla u_1,\nabla u_2\in L^{p,q}, p\in[3,\infty], q\in[2,\infty]$ and $3/p+2/q=1$. If this assumption holds than either the condition (A3) or the condition (A4) is satisfied. Thus, in this sense, Theorem 1.2 is a generalization of Theorem 1.2 from [1].

The proof of Theorem 1.2 will be based on Takahashi [12], and Theorem 1.1 will be a corollay of Theorem 1.2. Firstly, we present some auxiliary results.

2. Auxiliary results

We will use the following lemma which was proved in [12] and in [3, Theorem 3.2, Chap.III.3].

Lemma 2.1. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$, Γ be an open subset of ∂D, $r\in(1,\infty)$, $j\in N\cup\{0\}$. There exists a bounded linear operator $K=K_{j,r,D,\Gamma}$: $W^{j,r}_0(D)\to W^{j+1,r}_0(D)^3$ such that*

- (i) $\nabla\cdot Kg=g$ *for all* $g\in W^{j,r}_0(D)$ *such that* $\int_D g\,dx=0$,
- (ii) $\|\nabla^{j+1}Kg\|_r\le c\|\nabla^j g\|_r$ *for all* $g\in W^{j,r}_0(D)$, $c=c(j,r,D)$
- (iii) $\operatorname{supp}Kg\subset D\cup\Gamma$ *if* $\operatorname{supp}g\subset D\cup\Gamma$.

In Lemma 2.1, $W^{j,r}_0(D)$ is the completion of $C^\infty_0(D)$ with respect to the standard norm of the space $W^{j,r}(D)$. It is possible to show that $K_{j,r,D,\Gamma}(g)=K_{l,s,G,\Gamma}(g)$ if $g\in W^{j,r}_0(D)\cap W^{l,s}_0(D)$, where $r,s\in(1,\infty)$ and $j,l\in N\cup\{0\}$ and in the rest of the paper the operator $K_{j,r,D,\Gamma}$ is denoted by K.

For $l,l'\in(1,\infty)$ we define the Banach space

$$X^{l,l'}=\{v\in L^{l'}(0,T,D(A_l));\frac{\partial v}{\partial t}\in L^{l'}(0,T,L^l_\sigma(\Omega)),v(0)=0\}$$

with the norm

$$\|v\|_{X^{l,l'}}=\|A_lv\|_{l,l'}+\Big\|\frac{\partial v}{\partial t}\Big\|_{l,l'}.$$

We consider the Stokes problem

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla\phi = f \quad \text{in } Q_T, \tag{2.1}$$

$$\nabla\cdot u = 0 \quad \text{in } Q_T, \tag{2.2}$$

$$u = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.3}$$

$$u|_{t=0} = 0. \tag{2.4}$$

It was proved in [5, Theorem 2.8] that if $f \in L^{\beta,\beta'}$, where $\beta,\beta' \in (1,\infty)$, then there exists a unique weak solution $(u,\nabla\phi)$ of (2.1) - (2.4) such that

$$\|\frac{\partial u}{\partial t}\|_{\beta,\beta'} + \|A_\beta u\|_{\beta,\beta'} + \|\nabla\phi\|_{\beta,\beta'} \le c\|f\|_{\beta,\beta'},\ c = c(\beta,\beta'). \tag{2.5}$$

The following lemma was proved in [12] and [11]. It further improves the regularity of the velocity u.

Lemma 2.2. *Let $\beta,\beta' \in (1,\infty)$, $\gamma\in[\beta,\infty)$, $\gamma'\in[\beta',\infty)$ and*

$$\frac{2}{\beta'} + \frac{3}{\beta} = \frac{2}{\gamma'} + \frac{3}{\gamma} + 1. \tag{2.6}$$

Let $f \in L^{\beta,\beta'}$. If $(u,\nabla\phi)$ is a weak solution of (2.1)–(2.4) then $\nabla u \in L^{\gamma,\gamma'}$ and

$$\|\nabla u\|_{\gamma,\gamma'} \le c\|f\|_{\beta,\beta'}, \quad c = c(\beta,\beta',\gamma,\gamma'). \tag{2.7}$$

Lemma 2.3. *Let $u \in L^\infty(0,T,L^3_\omega(\Omega))$, $v\in L^s(0,T;W^{2,r}(\Omega)\cap W^{1,r}_0(\Omega))$, $r\in(1,3)$, $s\in(1,2)$. Then*

$$\|u\cdot\nabla v\|_{r,s} \le C\|u\|_{L^\infty(0,T,L^3_\omega(\Omega))}\|\cdot\|v\|_{L^s(0,T;W^{2,r}(\Omega))}. \tag{2.8}$$

Proof. We use the procedure used in [7, Lemma 2.7]. Let $1<r_0<r<r_1<3$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ for some $\theta\in(0,1)$. Let $\frac{1}{q_j} = \frac{1}{r_j} - \frac{1}{3}$, $j=0,1$. We can write for every $w\in D(\Delta_{r_j}) = W^{2,r_j}(\Omega)\cap W^{1,r_j}_0(\Omega)$:

$$\begin{aligned}\|u\cdot\nabla w\|_{r_j,\omega} &\le C\|u\|_{3,\omega}\|\nabla w\|_{q_j,\omega} \le C\|u\|_{3,\omega}\|\nabla w\|_{q_j}\\ &\le C\|u\|_{3,\omega}\|w\|_{W^{2,r_j}} \le C\|u\|_{3,\omega}\|\Delta_{r_j}w\|_{r_j}.\end{aligned}$$

If $\phi\in L^{r_j}$ and we put $w = \Delta^{-1}_{r_j}\phi$, we get

$$\|u\cdot\nabla(\Delta^{-1}_{r_j}\phi)\|_{r_j,\omega} \le C\|u\|_{3,\omega}\|\phi\|_{r_j}.$$

Therefore, the mapping $\phi\to u\cdot\nabla(\Delta^{-1}_{r_j}\phi)$ is a linear bounded operator from L^{r_j} into $L^{r_j}_\omega$ with the norm less than $C\|u\|_{3,\omega}$. By the use of the Marcinkiewicz interpolation theorem (see [8], p.106) we get that it is also a linear bounded operator from L^r into L^r with the norm less than $C\|u\|_{3,\omega}$, that is

$$\|u\cdot\nabla(\Delta^{-1}_r\phi)\|_r \le C\|u\|_{3,\omega}\|\phi\|_r$$

for every $\phi\in L^r$. If $w\in D(\Delta_r)$, then $\Delta_r w\in L^r$ and

$$\|u\cdot\nabla w\|_r = \|u\cdot\nabla(\Delta^{-1}_r(\Delta_r w))\|_r \le C\|u\|_{3,\omega}\|\Delta_r w\|_r \le C\|u\|_{3,\omega}\|w\|_{W^{2,r}(\Omega)}.$$

Inequality (2.8) now follows easily from the Hölder inequality. □

Lemma 2.4. *Let $\nabla u\in L^{p,q}(Q_T)$, $p\in(3/2,3]$, $q\in[2,\infty)$ and $3/p+2/q=2$. Let $v\in X^{r,s}$, $r\in(1,p)$, $s\in(1,q)$. Then*

$$\|v\cdot\nabla u\|_{r,s} \le C\|\nabla u\|_{p,q}\cdot\|v\|_{X^{r,s}}. \tag{2.9}$$

Proof. Since $v \in X^{r,s}$, it follows from Lemma 2.2 that $\nabla v \in L^{\frac{3qr}{2r+3q-qr}, \frac{qs}{q-s}}$ and

$$\|\nabla v\|_{\frac{3qr}{2r+3q-qr}, \frac{qs}{q-s}} \le c\|v\|_{X^{r,s}}.$$

The Sobolev inequality gives immediately that $\|v\|_{\frac{pr}{p-r}, \frac{qs}{q-s}} \le C\|v\|_{X^{r,s}}$. The Hölder inequality yields that $\|v \cdot \nabla u\|_{r,s} \le \|\nabla u\|_{p,q} \cdot \|v\|_{\frac{pr}{p-r}, \frac{qs}{q-s}}$ and (2.9) is the consequence of the last two inequalities. □

Lemma 2.5. *Let $u \in L^{p,q}(Q_T)$, $p \in (3, \infty]$, $q \in [2, \infty)$, $3/p + 2/q = 1$. Let $v \in X^{r,s}$, $r \in (1, p)$, $s \in (1, q)$. Then*

$$\|u \cdot \nabla v\|_{r,s} \le c\|u\|_{p,q} \cdot \|v\|_{X^{r,s}}. \tag{2.10}$$

Proof. Let $p < \infty$. Using Lemma 2.2,

$$\|\nabla v\|_{\frac{pr}{p-r}, \frac{qs}{q-s}} \le c\|v\|_{X^{r,s}}.$$

Since $\|u \cdot \nabla v\|_{r,s} \le \|u\|_{p,q} \cdot \|\nabla v\|_{\frac{pr}{p-r}, \frac{qs}{q-s}}$, (2.10) follows immediately. If $p = \infty$, then the proof proceeds analogically. □

Lemma 2.6. *Let $\nabla u \in L^{p,q}(Q_T)$, $p \in (3/2, 3]$, $q \in [2, \infty)$ and $3/p + 2/q = 2$. Let further $r \in (1, p)$, $s \in (2q/(q+2), q)$, $3/r + 2/s = 3$ and $w \in L^\rho(0, T; W^{2,h})$ for every $h \in (1, 3)$ and $\rho \in (1, \infty)$ such that $2/\rho + 3/h = 3$. Then*

$$w \cdot \nabla u \in L^{r,s}(Q_T). \tag{2.11}$$

Proof. If we choose $\rho = qs/(q-s)$ and $h = 3qs/(3qs - 2q + 2s)$, then $\rho \in (2, \infty)$, $h \in (1, 3/2)$, $3/h + 2/\rho = 3$ and $w \in L^\rho(0, T; W^{2,h})$. Consequently, by the Sobolev inequality

$$\|w\|_{\frac{pr}{p-r}, \frac{qs}{q-s}} \le C\|w\|_{L^\rho(0,T;W^{2,h})}$$

and (2.11) follows from the inequality $\|w \cdot \nabla u\|_{r,s} \le \|\nabla u\|_{p,q}\|w\|_{\frac{pr}{p-r}, \frac{qs}{q-s}}$. □

Lemma 2.7. *Let $u \in L^q(0, T; W_0^{1,p}(\Omega))$, $p \in (3/2, 3)$, $q \in (2, \infty)$ and $3/p + 2/q = 2$. Let further $r \in (1, 3)$, $s \in (1, q)$, $3/r + 2/s = 3$ and $w \in L^\rho(0, T; W^{2,h})$ for every $h \in (1, 3)$ and $\rho \in (1, \infty)$, such that $2/\rho + 3/h = 3$. Then*

$$u \cdot \nabla w \in L^{r,s}(Q_T). \tag{2.12}$$

Proof. Let us put $p' = 3p/(3-p)$. Then $u \in L^{p',q}(Q_T)$. Further,

$$\nabla w \in L^\rho(0, T; L^{h'})$$

for every $h' \in (3/2, \infty)$, $\rho \in (1, \infty)$ such that $2/\rho + 3/h' = 2$. If we choose $h' = p'r/(p'-r)$ and $\rho = qs/(q-s)$, (2.12) then follows immediately from the inequality $\|u \cdot \nabla w\|_{r,s} \le \|u\|_{p',q}\|\nabla w\|_{h',\rho}$. □

Remark 2.8. Lemma 2.7 holds also if $p > 3$ and $q = 2$. In this case we have $p' = \infty$ and $h' = r$.

We now denote

$$B_1(u, v) = u \cdot \nabla v = \Big(u_j \frac{\partial v_1}{\partial x_j}, u_j \frac{\partial v_2}{\partial x_j}, u_j \frac{\partial v_3}{\partial x_j}\Big),$$

$$B_2(u, v) = v \cdot \nabla u = \Big(v_j \frac{\partial u_1}{\partial x_j}, v_j \frac{\partial u_2}{\partial x_j}, v_j \frac{\partial u_3}{\partial x_j}\Big),$$

$$B_3(u, v) = B_4(u, v) = \Big(v_j \frac{\partial u_1}{\partial x_j}, v_j \frac{\partial u_2}{\partial x_j}, u_1 \frac{\partial v_3}{\partial x_1} + u_2 \frac{\partial v_3}{\partial x_2} - v_3\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)\Big).$$

Lemma 2.9. *Let $i \in \{1,2,3,4\}$, $l' \in (1,2)$, $l \in (1,3)$. Let us consider the following conditions:*

1. $u \in L^\infty(0,T; L^3_\omega(\Omega))$,
2. $\nabla u \in L^{3,2}(Q_T)$,
3. $u_1, u_2 \in L^q(0,T; W_0^{1,p}(\Omega))$, $p \in (3/2, \infty]$, $q \in (2,\infty]$, $3/p + 2/q \le 2$,
4. $u_1, u_2 \in L^q(0,T; W_0^{1,p}(\Omega))$, $p \in (3,\infty]$, $q = 2$.

If condition (1) is satisfied and if moreover $l \in (1,p)$ for $i = 3$, then the operator $v \mapsto B_i(u,v)$ is a linear bounded operator from $X^{l,l'}$ into $L^{l,l'}$ with the norm less than $C\|u\|_{L^\infty(0,T,L^3_\omega(\Omega))}$ for $i = 1$, $C\|\nabla u\|_{3,2}$ for $i = 2$ and $C(\|\nabla u_1\|_{p,q} + \|\nabla u_2\|_{p,q})$ for $i = 3, 4$.

Proof. If $i = 1$ then the proof follows immediately from Lemma 2.3. If $i = 2$ then the proof follows immediately from Lemma 2.4. If $i = 3$ and $i = 4$ the proof is the consequence of Lemma 2.4 and Lemma 2.5. □

For $i \in \{1,2,3,4\}$ let us consider the problem

$$\frac{\partial v}{\partial t} - \nu \Delta v + B_i(u,v) + \nabla P = g \quad \text{in } Q_T, \tag{2.13}$$

$$\nabla \cdot v = 0 \quad \text{in } Q_T, \tag{2.14}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.15}$$

$$v|_{t=0} = 0. \tag{2.16}$$

Lemma 2.10. *Let $i \in \{1,2,3,4\}$. Let $g \in L^{l,l'}$, $2/l' + 3/l = 3$, $l' \in (1,2)$, $l \in (3/2,3)$. Let the condition i) from Lemma 2.9 be fulfilled and $C\|u\|_{L^\infty(0,T;L^3_\omega(\Omega))} < \varepsilon$ for $i = 1$, $C\|\nabla u\|_{3,2} < \varepsilon$ for $i = 2$, $C(\|\nabla u_1\|_{p,q} + \|\nabla u_2\|_{p,q}) < \varepsilon$ and $l \in (3/2, p)$ for $i = 3$ and $C(\|\nabla u_1\|_{p,q} + \|\nabla u_2\|_{p,q}) < \varepsilon$ for $i = 4$, where ε is a sufficiently small positive number. Then there exists a unique $v \in X^{l,l'}$ and $\nabla P \in L^{l,l'}$, which solve the problem* (2.13)-(2.16).

Proof. The operator $v \to \frac{\partial v}{\partial t} + A_l v$ is one to one linear bounded operator from $X^{l,l'}$ onto $L^{l'}(0,T; L^l_\sigma)$. According to Lemma 2.9, the norm of the operator $v \to P^l_\sigma B_i(u,v)$ is sufficiently small. Accordingly, the operator $v \to \frac{\partial v}{\partial t} + A_l v + P^l_\sigma B_i(u,v)$ is one to one linear bounded operator from $X^{l,l'}$ onto $L^{l'}(0,T; L^l_\sigma)$. Therefore, there exists a unique $v \in X^{l,l'}$ such that

$$\frac{\partial v}{\partial t} + A_l v + P^l_\sigma B_i(u,v) = P^l_\sigma g$$

that is

$$P^l_\sigma \Big(\frac{\partial v}{\partial t} - \nu \Delta v + B_i(u,v) - g \Big) = 0$$

holds for almost every $t \in (0,T)$. The existence of P such that (2.13) follows from Helmholtz decomposition of the space L^l. □

3. Proof of Theorem 1.2

We suppose throughout this section that the assumptions of Theorem 1.2 are satisfied. Let $i \in \{1,2,3,4\}$ be fixed and the condition (Ai) from Theorem 1.2 be fulfilled. ϕ denotes the associated pressure to u. Let $\tilde r \in (0,r)$. Let us localize the problem (1.1)-(1.4) in a standard way: Let $\psi \in C^\infty(\overline{Q}_T)$ be a cut-off function such that $\psi(x,t) = 0$ if $(x,t) \in Q_T \setminus \overline{Q}_{2r/3+\tilde r/3}$, $\psi(x,t) = 1$ if $(x,t) \in Q_{r/3+2\tilde r/3}$ and

$\psi(x,t) \in [0,1]$ for every $(x,t) \in Q_T$. We set $w = K(\nabla \cdot (\psi u))$, $v = \psi u - w$, where $K = K_{D_r}$. Then v satisfies the following system of equations:

$$\frac{\partial v}{\partial t} - \nu \Delta v + B_i(u,v) + \nabla(\psi\phi) = h^i \quad \text{in } Q_T, \tag{3.1}$$

$$\nabla \cdot v = 0 \quad \text{in } Q_T, \tag{3.2}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3.3}$$

$$v|_{t=0} = 0, \tag{3.4}$$

where

$$h^1 = -\nu\Delta\psi u - 2\nu\nabla\psi\cdot\nabla u + \ u\cdot\nabla\psi u + \phi\nabla\psi - \frac{\partial w}{\partial t} + \nu\Delta w - u\cdot\nabla w + \frac{\partial \psi}{\partial t}u,$$

$$h^2 = -\nu\Delta\psi u - 2\nu\nabla\psi\cdot\nabla u + \phi\nabla\psi - \frac{\partial w}{\partial t} + \nu\Delta w - w\cdot\nabla u + \frac{\partial \psi}{\partial t}u,$$

$$h^3 = h^4 = (h^3_1, h^3_2, h^3_3),$$

$$h^3_i = -\nu\Delta\psi u_i - 2\nu\nabla\psi\cdot\nabla u_i + \phi\frac{\partial\psi}{\partial x_i} - \frac{\partial w_i}{\partial t} + \nu\Delta w_i - w\cdot\nabla u_i + \frac{\partial\psi}{\partial t}u_i, \quad i = 1,2,$$

$$\begin{aligned} h^3_3 = &- \nu\Delta\psi u_3 - 2\nu\nabla\psi\cdot\nabla u_3 + \phi\frac{\partial\psi}{\partial x_3} - \frac{\partial w_3}{\partial t} + \nu\Delta w_3 + \frac{\partial\psi}{\partial t}u_3 \\ &+ u_1\frac{\partial\psi}{\partial x_1}u_3 + u_2\frac{\partial\psi}{\partial x_2}u_3 - u_1\frac{\partial w_3}{\partial x_1} - u_2\frac{\partial w_3}{\partial x_2} + w_3\frac{\partial u_1}{\partial x_1} + w_3\frac{\partial u_2}{\partial x_2}. \end{aligned}$$

Remark 3.1. We can proceed in such a way that both supp w and supp v lie in $\overline{Q}_{3r/4+\tilde{r}/4}$. Therefore, it is possible to replace the function u in the term $B_i(u,v)$ and also in the right hand side h^i of (3.1) with a function ηu, where $\eta \in C^\infty(\overline{Q}_T)$ is such a cut-off function that $\eta(x,t) = 0$ if $(x,t) \in Q_T \setminus \overline{Q}_r$, $\eta(x,t) = 1$ if $(x,t) \in Q_{3r/4+\tilde{r}/4}$ and $\eta(x,t) \in [0,1]$ for every $(x,t) \in Q_T$. For the sake of simplicity we still write u instead of ηu.

We will show at first that

$$h^i \in L^{l,l'}, \text{ for some } l' \in (1,2),\ l \in (3/2,3) \text{ such that } \frac{2}{l'} + \frac{3}{l} = 3. \tag{3.5}$$

We will use the following global estimates for u and ϕ derived in [5], Theorem 3.1:

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{q,s} + \|\nabla^2 u\|_{q,s} + \|\nabla\phi\|_{q,s} < \infty, \quad s \in (1,2), q \in (1,3/2), \frac{2}{s} + \frac{3}{q} = 4, \tag{3.6}$$

$$\|\nabla u\|_{h,\rho} < \infty, \quad h \in (1,3), \rho \in (1,\infty), \frac{2}{\rho} + \frac{3}{h} = 3, \tag{3.7}$$

$$\|u\|_{h^*,\rho} < \infty, \quad h^* \in (3/2,\infty), \rho \in (1,\infty), \frac{2}{\rho} + \frac{3}{h^*} = 2, \tag{3.8}$$

and

$$\begin{gathered} \|\phi\|_{r,s} < \infty, \quad r \in (3/2,3), \quad s \in (1,2), \quad \frac{2}{s} + \frac{3}{r} = 3, \\ \text{if } \int_\Omega \phi(x,t)dx = 0 \text{ for every } t \in (0,T). \end{gathered} \tag{3.9}$$

We have immediately from (3.9) that $\phi\nabla\psi \in L^{l,l'}$. It follows from Lemma 2.1 that

$$\Big\|\frac{\partial w}{\partial t}\Big\|_{l,l'} = \Big\|\frac{\partial}{\partial t}(K(\nabla\psi\cdot u))\Big\|_{l,l'} = \Big\|K\Big(\frac{\partial}{\partial t}(\nabla\psi\cdot u)\Big)\Big\|_{l,l'} \le c\Big\|\frac{\partial}{\partial t}(\nabla\psi\cdot u)\Big\|_{q,l'},$$

where $1/q = 1/l + 1/3$. Since $2/l' + 3/q = 4$, we have $\partial w/\partial t \in L^{l,l'}$ by (3.6). Similarly, $\nu\Delta w \in L^{l,l'}$, as follows from Lemma 2.1 and (3.7).

Let us show that also the terms of the type $u\cdot\nabla w$ and $w\cdot\nabla u$ are from $L^{l,l'}$ for some $l' \in (1,2)$, $l \in (3/2,3)$ such that $2/l' + 3/l = 3$. By (3.7), $u \in L^\rho(0,T;W_0^{1,h}(\Omega))$, for every h, ρ such that $2/\rho + 3/h = 3$, $h \in (1,3)$, $\rho \in (1,\infty)$. Consequently, Lemma 2.1 gives that $w \in L^\rho(0,T;W_0^{2,h}(\Omega))$. Let us note that in the following paragraphs we use Remark 3.1.

If $i = 1$, it follows from Lemma 2.3 that

$$\|u\cdot\nabla w\|_{l,l',\Omega} \le C\|u\|_{L^\infty(t_0-r^2,t_0+r^2,L^3_\omega(D_r))}\| \cdot \|w\|_{L^{l'}(0,T,W^{2,l}(\Omega))} < \infty$$

and $u\cdot\nabla w \in L^{l,l'}$ for every l, l' from (3.5).

If $i = 2$, it follows from Lemma 2.6 that $w\cdot\nabla u \in L^{l,l'}$ for every l, l' from (3.5).

Let $i = 3$. We apply Lemma 2.6 and get: If moreover $p \ge 3$ then the terms $w\cdot\nabla u$ are in $L^{l,l'}$ for every l, l' from (3.5). If $p \in (3/2,3)$ and $q \in (2,\infty)$ then the terms $w\cdot\nabla u$ are in $L^{l,l'}$ for $l \in (1,p)$ and $l' \in (2q/(q+2), q)$. If $p \in (3/2,3)$ and $q = \infty$ then the terms $w\cdot\nabla u$ are in $L^{l,l'}$ for $l \in (1,p)$ and $l' \in (2p/(3p-3), 2)$. Similarly, using Lemma 2.7 we get that the terms $u\cdot\nabla w$ are in $L^{l,l'}$ for every l, l' from (3.5).

Finally, let $i = 4$. Then the terms $w\cdot\nabla u$ are in $L^{l,l'}$ for every l, l' from (3.5), as follows easily from Lemma 2.6. Similarly, the terms $u\cdot\nabla w$ are in $L^{l,l'}$ for every l, l' from (3.5) due to Lemma 2.7 and Remark 2.8.

The remaining terms in $h^i, i = 1,2,3,4$ belong obviously to the space $L^{l,l'}$ for every l, l' from (3.5) and (3.5) is proved.

Proof of Theorem 1.2. Let us fix now l, l' from (3.5) such that $h^i \in L^{l,l'}$. Let $\tilde v \in X^{l,l'}$ and P, $\nabla P \in L^{l,l'}$, solve the equations (3.1) - (3.4). The existence of this solution follows from Lemma 2.9, Lemma 2.10 and Remark 3.1, since the norm of the operator $B_i(u,\cdot)$ is or can be made sufficiently small due to the condition (a_i). Then $V = \tilde v - v$ and $p = P - \psi\phi$ solve the equations (2.13) - (2.16) with the right hand side 0 and $V \in X^{q,s}$ and $\nabla p \in L^{q,s}$, where q, s fulfil conditions from (3.6). Transferring now the term $B_i(u,V)$ to the right hand side and using (2.5) and Lemma 2.9, we obtain that

$$\begin{gathered}
\|V\|_{X^{q,s}} \le C\|u\|_{L^\infty(t_0-r^2,t_0+r^2;L^3_\omega(D_r))}\|V\|_{X^{q,s}} \quad \text{if } i = 1,\\
\|V\|_{X^{q,s}} \le C\|\nabla u\|_{3,2,Q_r}\|V\|_{X^{q,s}} \quad \text{if } i = 2,\\
\|V\|_{X^{q,s}} \le C(\|\nabla u_1\|_{p,q,Q_r} + \|\nabla u_2\|_{p,q,Q_r})\|V\|_{X^{q,s}} \quad \text{if } i = 3,\\
\|V\|_{X^{q,s}} \le C(\|\nabla u_1\|_{p,2,Q_r} + \|\nabla u_2\|_{p,2,Q_r})\|V\|_{X^{q,s}} \quad \text{if } i = 4.
\end{gathered}$$

While $C\|u\|_{L^\infty(t_0-r^2,t_0+r^2;L^3_\omega(D_r))} < 1$ due to the assumption (A1) in Theorem 1.2 (supposing that ε is sufficiently small), $C\|\nabla u\|_{3,2,Q_r}$, $C(\|\nabla u_1\|_{p,q,Q_r} + \|\nabla u_2\|_{p,q,Q_r})$ and $C(\|\nabla u_1\|_{p,2,Q_r} + \|\nabla u_2\|_{p,2,Q_r})$ can be made smaller than 1 by diminishing r. In any case we have $V \equiv 0$ and $v = \tilde v$. Therefore, v solves the equations (3.1) - (3.4) and v, h^i and $B_i(u,v)$ are from $L^{l,l'}$, where l, l' fulfil the conditions from (3.5) and $l \in (1,p)$ if $i = 3$. It follows from Lemma 2.2 that

$$\nabla v \in L^{\alpha,\alpha'}$$

for every $\alpha \in [l, \infty)$, $\alpha' \in [l', \infty)$ such that $2/\alpha' + 3/\alpha = 2$. Thus, by the choice $\alpha = l$ and $\alpha' = 2l/(2l-3)$, we have that

$$\nabla v \in L^{l, \frac{2l}{2l-3}}.$$

Since $v = 0$ on $\partial\Omega \times (0,T)$, we have immediately that $v \in L^{\frac{3l}{3-l}, \frac{2l}{2l-3}}$ and that

$$3\frac{3-l}{3l} + 2\frac{2l-3}{2l} = 1.$$

By the definition of v, $v = u$ in a space-time neighborhood of (x_0, t_0). We can now use Theorem 1.1 and the proof of Theorem 1.2 is complete. $\square$

Remark 3.2. The condition (A3) in Theorem 1.2 can be replaced by the condition

(A3') $\nabla u_1, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \frac{\partial u_3}{\partial x_2} \in L^{p,q}(Q_r)$, $p \in (3/2, \infty]$, $q \in (2, \infty]$, $3/p + 2/q \leq 2$

or by the more general condition

(A3") $\frac{\partial u_i}{\partial x_j} \in L^{p^i_j, q^i_j}(Q_r)$, $p^i_j \in (3/2, \infty]$, $q^i_j \in (2, \infty]$, $3/p^i_j + 2/q^i_j \leq 2$, for $i = 1, 2$ and $j = 1, 2, 3$.

Similarly, the condition (A4) from Theorem 1.2 can be replaced by the condition

(A4') $\nabla u_1, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \frac{\partial u_3}{\partial x_2} \in L^{p,q}(Q_r)$, $p \in (3, \infty]$, $q = 2$.

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