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EXISTENCE OF SOLUTIONS FOR AN ELLIPTIC EQUATION INVOLVING THE $p(x)$-LAPLACE OPERATOR

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Abstract. In this paper we study an elliptic equation involving the $p(x)$-Laplace operator on the whole space $\mathbb{R}^N$. For that equation we prove the existence of a nontrivial weak solution using as main argument the mountain pass theorem of Ambrosetti and Rabinowitz.

1. Introduction

In this paper we discuss the existence of solutions for the problem

$$\begin{gathered} -\Delta_{p(x)} u(x) + b(x)|u(x)|^{p(x)-2}u = f(x,u), \quad \text{for } x \in \mathbb{R}^N \\ u \in W_0^{1,p(x)}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $N \geq 3$, $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with $2 \leq \operatorname{ess\,inf}_{\mathbb{R}^N} p(x) < \operatorname{ess\,sup}_{\mathbb{R}^N} p(x) < N$, $b : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two functions which satisfy certain conditions. We denoted by $\Delta_{p(x)}$ the $p(x)$-Laplace operator given by

$$\Delta_{p(x)} u = \operatorname{div}\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big).$$

The study of equations involving $p(x)$-growth conditions, such as (1.1), has captured a special attention since there are some physical phenomena which can be modelled by such kind of equation. In that context we just remember their applications to the study of electrorheological fluids and in elastic mechanics (see Diening [3], Halsey [8], Ruzicka [15], Zhikov [16]).

On the other hand, we point out that equation (1.1) is related with stationary non-linear Schrödinger equations (see Rabinowitz [14] and Mihăilescu-Rădulescu [10] for more details).

Existence results for $p(x)$-Laplacian Dirichlet problems on bounded domains were studied in [6, 11, 12] while for the study of $p(x)$-Laplacian problems in $\mathbb{R}^N$ we refer to [4, 1].

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2. Preliminary results

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context we refer to the book of Musielak [13] and the papers of Kovacik and Rakosnik [9] and Fan et al. [4, 5, 7].

Set

$$L_+^\infty(\Omega) = \{h;\ h \in L^\infty(\Omega), \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in L_+^\infty(\Omega)$ we define

$$h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x).$$

For any $p(x) \in L_+^\infty(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \text{is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\big\{\mu > 0;\ \int_\Omega \big|\frac{u(x)}{\mu}\big|^{p(x)}\,dx \le 1\big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [9, Theorem 2.5], the Hölder inequality holds [9, Theorem 2.1], they are reflexive if and only if $1 < p^- \le p^+ < \infty$ [9, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [9, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [9, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \le r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\big|\int_\Omega uv\,dx\big| \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)|u|_{p(x)}|v|_{p'(x)} \tag{2.1}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}; \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow\ |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}; \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

We also consider the weighted variable exponent Lebesgue spaces. Let $b:\mathbb{R}^N\to\mathbb{R}$ be a measurable real function such that $b(x)>0$ a.e. $x\in\Omega$. We define

$$L^{p(x)}_{b(x)}(\Omega)=\{u:u \text{ is a measurable real-valued function such that}\\ \int_\Omega b(x)|u(x)|^{p(x)}dx<\infty\}.$$

The space $L^{p(x)}_{b(x)}(\Omega)$ endowed with the above norm is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is $\rho_{b(x);p(x)}:L^{p(x)}_{b(x)}(\Omega)\to\mathbb{R}$ defined by

$$\rho_{b(x);p(x)}(u)=\int_\Omega b(x)|u|^{p(x)}dx.$$

If $u\in L^{p(x)}_{b(x)}(\Omega)$, then the following relations hold

$$|u|_{(b(x),p(x))}>1\Rightarrow |u|^{p^-}_{(b(x),p(x))}\le\rho_{b(x);p(x)}(u)\le|u|^{p^+}_{(b(x),p(x))},$$
$$|u|_{(b(x),p(x))}<1\Rightarrow |u|^{p^+}_{(b(x),p(x))}\le\rho_{b(x);p(x)}(u)\le|u|^{p^-}_{(b(x),p(x))}.$$

We define also the variable Sobolev space

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):\ |\nabla u|\in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)}=|u|_{p(x)}+|\nabla u|_{p(x)}$$

or

$$|||u|||=\inf\big\{\mu>0;\ \int_\Omega\big(\big|\frac{\nabla u(x)}{\mu}\big|^{p(x)}+\big|\frac{u(x)}{\mu}\big|^{p(x)}\big)\,dx\le1\big\}.$$

We define also $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Assuming $p^->1$ the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Set

$$I_{p(x)}(u)=\int_\Omega\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)\,dx.$$

For all $u\in W_0^{1,p(x)}(\Omega)$ the following relations hold

$$|||u|||>1\ \Rightarrow\ |||u|||^{p^-}\le I_{p(x)}(u)\le|||u|||^{p^+};\tag{2.5}$$
$$|||u|||<1\ \Rightarrow\ |||u|||^{p^+}\le I_{p(x)}(u)\le|||u|||^{p^-}.\tag{2.6}$$

Finally, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. For the continuous embedding between variable exponent Lebesgue-Sobolev spaces we refer to [5, Theorem 1.1]: if $p:\Omega\to\mathbb{R}$ is Lipschitz continuous and $p^+<N$, then for any $q\in L^\infty_+(\Omega)$ with $p(x)\le q(x)\le\frac{Np(x)}{N-p(x)}$, there is a continuous embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$. In what concerns the compact embedding we refer to [5, Theorem 1.3]: if Ω is a bounded domain in $\mathbb{R}^N$, $p(x)\in C(\overline{\Omega})$, $p^+>N$, then for any $q(x)\in L^\infty_+(\Omega)$ with $\operatorname{ess\,inf}_{x\in\overline{\Omega}}\big(\frac{Np(x)}{N-p(x)}-q(x)\big)>0$ there is a compact embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.

3. Main result

In this paper we assume that b and f satisfy the hypotheses:

(B1) $b \in L^{\infty}_{\rm loc}(\mathbb{R}^N)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$, for any $x \in \mathbb{R}^N$;

(F1) $f \in C^1(\mathbb{R}^N \times \mathbb{R})$, with $f = f(x,z)$, $f(x,0) = 0$ and $\lim_{z\to 0} \frac{f_z(x,z)}{|z|^{p^+-2}} = 0$, for all $x \in \mathbb{R}^N$;

(F2) $p^+ < \frac{Np^-}{N-p^-}$ and there exist $a_1, a_2 > 0$ and $s \in (p^+ - 1, Np^-/(N-p^-) - 1)$ such that
$$|f_z(x,z)| \leq a_1|z|^{p^+-2} + a_2|z|^{s-1}, \quad \forall\, x \in \mathbb{R}^N, \forall\, z \in \mathbb{R};$$

(F3) there exists $\mu > p^+$ such that
$$0 < \mu F(x,z) := \mu \int_0^z f(x,t)dt \leq z f(x,z), \quad \forall\, x \in \mathbb{R}^N, \forall\, z \in \mathbb{R} \setminus \{0\}.$$

Let E be the space defined as the completion of $C_0^{\infty}(\mathbb{R}^N)$ with respect to the norm
$$\|u\|_1 = |\nabla u|_{p(x)} + |u|_{(b(x),p(x))}.$$

Remark 3.1. *Condition (B1) implies that $E \subset W_0^{1,p(x)}(\mathbb{R}^N)$.*

A simple calculation shows that the above norm is equivalent to
$$\|u\| = \inf\big\{\mu > 0;\ \int_{\Omega} \big(\big|\frac{\nabla u(x)}{\mu}\big|^{p(x)} + b(x)\big|\frac{u(x)}{\mu}\big|^{p(x)}\big)\, dx \leq 1\big\}.$$

Set
$$J(u) := \int_{\mathbb{R}^N} \Big(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\Big)\, dx.$$

Then, for all $u \in E$ the following relations hold:
$$\begin{aligned} &\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq J(u) \leq \|u\|^{p^+}, \\ &\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq J(u) \leq \|u\|^{p^-}. \end{aligned} \tag{3.1}$$

We say that $u \in E$ is a *weak solution* of (1.1) if
$$\int_{\mathbb{R}^N} \big(|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv\big)dx = \int_{\mathbb{R}^N} f(x,u)v dx,$$
for any $v \in C_0^{\infty}(\mathbb{R}^N)$.

The main result of this paper is given by the following theorem.

Theorem 3.2. *Assume conditions (B1) and (F1)-(F3) are fulfilled. Then problem* (1.1) *has a non-trivial weak solution.*

We point out the fact that the result of Theorem 3.2 extends the results from [14] and [10] where similar equations are studied in the linear case.

4. Proof of Main Theorem

The energy functional corresponding to problem (1.1) is defined as $I : E \to \mathbb{R}$,

$$I(u) := \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Similar arguments as those used in [4, Lemmas 3.1 and 3.2] assure that $I \in C^1(E, \mathbb{R})$ with

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + b(x)|u|^{p(x)-2} uv \right) dx - \int_{\mathbb{R}^N} f(x,u)vdx,$$

for any u, $v \in E$. Thus, we observe that the critical points of functional I are the weak solutions for equation (1.1).

Our idea is to prove Theorem 3.2 applying the mountain pass theorem (see e.g. [2]). With that end in view, we prove some auxiliary results which show that the functional I has a mountain pass geometry.

Lemma 4.1. *If (B1) and (F1)-(F3) hold, then there exist $\tau > 0$ and $a > 0$ such that for all $u \in E$ with $\|u\| = \tau$*

$$I(u) \geq a > 0.$$

Proof. Using (F1) and L'Hospital Theorem, we have

$$\lim_{z \to 0} \frac{F(x,z)}{z^{p^+}} = \lim_{z \to 0} \frac{f(x,z)}{p^+ \cdot z^{p^+-1}} = \lim_{z \to 0} \frac{f_z(x,z)}{p^+(p^+-1) \cdot z^{p^+-2}} = 0,$$

for all $x \in \mathbb{R}^N$. Thus,

$$\lim_{z \to 0} \frac{F(x,z)}{z^{p^+}} = 0. \tag{4.1}$$

Using (F2) we have

$$f_z(x,z) \leq |f_z(x,z)| \leq a_1 |z|^{p^+-2} + a_2 |z|^{s-1}.$$

By integrating, we obtain

$$f(x,z) \leq \frac{a_1}{p^+-1} |z|^{p^+-1} + \frac{a_2}{s} |z|^s.$$

We integrate again:

$$0 < F(x,z)| \leq A_1 |z|^{p^+} + A_2 |z|^{s+1}, \tag{4.2}$$

where A_1, A_2 are positive constants. Then

$$0 \leq \lim_{z \to \infty} \frac{F(x,z)}{z^{Np^-/(N-p^-)}} \leq \lim_{z \to \infty} \frac{A_1 |z|^{p^+} + A_2 |z|^{s+1}}{z^{Np^-/(N-p^-)}} = 0,$$

since $s \in (p^+ - 1, Np^-/(N-p^-) - 1)$. Therefore,

$$\lim_{z \to \infty} \frac{F(x,z)}{z^{Np^-/(N-p^-)}} = 0. \tag{4.3}$$

Using relations (4.1) and (4.3), we obtain

$$\forall\, \varepsilon > 0,\ \exists\, \delta_1 > 0 \text{ such that } \Big| \frac{F(x,z)}{z^{p^+}} \Big| < \varepsilon \text{ for all } z \text{ with } |z| < \delta_1;$$

$$\forall\, \varepsilon > 0,\ \exists\, \delta_2 > 0 \text{ such that } \Big| \frac{F(x,z)}{z^{Np^-/(N-p^-)}} \Big| < \varepsilon \text{ for all } z \text{ with } |z| > \delta_2.$$

Thus, for $\varepsilon > 0$ there exist $\delta_1,\ \delta_2 > 0$ such that

$$F(x,z) < \varepsilon \cdot |z|^{p^+}, \quad |z| < \delta_1$$

and

$$F(x,z) < \varepsilon \cdot |z|^{Np^-/(N-p^-)}, \quad |z| > \delta_2.$$

Relation (4.2) implies that there exists a constant $c > 0$ such that

$$F(x,z) \le c \quad \text{for all } z \text{ with } |z| \in [\delta_1, \delta_2].$$

We conclude that for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$F(x,z) \le \varepsilon |z|^{p^+} + c_\varepsilon |z|^{Np^-/(N-p^-)}. \tag{4.4}$$

Let us assume that $\|u\| < 1$. Then, using relations (3.1) and (4.4), we have

$$\begin{aligned}
I(u) &\ge \frac{1}{p^+} J(u) - \int_{\mathbb{R}^N} F(x,u)dx \\
&\ge \frac{1}{p^+} \|u\|^{p+} - \int_{\mathbb{R}^N} F(x,u)dx \\
&\ge \frac{1}{p^+} \|u\|^{p+} - \epsilon \int_{\mathbb{R}^N} |u|^{p^+} dx - c_\varepsilon \int_{\mathbb{R}^N} |u|^{Np^-/(N-p^-)} dx.
\end{aligned}$$

For $p(x) \le q(x) \le \frac{Np(x)}{N-p(x)}$ we have $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ continuous, so $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ continuous, thus $|u|_{L^{q(x)}} \le c\|u\|_E$. Choosing $q(x) = p^+$ and then $q(x) = \frac{Np^-}{N-p^-}$ we obtain

$$|u|_{p^+} \le c_1 \|u\| \Leftrightarrow \Big(\int_{\mathbb{R}^N} |u|^{p^+} dx \Big)^{\frac{1}{p^+}} \le c_1 \|u\|;$$

$$|u|_{Np^-/(N-p^-)} \le c_2 \|u\| \Leftrightarrow \Big(\int_{\mathbb{R}^N} |u|^{Np^-/(N-p^-)} dx \Big)^{\frac{N-p^-}{Np^-}} \le c_2 \|u\|\,.$$

Therefore,

$$\begin{aligned}
I(u) &\ge \frac{1}{p^+} \|u\|^{p+} - \varepsilon c_1 \|u\|^{p+} - c_2 \cdot c_\varepsilon \|u\|^{Np^-/(N-p^-)} \\
&\ge \|u\|^{p+} \Big[(\frac{1}{p^+} - \varepsilon c_1) - c_2 \cdot c_\varepsilon \|u\|^{Np^-/(N-p^-)-p^+} \Big] \ge a > 0,
\end{aligned}$$

for some fixed $\varepsilon \in (0, \frac{1}{c_1 p^+})$ and $a, \|u\|$ sufficiently small. $\square$

Lemma 4.2. *Assume conditions (B1), (F1)-(F3) hold. Then there exists $e \in E$ with $\|e\| > \tau$ (τ given in Lemma 4.1) such that $I(e) < 0$.*

Proof. Denote

$$h(t) = \frac{F(x,tz)}{t^\mu}, \quad \forall\, t > 0.$$

Then using (F3) we get

$$h'(t) = \frac{1}{t^{\mu+1}} [tzf(x,tz) - \mu F(x,tz)] \ge 0, \quad \forall\, t > 0.$$

Thus, we deduce that for any $t \ge 1$, $F(x,tz) \ge t^\mu F(x,z)$.

Choosing $u \in E$ with $\|u\| > 1$ and $\int_{\mathbb{R}^N} F(x,u)dx > 0$ fixed and $t > 1$, we have

$$\begin{aligned} I(tu) &= \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big(|\nabla(tu)|^{p(x)} + b(x)|tu|^{p(x)}\Big)\, dx - \int_{\mathbb{R}^N} F(x,tu)dx \\ &= \int_{\mathbb{R}^N} \frac{1}{p(x)} t^{p(x)} \Big(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\Big)\, dx - \int_{\mathbb{R}^N} F(x,tu)dx \\ &\le \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} \Big(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\Big)\, dx - \int_{\mathbb{R}^N} F(x,tu)dx \\ &\le \frac{t^{p^+}}{p^-} \|u\|^{p^+} - t^{\mu} \int_{\mathbb{R}^N} F(x,u)dx. \end{aligned}$$

But $\mu > p^+$, therefore $I(tu) \to -\infty$ when t approaches $+\infty$, which concludes our lemma. $\square$

Proof of Theorem 3.2. We set

$$\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ \gamma(1) = e\},$$

where $e \in E$ is determined by Lemma 4.2, and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

According to Lemma 4.2 we know that $\|e\| > \tau$, so every path $\gamma \in \Gamma$ intersects the sphere $\|w\| = \tau$. Then Lemma 4.1 implies

$$c \ge \inf_{\|u\| = \tau} I(u) \ge a, \tag{4.5}$$

with the constant $a > 0$ in Lemma 4.1, thus $c > 0$.

By the mountain-pass theorem (see, e.g., [2]) we obtain a sequence $(u_n)_n \subset E$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0. \tag{4.6}$$

We claim that $(u_n)_n$ is bounded in E. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \to \infty$. Using (4.6) it follows that for n large enough, we have

$$c + 1 + \|u_n\| \ \ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle. \tag{4.7}$$

Since

$$\begin{gathered} I(u_n) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\Big)\, dx - \int_{\mathbb{R}^N} F(x,u_n)dx, \\ \langle I'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\Big)\, dx - \int_{\mathbb{R}^N} f(x,u_n)u_n dx, \end{gathered}$$

using (4.7) we obtain

$$c + 1 + \|u_n\| \ \ge \Big(\frac{1}{p^+} - \frac{1}{\mu}\Big) J(u_n) - \int_{\mathbb{R}^N} \Big[F(x,u_n) - \frac{1}{\mu} f(x,u_n)u_n\Big]\, dx.$$

By (F3) we have

$$\int_{\mathbb{R}^N} \big[F(x,u_n) - \frac{1}{\mu} f(x,u_n)u_n\big] dx \le 0.$$

The above inequalities combined with relation (3.1) yield

$$c + 1 + \|u_n\| \ \ge \Big(\frac{1}{p^+} - \frac{1}{\mu}\Big) J(u_n) \ge \Big(\frac{1}{p^+} - \frac{1}{\mu}\Big)\|u_n\|^{p^-}.$$

We obtain

$$c+1+\|u_n\| \ \geq (\frac{1}{p^+}-\frac{1}{\mu})\|u_n\|^{p^-}. \tag{4.8}$$

Now dividing by $\|u_n\|$ in (4.8) and passing to the limit as $n\to\infty$ we obtain a contradiction. So, up to a subsequence, $(u_n)_n$ converges weakly in E to some $u\in E$. If Ω is bounded then there exists a compact embedding $E(\Omega)\hookrightarrow L^{\frac{Np^-}{N-p^-}}(\Omega)$. Then $(u_n)_n$ converges strongly in $L^{\frac{Np^-}{N-p^-}}(\Omega)$, for all Ω bounded domains in $\mathbb{R}^N$. If we prove that

$$\langle I'(u_n),\varphi\rangle\to\langle I'(u),\varphi\rangle,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^N). \tag{4.9}$$

Then, by (4.6), u is a weak solution of (1.1), since $C_0^\infty(\Omega)$ is dense in E. To do this, let $\varphi\in C_0^\infty(\mathbb{R}^N)$ be fixed. We set $\Omega=\operatorname{supp}(\varphi)$.

To prove (4.9), first we prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)\varphi dx=\int_{\mathbb{R}^N}f(x,u)\varphi dx.$$

A simple calculation implies

$$\begin{aligned}
\Big|\int_\Omega(f(x,u_n)-f(x,u))\varphi(x)dx\Big| &\leq \int_\Omega|f(x,u_n)-f(x,u)|\cdot|\varphi(x)|dx\\
&\leq \|\varphi\|_{L^\infty(\Omega)}\int_\Omega|f(x,u_n)-f(x,u)|dx\\
&= \|\varphi\|_{L^\infty(\Omega)}\int_\Omega\Big|\frac{f(x,u_n)-f(x,u)}{u_n-u}\Big|\cdot|u_n-u|dx\\
&\leq \|\varphi\|_{L^\infty(\Omega)}\int_\Omega|f_z(x,v_n)|\cdot|u_n-u|dx,
\end{aligned}$$

where $v_n\in[u_n,u]$ (or $[u,u_n]$). Using (F2), we obtain

$$\begin{aligned}
&\Big|\int_\Omega(f(x,u_n)-f(x,u))\varphi(x)dx\Big|\\
&\leq \|\varphi\|_{L^\infty(\Omega)}\int_\Omega|a_1|v_n|^{p^+-2}+a_2|v_n|^{s-1}|\cdot|u_n-u|dx\\
&\leq \|\varphi\|_{L^\infty(\Omega)}\cdot\Big[a_1\int_\Omega|v_n|^{p^+-2}\cdot|u_n-u|dx+a_2\int_\Omega|v_n|^{s-1}\cdot|u_n-u|dx\Big].
\end{aligned}$$

We have $\frac{1}{p^+-1}+\frac{p^+-2}{p^+-1}=1$ and $\frac{1}{s}+\frac{s-1}{s}=1$. Using Hölder inequality,

$$\begin{aligned}
&\Big|\int_\Omega(f(x,u_n)-f(x,u))\varphi(x)dx\Big|\\
&\leq \|\varphi\|_{L^\infty(\Omega)}\cdot[a_1\|v_n\|^{p^+-2}_{L^{p^+-1}(\Omega)}\cdot\|u_n-u\|_{L^{p^+-1}(\Omega)}+a_2\|v_n\|^{s-1}_{L^s(\Omega)}\cdot\|u_n-u\|_{L^s(\Omega)}].
\end{aligned}$$

Taking into account that $u_n\to u$ strongly in $L^i(\Omega)$, for all $i\in[p^+-1,\frac{Np^-}{N-p^-}]$ and remarking that for all $x\in\Omega$ and for all $n\geq1$ there exists $\lambda_n(x)\in[0,1]$ such that $v_n(x)=\lambda_n(x)u_n(x)+[1-\lambda_n(x)]u(x)$ we deduce

$$\int_\Omega|v_n-u|^sdx=\int_\Omega|\lambda_n(x)|^s\cdot|u_n-u|^sdx\leq\int_\Omega|u_n-u|^sdx\to0,\ \text{as } n\to\infty.$$

It results that

$$\int_\Omega|v_n|^sdx\to\int_\Omega|u|^sdx\ ,\ \text{as } n\to\infty.$$

From the above considerations, we obtain

$$\Big|\int_\Omega (f(x,u_n)-f(x,u))\varphi(x)dx\Big| \to 0,\ \text{as } n\to\infty.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in E, the above relation implies

$$\lim_{n\to\infty}\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx=0.$$

Next, since $(u_n)_n$ converges weakly to u in E, it follows that

$$\lim_{n\to\infty}\int_\Omega f(x,u)(u_n-u)dx=0.$$

Thus, actually we find

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0.$$

On the other hand, we have

$$\lim_{n\to\infty}\langle I'(u_n),u_n-u\rangle=0.$$

Combining the last two relations we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)+b(x)|u_n|^{p(x)-2}u_n(u_n-u)\Big)\,dx=0. \tag{4.10}$$

Since relation (4.10) holds true and $(u_n)_n$ converges weakly to u in E, by [4, Lemma 3.1], we deduce that $(u_n)_n$ converges strongly to u in E. Then since $I\in C^1(E,\mathbb{R})$ we conclude

$$I'(u_n)\to I'(u), \tag{4.11}$$

as $n\to\infty$.

Relations (4.6) and (4.11) show that $I'(u)=0$ and thus u is a weak solution for (1.1). Moreover, by relation (4.6) it follows that $I(u)>0$ and thus, u is a nontrivial weak solution for (1.1). The proof of Theorem 3.2 is complete. □

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Corrigendum posted December 1, 2006

The author would like to thank Professor Xianling Fan for pointing out an error that occurred in the original paper. More exactly, condition (F2) must be replaced by

(F2) $p^+ < \frac{Np^-}{N-p^-}$ and there exist $s \in (p^+-1, Np^-/(N-p^-)-1)$, $\theta \in (s, Np^-/(N-p^-))$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\theta/(\theta-p^++1)}(\mathbb{R}^N)$, $g_2 \in L^\infty(\mathbb{R}^N) \cap L^{\theta/(\theta-s)}(\mathbb{R}^N)$, with $g_1(x), g_2(x) \geq 0$ such that
$$|f_z(x,z)| \leq g_1(x)|z|^{p^+-2} + g_2(x)|z|^{s-1}, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, z \in \mathbb{R}.$$

This new condition was inspired from the paper by Fan and Han [4], and implies the old condition.

Remark. Condition (F2) implies that there exist $a_1, a_2 > 0$ and $s \in (p^+-1, Np^-/(N-p^-)-1)$ such that
$$|f_z(x,z)| \leq a_1|z|^{p^+-2} + a_2|z|^{s-1}, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, z \in \mathbb{R}.$$

After this correction, the proof of Theorem 3.2 will change as well. At the end of the proof, from “If Ω is bounded ...” on page 8, line 5, to “$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx = 0.$”, page 9, line 8, will be replaced by:

Next, since $(u_n)_n$ converges weakly to u in E, it follows that
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u)(u_n-u)dx = 0.$$
Using condition (F2) and [4, Lemma 3.2], we find
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)dx = 0.$$
Combining the above two relations we obtain
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (f(x,u_n)-f(x,u))(u_n-u)dx = 0.$$

End of corrigendum.

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