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EXISTENCE RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS IN BOUNDED DOMAINS OF $\mathbb{R}^n$

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Abstract. We establish existence results for the boundary-value problem $\Delta u + f(., u) = 0$ in a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, where f satisfies some appropriate conditions related to a Kato class. The proofs are based on various techniques related to potential theory.

1. Introduction

Let Ω be a $C^{1,1}$ bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. In this paper we study the existence and the asymptotic behaviour of bounded solutions to the nonlinear elliptic boundary-value problem

$$\begin{gathered} \Delta u + f(., u) = 0 \quad \text{in } \Omega \\ u > 0, \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where g is a nonnegative continuous function on $\partial\Omega$ and f satisfies some convenient conditions. The question of existence of solutions of (1.1) has been studied by several authors in both bounded and unbounded domains with various nonlinearities; see for example [2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21] and references therein. Note that solutions of these problems are understood in distributional sense.

Our tools are based essentially on some inequalities satisfied by the Green function $G(x, y)$ of $(-\Delta)$ in Ω which allow to some properties of functions belonging to the Kato class $K(\Omega)$ which contains properly the classical one; see [1, 4]. The class $K(\Omega)$ has been introduced in [15], for $n \geq 3$ and [12, 20] for $n = 2$ as follows.

We denote by $\delta(x)$ the Euclidian distance between x and $\partial\Omega$.

Definition 1.1. A Borel measurable function q in Ω belongs to the Kato class $K(\Omega)$ if q satisfies

$$\lim_{\alpha \to 0} \big(\sup_{x \in \Omega} \int_{\Omega \cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) |q(y)| dy \big) = 0. \tag{1.2}$$

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For the sake of simplicity we set Hg the bounded continuous solution of the Dirichlet problem

$$\begin{gathered} \Delta u = 0 \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega, \end{gathered}$$

where g is a nonnegative continuous function on $\partial\Omega$. We also refer to Vf the potential of a measurable nonnegative function f, defined on Ω by

$$Vf(x) = \int_\Omega G(x,y) f(y) dy.$$

Our plan in this paper is as follows. The section 2 is devoted to collect some preliminary results about the Green function $G(x,y)$ and the properties of the Kato class $K(\Omega)$.

In section 3, we establish an existence result for (1.1) where the combined effects of a singular and a sublinear term in the nonlinearity f are considered. Our motivation in this section comes from paper [17], where Shi and Yao investigated the existence of nonnegative solutions for the elliptic problem

$$\begin{gathered} \Delta u + K(x) u^{-\gamma} + \lambda u^\alpha = 0 \quad \text{in } \Omega \\ u(x) > 0 \quad \text{in } \Omega \\ , u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where γ and α in $(0,1)$ are two constants, λ is a real parameter and K is in $C^{0,\beta}(\overline{\Omega})$. Using this result. Sun and Li [19] gave a similar result in $\mathbb{R}^n$ $(n \geq 2)$. In fact they proved an existence result for the problem

$$\begin{gathered} \Delta u + p(x) u^{-\gamma} + q(x) u^\alpha = 0 \quad \text{in } \mathbb{R}^n \\ u(x) > 0, \quad x \in \mathbb{R}^n \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered}$$

where γ and α in $(0,1)$ are two constants and p, q are two nonnegative functions in $C^\beta_{\rm loc}(\mathbb{R}^n)$ such that $p + q \neq 0$.

The pure singular elliptic equation

$$\Delta u + p(x) u^{-\gamma} = 0, \quad \gamma > 0,\ x \in D \subseteq \mathbb{R}^n \tag{1.3}$$

has been extensively studied for both bounded and unbounded domains D in $\mathbb{R}^n (n \geq 2)$. We refer to [5, 6, 7, 9, 10] and references therein) for various existence and uniqueness results related to solutions for equation (1.3).

For more general situations Mâagli and Zribi showed in [14] that the problem

$$\begin{gathered} \Delta u + \varphi(., u) = 0, \quad x \in D \\ u = 0 \quad \text{on } \partial D \\ \lim_{|x| \to \infty} u(x) = 0, \quad \text{if } D \text{ is unbounded} \end{gathered}$$

admits a unique positive solution if φ is a nonnegative measurable function on $(0, \infty)$, which is nonincreasing and continuous with respect to the second variable and satisfies

- (H0) For all $c > 0$, $\varphi(., c)$ is in $K_n^\infty(D)$, where $K_n^\infty(D)$ is the classical Kato class; see [21].

On the other hand, the problem (1.1) with a sublinear term $f(.,u)$ have been studied in $\mathbb{R}^n$ by Brezis and Kamin in [3]. Indeed, the authors proved the existence and the uniqueness of a positive solution for the problem

$$\begin{gathered} \Delta u + \rho(x)u^{\alpha} = 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0, \end{gathered}$$

with $0 < \alpha < 1$ and ρ is a nonnegative measurable function satisfying some appropriate conditions.

Thus a natural question to ask is for more general singular and sublinear terms combined in the nonlinearity, whether or not (1.1) has a solution which we aim to study in this section. In fact we are interested in solving the following problem (in the sense of distributions)

$$\begin{gathered} \Delta u + \varphi(.,u) + \psi(.,u) = 0, \quad \text{in } \Omega \\ u > 0, \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.4}$$

Here φ and ψ are required to satisfy the following hypotheses:

(H1) φ is a nonnegative Borel measurable function on $\Omega \times (0,\infty)$, continuous and nonincreasing with respect to the second variable.

(H2) For all $c > 0$, $x \to \varphi(x, c\delta(x))$ is in $K(\Omega)$.

(H3) ψ is a nonnegative Borel measurable function on $\Omega\times(0,\infty)$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function p and a nonnegative function $q \in K(\Omega)$ satisfying for $x \in \Omega$ and $t > 0$,
$$p(x)h(t) \le \psi(x,t) \le q(x)f(t), \tag{1.5}$$
where h is a measurable nondecreasing function on $[0,\infty)$ satisfying
$$\lim_{t\to 0^+} \frac{h(t)}{t} = +\infty \tag{1.6}$$
and f is a nonnegative measurable function locally bounded on $[\ 0,\infty)$ satisfying
$$\limsup_{t\to\infty} \frac{f(t)}{t} < \|Vq\|_{\infty}. \tag{1.7}$$

Using a fixed point argument, we shall prove the following existence result.

Theorem 1.2. *Assume (H1)–(H3). Then the problem* (1.4) *has a positive solution* $u \in C_b(\Omega)$ *such that for each* $x \in \Omega$,
$$a\delta(x) \le u(x) \le V(\varphi(.,a\delta))(x) + bVq(x),$$
where a, b *are positive constants.*

Typical examples of nonlinearities satisfying (H1)-(H3) are:

$$\begin{gathered} \varphi(x,t) = p(x)(\delta(x))^{\gamma}t^{-\gamma}; \quad \gamma \ge 0, \\ \psi(x,t) = q(x)t^{\alpha}\log(1+t^{\beta}), \quad \alpha,\beta \ge 0 \end{gathered}$$

such that $\alpha + \beta < 1$, where p and q are two nonnegative functions in $K(\Omega)$.

In this section, using different techniques from those used by Shi and Yao [17], we improve their results in the sense of distributional solutions.

In section 4, we consider the nonlinearity $f(x,t) = -\varphi(x,t)$ and we suppose that g is nontrivial, then using a potential theory approach we investigate an existence result and an uniqueness result for the problem

$$\begin{gathered} \Delta u - \varphi(.,u) = 0 \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega, \end{gathered} \tag{1.8}$$

where φ is required to satisfy the following three conditions:

- (H4) φ is a nonnegative measurable function on $\Omega \times [0,\infty)$, continuous and nondecreasing with respect to the second variable.
- (H5) $\varphi(.,0) = 0$.
- (H6) For all $c > 0$, $\varphi(.,c)$ is in $K(\Omega)$.

Our main result is the following.

Theorem 1.3. *Assume (H4)-(H6). Then the problem* (1.8) *has a unique positive solution u such that $0 < u(x) \leq Hg(x)$ for each $x \in \Omega$.*

Note that if $q \in K(\Omega)$ and $\varphi(x,t) \leq q(x)t$ locally on t, then the solution u satisfies $cHg(x) \leq u(x) \leq Hg(x)$, for $c \in (0,1)$.

This result follows up the one of Lair and Wood in [9], who have considered the equation

$$\Delta u = q(x) f(u),$$

in both bounded and unbounded domains of $\mathbb{R}^n$ $(n \geq 2)$ in the case $f(u) = u^\gamma$, $0 < \gamma \leq 1$. They studied the existence and nonexistence of positive large solutions and positive bounded ones under adequate hypothesis on q. The result of Lair and Wood have been generalized later by Bachar and Zeddini [2] to more general functions f and q satisfying some restrictive conditions.

To simplify our statements, we define some convenient notation:
(i) $B(\Omega)$ denotes the set of Borel measurable functions in Ω and $\mathcal{B}^+(\Omega)$ the set of nonnegative functions.
(ii) $C_0(\Omega) := \{w \in C(\Omega) : \lim_{x \to \partial\Omega} w(x) = 0\}$. We recall that this space is Banach with the uniform norm

$$\|w\|_\infty = \sup_{x \in \Omega} |w(x)|.$$

(iii) For $q \in \mathcal{B}(\Omega)$, we put

$$\|q\| := \sup_{x \in \Omega} \int_\Omega \frac{\delta(y)}{\delta(x)} G(x,y) |q(y)| dy.$$

(iv) Let f and g be two nonnegative functions on a set S. We call $f \preceq g$, if there is $c > 0$ such that

$$f(x) \leq c g(x) \quad \text{for all } x \in S.$$

We call $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c} g(x) \leq f(x) \leq c g(x) \quad \text{for all } x \in S.$$

2. Properties of the Green function and the Kato class

The existence results to prove, suggest collecting some estimates on the Green function G and some properties of functions belonging to the Kato class $K(\Omega)$. The proofs of the following estimates and inequalities of G can be found in [15] for $n \geq 3$ and [20] for $n = 2$.

Proposition 2.1. *For each $x, y \in \Omega$, we have*

$$G(x,y) \sim \begin{cases} \frac{\delta(x)\delta(y)}{|x-y|^{n-2}\left(|x-y|^2+\delta(x)\delta(y)\right)} & \text{if } n \geq 3, \\ \log\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \text{if } n = 2. \end{cases} \tag{2.1}$$

Corollary 2.2. *For $x, y \in \Omega$,*

$$\delta(x)\delta(y) \preceq G(x,y). \tag{2.2}$$

Theorem 2.3 (3G-Theorem)**.** *There exists $C_0 > 0$ depending only on Ω, such that for $x, y, z \in \Omega$, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0\big[\frac{\delta(z)}{\delta(x)}G(x,z) + \frac{\delta(z)}{\delta(y)}G(y,z)\big]. \tag{2.3}$$

To recall some properties of the class $K(\Omega)$, we first give the following examples:

(1) By [15, Proposition 4], the function $q(x) = 1/(\delta(x))^\lambda$ is in $K(\Omega)$ if and only if $\lambda < 2$.
(2) By [18, Proposition 3], if $p > n/2$ and $\lambda < 2 - \frac{n}{p}$, then $L^p(\Omega)/(\delta(.))^\lambda \subset K(\Omega)$.

The proof of the following Proposition can be found in [15, 20].

Proposition 2.4. *Let q be a nonnegative function in $K(\Omega)$. Then*

(i) $\|q\| < \infty$.
(ii) *The function $x \mapsto \delta(x)q(x)$ is in $L^1(\Omega)$.*
(iii) *We have*

$$\delta(x) \preceq Vq(x). \tag{2.4}$$

For a fixed nonnegative function q in $K(\Omega)$, we put

$$\mathcal{M}_q := \{\varphi \in B(\Omega), \quad |\varphi| \preceq q\}.$$

Proposition 2.5. *Let q be a nonnegative function in $K(\Omega)$, then the family of functions*

$$V(\mathcal{M}_q) = \{V\varphi : \varphi \in \mathcal{M}_q\}$$

is uniformly bounded and equicontinuous in $C_0(\Omega)$, and consequently it is relatively compact in $C_0(\Omega)$.

Proof. The result holds by similar arguments as in [15, proposition 3] and [20, Proposition 8]. □

In the sequel, we use the notation

$$\alpha_q := \sup_{x,y\in\Omega} \int_\Omega \frac{G(x,z)G(z,y)}{G(x,y)}|q(z)|dz.$$

Proposition 2.6. *Let q be a function in $K(\Omega)$ and v be a nonnegative superharmonic function in Ω. Then for each $x \in \Omega$,*

$$\int_\Omega G(x,y)v(y)|q(y)|dy \le \alpha_q v(x) \tag{2.5}$$

and consequently, $\|q\| \le \alpha_q \le 2C_0\|q\|$, where C_0 is the constant given in (2.3).

For the proof of the above proposition, we refer the reader to [18, Proposition 2].

Corollary 2.7. *Let q be a nonnegative function in $K(\Omega)$ and v be a nonnegative superharmonic function in Ω, then for each $x \in \Omega$ such that $v(x) < \infty$, we have*

$$\exp(-\alpha_q)v(x) \le (v - V_q(qv))(x) \le v(x).$$

Proof. The upper inequality is trivial. For the lower one, we consider the function $\gamma(\lambda) = v(x) - \lambda V_{\lambda q}(qv)(x)$ for $\lambda \ge 0$. The function γ is completely monotone on $[0,\infty)$ and so $\log\gamma$ is convex in $[0,\infty)$. This implies

$$\gamma(0) \le \gamma(1)\exp(-\frac{\gamma'(0)}{\gamma(0)}).$$

That is,

$$v(x) \le (v - V_q(qv))(x)\exp(\frac{V(qv)(x)}{v(x)}).$$

So, the result holds by (2.5). □

3. First existence result

Proof of Theorem 1.2. Assume (H1)-(H3). Using the Schauder fixed point theorem, we are going to construct a solution to problem (1.4). We note that by (2.2) there exists a constant $\alpha_1 > 0$ such that for each $x, y \in \Omega$

$$\alpha_1\delta(x)\delta(y) \le G(x,y). \tag{3.1}$$

Now, using (H3), there exists a compact K of Ω such that

$$0 < \alpha := \int_K \delta(y)p(y)dy < \infty.$$

We put $\beta := \min\{\delta(x) : x \in K\}$. Then from (1.6), we conclude that there exists $a > 0$ such that

$$\alpha_1\alpha h(a\beta) \ge a. \tag{3.2}$$

Furthermore, since $q \in K(\Omega)$, then by Proposition 2.5 we have obviously that $\|Vq\|_\infty < \infty$. So taking $0 < \eta < 1/\|Vq\|_\infty$, we deduce by (1.7) that there exists $\rho > 0$ such that for $t \ge \rho$ we have $f(t) \le \eta t$. Put $\gamma = \sup_{0\le t\le\rho} f(t)$. So we have that

$$0 \le f(t) \le \eta t + \gamma, \quad t \ge 0. \tag{3.3}$$

Next by (2.4), we note that there exists a constant $\alpha_2 > 0$ such that

$$\alpha_2\delta(x) \le Vq(x), \quad \forall x \in \Omega. \tag{3.4}$$

¿From (H2) and Proposition 2.5, we have that $\|V\varphi(.,a\delta)\|_\infty < \infty$. Hence, put

$$b = \max\{\frac{a}{\alpha_2}, \frac{\eta\|V\varphi(.,a\delta)\|_\infty + \gamma}{1 - \eta\|Vq\|_\infty}\}$$

and consider the closed convex set

$$\Lambda = \{u \in C_0(\Omega) : a\delta(x) \le u(x) \le V\varphi(., a\delta)(x) + bVq(x), \forall x \in \Omega\}.$$

Obviously, by (3.4) we have that the set Λ is nonempty. Define the integral operator T on Λ by

$$Tu(x) = \int_\Omega G(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy, \quad \forall x \in \Omega.$$

Let us prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$ and $x \in \Omega$, then by (H1), (H3) and (3.3) we have

$$\begin{aligned}
Tu(x) &\le V\varphi(., a\delta)(x) + \int_\Omega G(x,y)q(y)f(u(y))dy \\
&\le V\varphi(., a\delta)(x) + \int_\Omega G(x,y)q(y)[\eta u(y) + \gamma]dy \\
&\le V\varphi(., a\delta)(x) + \int_\Omega G(x,y)q(y)[\eta(\|V\varphi(., a\delta)\|_\infty + b\|Vq\|_\infty) + \gamma]dy \\
&\le V\varphi(., a\delta)(x) + bVq(x).
\end{aligned}$$

Moreover from the monotonicity of h, (3.1) and (3.2), we have

$$\begin{aligned}
Tu(x) &\ge \int_\Omega G(x,y)\psi(y,u(y))dy \\
&\ge \alpha_1\delta(x)\int_\Omega \delta(y)p(y)h(a\delta(y))dy \\
&\ge \alpha_1\delta(x)h(a\beta)\int_K \delta(y)p(y)dy \\
&\ge \alpha_1\alpha h(a\beta)\delta(x) \\
&\ge a\delta(x).
\end{aligned}$$

On the other hand, we have that for each $u \in \Lambda$,

$$\varphi(., u) \le \varphi(., a\delta) \text{ and } \psi(., u) \le [\eta(\|V\varphi(., a\delta)\| + b\|Vq\|_\infty) + \gamma]q. \tag{3.5}$$

This implies by Proposition 2.5 that $T\Lambda$ is relatively compact in $C_0(\Omega)$. In particular, we deduce that $T\Lambda \subset \Lambda$.

Next, we prove the continuity of T in Λ. Let $(u_k)_k$ be a sequence in Λ which converges uniformly to a function u in Λ. Then since φ and ψ are continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\forall x \in \Omega, \quad Tu_k(x) \to Tu(x) \quad \text{as } k \to \infty.$$

Now, since $T\Lambda$ is relatively compact in $C_0(\Omega)$, then we have the uniform convergence. Hence T is a compact operator mapping from Λ to itself. So the Schauder fixed point theorem leads to the existence of a function $u \in \Lambda$ such that

$$u(x) = \int_\Omega G(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy, \quad \forall x \in \Omega. \tag{3.6}$$

Finally, we need to prove that u is solution of the problem (1.4). Since q and $\varphi(., a\delta)$ are in $K(\Omega)$, we deduce by (3.5) and Proposition 2.4, that $y \mapsto \varphi(y,u(y)) + \psi(y,u(y)) \in L^1(\Omega)$. Moreover, since $u \in C_0(\Omega)$, we deduce from (3.6), that

$V(\varphi(.,u)+\psi(.,u)) \in L^1(\Omega)$. Hence u satisfies in the sense of distributions the elliptic equation

$$\Delta u + \varphi(.,u) + \psi(.,u) = 0, \quad \text{in } \Omega.$$

This completes the proof. $\square$

Example 3.1. Let $\alpha, \beta \geq 0$ such that $0 \leq \alpha + \beta < 1$, $\gamma > 0$ and $p, q \in K^+(\Omega)$. Then the problem

$$\begin{gathered} \Delta u + p(x)(u(x))^{-\gamma}(\delta(x))^{\gamma} + q(x)(u(x))^{\alpha}\log(1+(u(x))^{\beta}) = 0, \quad \text{in } \Omega \\ u > 0, \quad \text{in } \Omega \end{gathered} \tag{3.7}$$

has a solution $u \in C_0(\Omega)$ satisfying $a\delta(x) \leq u(x) \leq Vp(x) + bVq(x)$, where $a, b > 0$.

Remark 3.2. Taking in Example 3.1 $\lambda < 2$,

$$p(x) = q(x) = \frac{1}{(\delta(x))^{\lambda}},$$

we deduce from [15] that the solution of (3.7) satisfies the following:

(i) $u(x) \preceq (\delta(x))^{2-\lambda}$, if $1 < \lambda < 2$.
(ii) $u(x) \preceq \delta(x)\log\frac{(\sqrt{5}+1)d}{2\delta(x)}$, if $\lambda = 1$,
(iii) $u(x) \preceq \delta(x)$, if $\lambda < 1$, where $d = \operatorname{diam}(\Omega)$.

Note that in Example 3.1, we have the result obtained by Shi and Yao [17].

4. Second existence result

In this section, we shall prove Theorem 1.3. The proof is based on a comparison principle given by the following Lemma. For $u \in B(\Omega)$, put $u^+ = \max(u, 0)$.

Lemma 4.1. *Let φ and ψ satisfying (H4)-(H6). Assume that $\varphi \leq \psi$ on $\Omega \times \mathbb{R}_+$ and there exist continuous functions u, v on Ω satisfying*

(a) $\Delta u - \varphi(., u^+) \leq \Delta v - \psi(., v^+)$ *in Ω (in the distributional sense)*
(b) $u, v \in C_b(\Omega)$
(c) $u \geq v$ *on $\partial\Omega$.*

Then $u \geq v$ in Ω.

Proof. Suppose that the open set $D = \{x \in \Omega : u(x) < v(x)\}$ is nonempty. Put $z = u - v$. Then z satisfies

$$\begin{gathered} \Delta z = \varphi(., u^+) - \psi(., v^+) \\ = (\varphi(., u^+) - \psi(., u^+)) + (\psi(., u^+) - \psi(., v^+)) \leq 0 \quad \text{in } D \\ z \geq 0 \quad \text{on } \partial D \\ z \in C_b(D). \end{gathered}$$

Hence from the maximum principle, we conclude that $z \geq 0$ in D. Therefore, we get a contradiction with the definition of D. This completes the proof. $\square$

In the sequel, we recall that for each function $q \in \mathcal{B}^+(\Omega)$ such that $Vq < \infty$, we denote by V_q the unique kernel which satisfies the following resolvent equation (see [11, 16]):

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{4.1}$$

So for each $u \in \mathcal{B}(\Omega)$ such that $V(q|u|) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{4.2}$$

Proof of Theorem 1.3. As consequence of the comparison principle in Lemma 4.1, we deduce that problem (1.8) has at most one solution. The existence of a such solution is assured by the Schauder fixed point Theorem. Indeed, we consider the convex set

$$\Lambda = \{u \in C_b(\Omega) : u \leq \|g\|_\infty\}.$$

We define the integral operator T on Λ by

$$Tu(x) = Hg(x) - V(\varphi(., u^+))(x).$$

Since $Hg(x) \leq \|g\|_\infty$, for $x \in \Omega$, we deduce that for each $u \in \Lambda$,

$$Tu \leq \|g\|_\infty \quad \text{in } \Omega.$$

Furthermore, putting $q = \varphi(., \|g\|_\infty)$, we have by (H4) and (H6) that q is in $K(\Omega)$ and $V(\varphi(., u^+))$ is in $V(\mathcal{M}_q)$. This together with the fact that Hg is in $C_b(\Omega)$ imply by Proposition 2.5 that $T\Lambda$ is relatively compact in $C_b(\Omega)$ and in particular $T\Lambda \subset \Lambda$.

¿From the continuity of φ with respect to the second variable, we deduce that T is continuous in Λ and so it is a compact operator from Λ to itself. Then by the Schauder fixed point Theorem, we deduce that there exists a function $u \in \Lambda$ satisfying

$$u(x) = Hg(x) - V(\varphi(., u^+))(x).$$

Finally, since $\varphi(., u^+) \in \mathcal{M}_q$, we conclude by Proposition 2.4 that u satisfies in the sense of distributions the following

$$\begin{gathered}\Delta u - \varphi(., u^+) = 0 \\ \lim_{x \to \partial\Omega} u(x) = g.\end{gathered}$$

Hence by (H5) and Lemma 4.1, we conclude that $u \geq 0$ in Ω and so it is a solution of (1.8). □

Corollary 4.2. *Suppose that φ satisfies (H4)-(H6) and g is a nontrivial nonnegative continuous function in $\partial\Omega$. Suppose that there exists a function $q \in K(\Omega)$ such that*

$$0 \leq \varphi(x, t) \leq q(x)t \quad \text{on } \Omega \times [0, \|g\|_\infty]. \tag{4.3}$$

Then the solution u of (1.8) given by Theorem 1.3 satisfies

$$e^{-\alpha_q} Hg(x) \leq u(x) \leq Hg(x).$$

Proof. Since u satisfies the integral equation

$$u(x) = Hg(x) - V(\varphi(., u))(x),$$

using (4.1), we obtain

$$\begin{aligned} u - V_q(qu) &= (Hg - V_q(qHg)) - (V(\varphi(., u)) - V_q(qV(\varphi(., u))) \\ &= (Hg - V_q(qHg)) - V_q(\varphi(., u)). \end{aligned}$$

That is,

$$u = (Hg - V_q(qHg)) + V_q(qu - \varphi(., u)).$$

Now since $0 < u \leq \|g\|_\infty$ then by (4.3), we have that $u \geq Hg - V_q(qHg)$. Consequently, the result holds from Corollary 2.7. □

Example 4.3. Let g be a nontrivial nonnegative continuous function in $\partial\Omega$. Let $\sigma > 0$ and $q \in K^+(\Omega)$. Then the problem (in the sense of distributions)

$$\Delta u - q(x)u^\sigma = 0, \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

has a positive bounded continuous solution u satisfying, in Ω,

$$0 \le Hg(x) - u(x) \le \|g\|_\infty^\sigma Vq(x).$$

Furthermore, if $\sigma \ge 1$, by Corollary 4.2, for each $x \in \Omega$,

$$e^{-\alpha_q} Hg(x) \le u(x) \le Hg(x).$$

References

[1] M. Aizenman, B. Simon; *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure App. Math 35 (1982) 209-271.

[2] I. Bachar, N. Zeddini; *On the existence of positive solutions for a class of semilinear elliptic equations*, Nonlinear Anal. 52 (2003) 1239-1247.

[3] H. Brezis, S. Kamin; *Sublinear elliptic equations in* $\mathbb{R}^n$, Manus. Math. 74, (1992) 87-106.

[4] K. L. Chung, Z. Zhao; *From Brownian motion to Schrödinger's equation*, Springer Verlag (1995).

[5] J. I. Diaz, J. M. Morel, L. Oswald; *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations 12, (1987) 1333-1344.

[6] A. Edelson, *Entire solutions of singular elliptic equations*, J. Math. Anal. appl. (1989) 139, 523-532.

[7] T. Kusano, C. A. Swanson; *Entire positive solutions of singular semilinear elliptic equations*, Japan J. Math. 11 (1985) 145-155.

[8] A. V. Lair, A. W. Shaker; *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 211 (2002) 230-246.

[9] A. V. Lair, A. W. Wood; *Large solutions of sublinear elliptic equations*, Nonlinear Anal. 39 (2000) 745-753.

[10] A. C. Lazer, P. J. Mckenna; *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Mat. Soc. 111 (1991) 721-730.

[11] H. Mâagli; *Perturbation semi-linéaire des ré solvantes et des semi-groupes*, Potential Ana. 3, (1994) 61-87.

[12] H. Mâagli, L. Mâatoug; *Singular solutions of a nonlinear equation in bounded domains of* $\mathbb{R}^2$, J. Math. Anal. Appl. 270 (1997) 371-385.

[13] H. Mâagli, S. Masmoudi; *Positive solutions of some nonlinear elliptic problems in unbounded domain*, Ann. Aca. Sci. Fen. Math. 29, (2004) 151-166.

[14] H. Mâagli, M. Zribi; *Existence and Estimates of solutions for singular nonlinear elliptic problems*, J.Math.Anal.Appl. Vol 263 no. 2 (2001) 522-542.

[15] H. Mâagli, M. Zribi; *On a new Kato class and singular solutions of a nonlinearelliptic equation in bounded domains of* $\mathbb{R}^n$. To appear in Positivity (Articles in advance)

[16] J. Neveu; *Potentiel markovian reccurent des chaînes de Harris*, Ann. Int. Fourier 22 (2), (1972) 85-130.

[17] J. P. Shi, M. X. Yao; *On a singular semilinear elliptic problem*, Proc. Roy. Soc. Edinburg 128A (1998) 1389-1401.

[18] F. Toumi, *Existence of positive solutions for nonlinear boundary-value problems in bounded domains of* $\mathbb{R}^n$. To appear in Abstract and Applied Analysis (Articles in advance).

[19] S. Yijing, L. Shujie; *Structure of ground state solutions of singular semilinear elliptic equations*, Nonlinear Analysis 55 (2003) 399-417.

[20] N. Zeddini, *Positive solutions for a singular nonlinear problem on a bounded domain in* $\mathbb{R}^2$, Potential Analysis 18 (2003) 97-118.

[21] Qi S. Zhang, Z. Zhao; *Singular solutions of semilinear elliptic and parabolic equations*, Math. Ann. 310 (1998) 777-794.

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