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POSITIVE SOLUTIONS FOR SINGULAR NONLINEAR BEAM EQUATION

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Abstract. In paper, we study the existence of solutions for the singular p-Laplacian equation

$$\begin{gathered}\big(|u''|^{p-2}u''\big)'' - f(t,u) = 0, \quad t \in (0,1)\\ u(0) = u(1) = 0,\\ u''(0) = u''(1) = 0,\end{gathered}$$

where $f(t,u)$ is singular at $t = 0, 1$ and at $u = 0$. We prove the existence of at least one solution.

1. Introduction

In this paper, we establish the existence of solutions to the singular boundary-value problem

$$\begin{gathered}\big(|u''|^{p-2}u''\big)'' - f(t,u) = 0, \quad t \in (0,1)\\ u(0) = u(1) = 0,\\ u''(0) = u''(1) = 0,\end{gathered} \tag{1.1}$$

where $p > 1$ and $f(t,u)$ has singularity at $t = 0, 1$ and at $u = 0$. For convenience, we denote $\varphi_p(s) = |s|^{p-2}s$, for $p > 1$.

Equation (1.1) occurs in the following models of beams [4]: Beams with small deformations (also called geometric linearity); beams of a material which satisfies a nonlinear power-like stress-strain law; beams with two-sided links (for example, springs) which satisfy a nonlinear power-like elasticity law. The best known setting is the boundary-value problem, for $p = 2$,

$$u^{(4)} - f(t,u(t)) = 0, \quad t \in (0,1)\,.$$

This model describes deformations of an elastic beams with the boundary conditions reflecting both ends simply supported, also for one end simply supported and the other end clamped by sliding clamps. Vanishing moments and shear forces at the tail ends are frequently included in the boundary conditions; see for example Gupta [7] and its references. One derivation of lines is used in the description over regions of certain partial differential equations describing the deflection of an elastic beam.

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Agarwal et al. [2, 3] consider the boundary-value problem

$$(-1)^n x^{(2n)}(t) = \mu f(t, x(t), \dots, x^{(2n-2)}(t)),$$
$$x^{(2j)}(0) = x^{(2j)}(T) = 0,\ 0 \le j \le n-1$$

under the critical condition:

(A) For a.e. $t \in [0,T]$ and for each $(x_0, \dots, x_{2n-2}) \in D$ (defined in [2])

$$f(t, x_0, \dots, x_{2n-2}) \le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|^{\alpha_j}$$

where ϕ, $h_j \in L^1(0,T)$ and $q_j \in L^\infty(0,T)$ are nonnegative, $\omega_j : (0,\infty) \to (0,\infty)$ are nonincreasing, $\alpha_j \in (0,1)$ and

$$\int_0^T \omega_j(s)ds < \infty, \quad \omega_j(uv) \le \Lambda\omega_j(u)\omega_j(v)$$

for $0 \le j \le 2n-2$ and $u, v \in (0,\infty)$ with a positive constant Λ.

Closely related to the results of this paper is the recent work by Agarwal, Lü and O'Regan [1]. There the authors consider positive solutions for the boundary-value problem

$$\big(|u''|^{p-2}u''\big)'' - \lambda q(t) f(u(t)) = 0,$$

where the nonlinearity f is nonsingular. In this paper consider nonlinearity f may be singular. We point out a sufficient condition for problem (1.1) has a positive solution, but it doesn't satisfies the condition (A), for example

$$f(t,u) = \frac{t^\alpha(1+t)^\alpha}{u^\beta}$$

where $\alpha + 1 > \beta > 0$.

Singular nonlinear two point boundary-value problems arise naturally in applications and usually, only positive solutions are meaningful. By a positive solution of (1.1), we mean a function $u \in C^{(2)}[0,1]$ with $\varphi_p(u'') \in C^{(2)}(0,1)$ satisfying (1.1).

We next give definitions and some properties of cones in Banach spaces. After that, we state a fixed point theorem for operators that are decreasing with respect to a cone [5, 6].

Let B be a Banach space, and K a closed, nonempty subset of B. K is a cone provided (i) $\alpha u + \beta v \in K$, for all $u, v \in K$ and all $\alpha, \beta \ge 0$ and (ii) $u, -u \in K$ imply $u = 0$.

Given a cone K, a partial order, $\le$, is induced on B by $x \le y$, for $x, y \in B$ if $y - x \in K$. (For clarity, we may sometimes write $x \le y$ (wrt K). If $x, y \in B$ with $x \le y$, let $\langle x, y\rangle$ denote the closed order interval between x and y given by, $\langle x, y\rangle = \{z \in B | x \le z \le y\}$. A cone K is normal in B provided, there exists $\delta > 0$, such that $\|e_1 + e_2\| \ge \delta$, for all $e_1, e_2 \in K$, with $\|e_1\| = \|e_2\| = 1$.

The following fixed point theorem can be found in [5, 6].

Theorem 1.1. *Let B be a Banach space, K a normal cone in B, $E \subseteq K$ such that, if $x, y \in E$ with $x \le y$, then $\langle x, y\rangle \subseteq E$, and let $T : E \to K$ be a continuous mapping that is decreasing with respect to K, and which is compact on any closed order interval contained in E. Suppose there exists $x_0 \in E$ such that $T^2(x_0) = T(Tx_0)$ is defined, and furthermore, Tx_0, T^2x_0 are order comparable to x_0. If, either*

(I) *$Tx_0 \le x_0$ and $T^2x_0 \le x_0$, or $x_0 \le Tx_0$ and $x_0 \le T^2x_0$, or*

(II) *The complete sequence of iterates* $\{T^n x_0\}_{n=0}^{\infty}$ *is defined, and there exists* $y_0 \in E$ *such that* $Ty_0 \in E$ *and* $y_0 \leq T^n x_0$, *for all* $n \geq 0$, *then* T *has a fixed point in* E.

2. Main Theorem

Theorem 2.1. *Assume the following conditions hold:*

(a) $f(t,u) : (0,1) \times (0,\infty) \to (0,\infty)$ *is continuous,*
(b) $f(t.u)$ *is decreasing in* u, *for each fixed* $t \in (0,1)$,
(c) $\int_0^1 f(t,u)dt < \infty$, *for each fixed* u,
(d) $\lim_{u\to 0^+} f(t,u) = \infty$ *uniformly on compact subsets of* $(0,1)$,
(e) $\lim_{u\to\infty} f(t,u) = 0$ *uniformly on compact subsets of* $(0,1)$.
(f) *for each* $\tau > 0$, $0 < \int_0^1 f(t, g_\tau(t))dt < \infty$, *where* $g_\tau(x) = \tau g(x)$ *and*

$$g(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2}, \\ (1-t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then the boundary-value problem (1.1) *has a positive solution* $u \in C^{(2)}[0,1]$ *with* $\varphi_p(u'') \in C^{(2)}(0,1)$.

Before the proof of Theorem 2.1, We give some Lemmas which we will uses in its proof.

Lemma 2.2. *If* $u \in C^{(2)}[0,1]$, $\varphi_p(u'') \in C^{(2)}(0,1)$ *such that* $(|u''|^{p-2}u'')'' > 0$ *on* $(0,1)$, *and* $u(0) = u(1) = u''(0) = u''(1) = 0$, *then*

$$u(t) \geq \frac{1}{4} \max_{0\leq t\leq 1} |u(t)|, \quad \frac{1}{4} \leq t \leq \frac{3}{4}. \tag{2.1}$$

The proof of the above lemma is easy; so we omit it.

Recall that the Green function for the problem

$$\begin{gathered} u''(t) = 0, \quad 0 \leq t \leq 1, \\ u(0) = u(1) = 0 \end{gathered}$$

is defined as

$$G(t,s) = \begin{cases} (1-s)t, & 0 \leq t \leq s \leq 1, \\ (1-t)s, & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

A direct calculation shows that

$$G(t,s) \leq G(s,s) \quad \text{for } (t,s) \in [0,1] \times [0,1],$$

$$G(t,s) \leq \frac{1}{4} \quad \text{for } (t,s) \in [0,1] \times [0,1], \tag{2.3}$$

$$G(t,s) \geq \frac{1}{4} G(s,s) \geq \frac{3}{64} \quad \text{for } (t,s) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}] \tag{2.4}$$

Lemma 2.3. *If* $u \in C^{(2)}[0,1]$, $\varphi_p(u'') \in C^{(2)}(0,1)$ *such that* $(|u''|^{p-2}u'')'' > 0$ *on* $(0,1)$, *and* $u(0) = u(1) = u''(0) = u''(1) = 0$, *then* $u(t) \geq 0$ *on* $[0,1]$.

Proof. Let $v = u''$. Then

$$\begin{gathered} (\varphi_p(v))''(t) > 0, \\ v(0) = v(1) = 0. \end{gathered}$$

This implies

$$(\varphi_p(v))''(t) > 0,$$
$$(\varphi_p(v))(0) = (\varphi_p(v))(1) = 0.$$

By the convexity of $\varphi_p(v)$, we obtain $(\varphi_p(v))(t) \le 0$, for $0 \le t \le 1$. So $v(t) \le 0$, i.e.

$$u'' \le 0 \quad \text{on } [0,1],$$
$$u(0) = u(1) = 0.$$

Then by the concavity of u, we have $u \ge 0$ on $[0,1]$. □

It follows from Lemma 2.3 and Rolle's theorem, that $u(t)$ has an extreme point, say at $t_0 \in [0,1]$. Then we define a piecewise polynomial function,

$$p(t) = \begin{cases} \frac{|u|_\infty}{t_0} t, & 0 \le t \le t_0, \\ \frac{|u|_\infty}{1-t_0}(1-t), & t_0 \le t \le 1, \end{cases} \tag{2.5}$$

where $|u|_\infty = \sup_{0 \le t \le 1} |u(t)| = u(t_0)$. Then we have the following Lemma.

Lemma 2.4. *Assume $u \in C^{(2)}[0,1]$. Let $\varphi_p(u'')$ be a function in $C^{(2)}(0,1)$ such that $(\varphi_p(u''(t)))'' > 0$, $0 < t < 1$, and $u(0) = u(1) = u''(0) = u''(1) = 0$. Then $u(t) \ge p(t)$, on $[0,1]$, where $p(t)$ is defined by (2.5).*

Lemma 2.5. *Assume $u \in C^{(2)}[0,1]$. Let $\varphi_p(u'')$ be a function in $C^{(2)}(0,1)$ be such that $(|u''|^{p-2}u'')'' > 0$ on $(0,1)$ and $u(0) = u(1) = u''(0) = u''(1) = 0$. Then, there exists $\tau > 0$ such that $u(t) \ge g_\tau(t)$ on $[0,1]$.*

The proof of the above lemma is easy; so we omit it. Our next work is applying Theorem 1.1 to a sequence of operators that are decreasing with respect to a cone. The obtained fixed points provide a sequence of iterates which converges to a solution of (1.1). Positivity of solutions and Lemmas 2.2–2.4 are fundamental in this construction.

Let B be the Banach space $C[0,1]$ with the norm $\|u\| = |u|_\infty$. Let

$$K = \{u \in B : u(t) \ge 0, \text{ on } [0,1]\},$$

which is a normal cone in B. Let $D \subseteq K$ be defined by

$$D = \{\varphi \in B : \text{there exist } \tau(\varphi) > 0 \text{ such that } g_\tau(t) \le \varphi(t) \text{ on } [0,1]\}.$$

Define $T : D \to K$ by

$$T\varphi(t) = \int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s) f(s,\varphi(s))ds\Big)dx, \quad 0 \le t \le 1.$$

If $\varphi(t) > 0$ for $t \in [0,1]$, by assumption (a), we know $f(t,\varphi(t)) > 0$. If $\varphi(t) \in D$, then

$$(\varphi_p((T\varphi)''(t)))'' = f(t,\varphi(t)) > 0.$$

Note that $T\varphi(t)$ satisfies the boundary condition of (1.1). Lemma 2.5 yields that $T\varphi(t) \in D$. So $T : D \to D$. Moreover, if $\varphi(t)$ is a positive solution of (1.1), then by Lemma 2.5 $\varphi(t) \in D$ and $T\varphi(t) = \varphi(t)$. Next we prove that all of the solutions of (1.1) which belong to D have a priori bounds.

Lemma 2.6. *Assume that conditions (a)-(f) are satisfied. Then there exist an $R > 0$ such that $\|\varphi\| = |\varphi|_\infty \le R$, for all solutions, φ, of (1.1) that belong to D.*

Proof. Suppose that the conclusion is false. Then there exists a sequence, $\{\varphi_n\} \subset D$, of solutions of (1.1) such that $\lim_{n\to\infty} |\varphi_n| = \infty$. Without out loss of generality, we may assume that, for each $n \ge 1$,

$$|\varphi_n|_\infty \le |\varphi_{n+1}|_\infty. \tag{2.6}$$

For each $n \ge 1$, let $t_n \in (0,1)$ be the unique point such that

$$0 < \varphi_n(t_n) = |\varphi_n|_\infty.$$

Then we have $\varphi_n(t_n) \ge \varphi_{n-1}(t_{n-1}) \ge \dots \ge \varphi_1(t_1)$. Let $\tau = \frac{1}{4}\varphi_1(t_1)$ and

$$g_\tau(t) = \begin{cases} \tau t & \text{for } t \in [0, \frac{1}{2}], \\ \tau(1-t) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \tag{2.7}$$

By the inequality (2.1), we obtain

$$\varphi_n(t) \ge \frac{|\varphi_n|_\infty}{4} = \frac{\varphi_n(t_n)}{4} \ge \frac{1}{4}\varphi_1(t_1) \ge g_\tau(t) \quad \text{for } t \in [\frac{1}{4}, \frac{3}{4}].$$

Next, we claim that

$$\varphi_n(t) \ge g_\tau(t) \quad \text{for } t \in [0, \frac{1}{4}].$$

Let p_n be the corresponding piecewise polynomial defined by (2.5) relative to φ_n and t_n. There two case for t_n
Case 1. $t_n \ge \frac{1}{4}$. Then, for $0 \le t \le \frac{1}{4}$,

$$p_n(t) = \frac{|\varphi_n|_\infty}{t_n} t \ge |\varphi_n|_\infty t \ge \frac{|\varphi_1|_\infty}{4} t \ge g_\tau(t).$$

Case 2. $t_n < \frac{1}{4}$. Then, for $0 \le t \le t_n$, as the proof of Case 1, we have

$$p_n(t) \ge g_\tau(t) \quad \text{for } t \in [0, t_n].$$

On the other hand, on $[t_n, \frac{1}{4}]$,

$$p_n(t) = \frac{|\varphi_n|_\infty}{1-t_n}(1-t) \ge |\varphi_n|_\infty (1-t) \ge |\varphi_1|_\infty t \ge g_\tau(t).$$

Thus, again for $0 \le t \le \frac{1}{4}$,

$$p_n(t) \ge g_\tau(t).$$

Using analogous methods, we have $p_n(t) \ge g_\tau(t)$ for $t \in [\frac{3}{4}, 1]$. In conclusion,

$$p_n(t) \ge g_\tau(t) \quad \text{for } t \in [0,1]$$

which implies

$$\varphi_n(t) \ge g_\tau(t) \quad \text{for } t \in [0,1] \text{ and } n \ge 1. \tag{2.8}$$

Assumptions (b) and (f) yield, for $0 \le t \le 1$ and all $n \ge 1$,

$$\begin{aligned}
\varphi_n(t) &= (T\varphi_n)(t) \\
&= \int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s) f(s,\varphi_n(s)) ds\Big) dx \\
&\le \int_0^1 \frac{1}{4}\varphi_p^{-1}\Big(\int_0^1 \frac{1}{4} f(s,\varphi_n(s)) ds\Big) dx \\
&\le \int_0^1 \frac{1}{4}\varphi_p^{-1}\Big(\int_0^1 \frac{1}{4} f(s,g_\tau(s)) ds\Big) dx = N
\end{aligned}$$

for some $0 < N < \infty$. In particular, $|\varphi_n|_\infty \le N$ for all $n \ge 1$ which contradicts $\lim_{n\to\infty} |\varphi_n|_\infty = \infty$. The proof is complete. □

Our next step in obtaining solutions of (1.1) is to construct a sequence of nonsingular perturbations of f. For each $n \ge 1$, define $\psi_n : [0,1] \to [0,1]$ by

$$\psi_n(t) = \int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s)f(s,n)ds\Big)dx.$$

Because φ_p^{-1} is increasing and conditions (a)–(g), for $n \ge 1$,

$$0 < \psi_{n+1}(t) \le \psi_n(t) \quad \text{for } t \in (0,1),$$

and

$$\lim_{n\to\infty} \psi_n(t) = 0 \quad \text{uniformly on } \ [0,1]. \tag{2.9}$$

Now define a sequence of functions $f_n : (0,1) \times [0,\infty) \to (0,\infty)$, $n \ge 1$, by

$$f_n(t,u) = f(t, \max\{u, \psi_n(t)\}). \tag{2.10}$$

Then, for each $n \ge 1$, f_n is continuous, nonsingular and satisfies (b). Furthermore, for $n \ge 1$,

$$f_n(t,u) \le f(t,u) \quad \text{on } (0,1) \times (0,\infty),$$

$$f_n(t,u) \le f(t,\psi_n) \quad \text{on } (0,1) \times (0,\infty) \tag{2.11}$$

Proof of Theorem 2.1. We begin by defining a sequence of operators $T_n : K \to K$, $n \ge 1$ by

$$T_n\varphi(t) = \int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s)f_n(s,\varphi(s))ds\Big)dx.$$

Note that, for $n \ge 1$ and $\varphi \in K$, we have

$$\begin{gathered}
(\varphi_p((T_n\varphi)''))'' = f_n(t,\varphi(t)) > 0 \quad \text{for } t \in (0,1),\\
T_n\varphi(0) = T_n\varphi(1) = 0,\\
(T_n\varphi)''(0) = (T_n\varphi)''(1) = 0.
\end{gathered}$$

and $T_n\varphi > 0$ on $(0,1)$. In particular, $T_n\varphi \in D$. Since each f_n satisfies (b), it follows that if $\varphi_1, \varphi_2 \in K$ with $\varphi_1 \le \varphi_2$, then for $n \ge 1$, $T_n\varphi_2 \le T_n\varphi_1$; that is, each T_n is decreasing with respect to K. It is also clear that $0 \le T_n(0)$ and $0 \le T_n^2(0)$, for each n.

By Theorem 1.1, for each n, there exists a $\varphi_n \in K$, satisfies $T_n\varphi_n = \varphi_n$, and φ_n satisfies the boundary condition of (1.1).

In addition, by (2.11) we have $T_n\varphi \le T\Psi_n$, for each $\varphi \in K$ and $n \ge 1$. Thus

$$\varphi_n = T_n\varphi_n \le T\Psi_n, \ n \ge 1. \tag{2.12}$$

By essentially the same argument as in Lemma 2.6, there exist an $R > 0$, such that, for each $n \ge 1$

$$\varphi_n \le R \tag{2.13}$$

Our next claim is that there exist a $\kappa > 0$ such that $\kappa \le |\varphi_n|_\infty$ for all $n \ge 1$. We assume this claim to be false. Then, by passing to a subsequence and relabelling, we assume with no loss of generality that

$$\lim_{n\to\infty} \varphi_n(t) = 0, \ \text{uniformly on } [0,1]. \tag{2.14}$$

By condition (d), there exists a $\delta > 0$ such that, for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $0 < u < \delta$, $f(t,u) > 1$. By (2.14), there exist an $n_0 \geq 1$ such that for $n \geq n_0$,

$$0 < \varphi_n(t) < \frac{\delta}{2} \quad \text{for} \quad t \in (0,1).$$

Also from (2.9), there exist an $n_1 \geq n_0$ such that, for $n \geq n_1$,

$$0 < \psi_n(t) < \frac{\delta}{2}, \text{ for } t \in (0,1).$$

Thus for $n \geq n_1$ and $\frac{1}{4} \leq t \leq \frac{3}{4}$,

$$\begin{aligned}
\varphi_n(t) &= T_n\varphi_n(t) \\
&= \int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s)f_n(s,\varphi_n(s))ds\Big)dx \\
&\geq \int_{\frac{1}{4}}^{3/4} G(t,x)\varphi_p^{-1}\Big(\int_{\frac{1}{4}}^{3/4} G(x,s)f_n(s,\varphi_n(s))ds\Big)dx \\
&\geq \frac{1}{2} \times \frac{3}{64}\varphi_p^{-1}\Big(\int_{1/4}^{3/4} \frac{3}{64} f(s, \max\{\varphi_n(s), \psi_n(s)\})ds\Big) \\
&\geq \frac{1}{2} \times \frac{3}{64}\varphi_p^{-1}\Big(\int_{1/4}^{3/4} \frac{3}{64} f(s, \frac{\delta}{2})ds\Big) \\
&\geq \kappa > 0.
\end{aligned}$$

This contradicts the uniform limit (2.14). Our claim is verified. That is there exists a $\kappa > 0$ such that

$$\kappa \leq |\varphi_n|_\infty \leq R \quad \text{for all } n$$

Applying Lemma 2.2,

$$\varphi_n(t) \geq \frac{1}{4}|\varphi_n|_\infty \geq \frac{\kappa}{4}, \quad t \in [\frac{1}{4}, \frac{3}{4}],\ n \geq 1.$$

Let $\tau = \kappa/4$. Using a mimic methods in the proof of Lemma 2.6, we have

$$g_\tau(t) \leq \varphi_n(t) \quad \text{on } [0,1], \text{ for } n \geq 1$$

By (2.13), we now have

$$g_\tau(t) \leq \varphi_n(t) \leq R \quad \text{for all } n \geq 1;$$

that is, the sequence $\{\varphi_n(t)\}$ belongs to the closed order interval $\langle g_\tau, R\rangle \subset D$.

When restricted to this closed order interval, T is a compact mapping, and so, there is a subsequence of $\{T\varphi_n(t)\}$ which converges to some $\varphi^* \in K$. We relabel the subsequence as the original sequence so that

$$\lim_{n\to\infty} \|T\varphi_n - \varphi^*\| = 0. \tag{2.15}$$

The final part of the proof is to establish that

$$\lim_{n\to\infty} \|T\varphi_n - \varphi_n\| = 0.$$

Let $C = \frac{1}{4}\int_0^1 f(s, g_\tau(s))ds$. Then

$$\int_0^1 G(x,s)f(s,\varphi_n(s))ds \leq C \quad \text{for all } n \geq 1.$$

By the uniformly continuous of φ_p^{-1} on $[0,C]$, let $\varepsilon>0$ be given, there exists $\delta>0$, such that if $s_1,s_2\in[0,C]$ and $|s_1-s_2|<\delta$, we have

$$|\varphi_p^{-1}(s_1)-\varphi_p^{-1}(s_2)|<\varepsilon.$$

By the integrability condition (f), for above δ, there exists $0<\delta_1<1$, such that

$$\int_0^{\delta_1} f(s,g_\tau(s))ds+\int_{1-\delta_1}^1 f(s,g_\tau(s))ds\le\delta. \tag{2.16}$$

Further, by (2.9) there exists an n_0 such that, for $n\ge n_0$,

$$\psi_n(t)\le g_\tau(t)\le\varphi_n(t)\quad\text{on } [\delta_1,1-\delta_1].$$

from the definition of (2.10), we know

$$f_n(s,\varphi_n(s))=f(s,\varphi_n(s)),\ \text{for } s\in[\delta_1,1-\delta_1] \text{ and } n\ge n_0.$$

Thus, for $t\in[0,1]$ and $n\ge n_0$, by (2.16),

$$\begin{aligned}
&\int_0^1 G(x,s)f(s,\varphi_n(s))ds-\int_0^1 G(x,s)f_n(s,\varphi_n(s))ds\\
&=\int_0^{\delta_1} G(x,s)f(s,\varphi_n(s))ds+\int_{1-\delta_1}^1 G(x,s)f(s,\varphi_n(s))ds\\
&\quad-(\int_0^{\delta_1} G(x,s)f_n(s,\varphi_n(s))ds+\int_{1-\delta_1}^1 G(x,s)f_n(s,\varphi_n(s))ds)\\
&\le\int_0^{\delta_1} f(s,g_\tau(s))ds+\int_{1-\delta_1}^1 f(s,g_\tau(s))ds\le\delta.
\end{aligned}$$

So

$$\Big|\varphi_p^{-1}\Big(\int_0^1 G(x,s)f(s,\varphi_n(s))ds\Big)-\varphi_p^{-1}\Big(\int_0^1 G(x,s)f_n(s,\varphi(s))ds\Big))\Big|\le\varepsilon.$$

Then for $n\ge n_0$, we have

$$\begin{aligned}
|T\varphi_n(t)-\varphi_n(t)|&=\Big|\int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s)f(s,\varphi_n(s))ds\Big)dx\\
&\quad-\int_0^1 G(t,x)\varphi_p^{-1}\Big(\int_0^1 G(x,s)f_n(s,\varphi_n(s))ds\Big)dx\Big|\\
&=\int_0^1 G(t,x)\Big|\varphi_p^{-1}\Big(\int_0^1 G(x,s)f(s,\varphi_n(s))ds\Big)\\
&\quad-\varphi_p^{-1}\Big(\int_0^1 G(x,s)f_n(s,\varphi_n(s))ds\Big)\Big|dx\\
&\le\frac14\varepsilon<\varepsilon
\end{aligned}$$

In particular,

$$\lim_{n\to\infty}\|T\varphi_n(t)-\varphi_n(t)\|=0.$$

Then in conjunction with (2.15) we can easily obtain

$$\lim_{n\to\infty}\|\varphi_n-\varphi^*\|=0,$$

and this implies $\varphi^*\in\langle g_\tau,K\rangle\subset D$ and

$$\varphi^*=\lim_{n\to\infty}T\varphi_n=T\big(\lim_{n\to\infty}\varphi_n\big)=T\varphi^*,$$

which is sufficient for the conclusion of the Theorem 2.1. □

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