Electronic Journal of Differential Equations, Vol. 2007(2007), No. 113, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A BLOW-UP RESULT FOR A VISCOELASTIC SYSTEM IN $\mathbb{R}^n$

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Abstract. In this paper we consider a coupled system of nonlinear viscoelastic equations. Under suitable conditions on the initial data and the relaxation functions, we prove a finite-time blow-up result.

1. Introduction

In [7], Messaudi considered the following initial-boundary value problem

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive non-increasing function. He showed, under suitable conditions on g, that solutions with initial negative energy blow up in finite time if $p > m$ and continue to exist if $m \geq p$. This result has been later pushed, by the same author [11], to certain solutions with positive initial energy. A similar result been also obtained by Wu [15] using a different method.

In the absence of the viscoelastic term $(g = 0)$, problem (1.1) has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$u_{tt} - \Delta u + au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0,\infty) \tag{1.2}$$

$m, \gamma \geq 0$, it is well known that, for $a = 0$, the source term $bu|u|^\gamma$, $(\gamma > 0)$ causes finite time blow up of solutions with negative initial energy (see [1]). The interaction between the damping and the source terms was first considered by Levine [4], and in [5] the linear damping case $(m = 0)$. He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [2] extended Levine's result to the nonlinear damping case $(m > 0)$. In their work, the authors introduced a different method and showed that solutions with negative energy continue to exist globally "in time" if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended

2000 *Mathematics Subject Classification.* 35B05, 35L05, 35L15, 35L55.
Key words and phrases. Blow up; coupled system; relaxation function; viscoelastic.

Submitted February 14, 2007. Published August 18, 2007.
Supported by Project SABIC-2006/01 from KFUPM.

to solutions with negative initial energy by Messaoudi [8] and others. For results of same nature, we refer the reader to Levine and Serrin [3], and Vitillaro [12], Messaoudi and Said-Houari [10].

For problem (1.2) in $\mathbb{R}^n$, we mention, among others, the work of Levine Serrin and Park [6], Todorova [12, 13], Messaoudi [9], and Zhou [16].

In this work, we are concerned with the Cauchy problem

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f_1(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds = f_2(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \mathbb{R}^n \end{gathered} \tag{1.3}$$

where $g, h,\ u_0,\ u_1,\ v_0,\ v_1$ are functions to be specified later. This type of problems arises in viscoelasticity and in systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model. Our aim is to extend the result of [16], established for the wave equation, to our problem. To achieve this goal some conditions have to be imposed on the relaxation functions g and h.

2. Preliminaries

In this section we present some material needed in the proof of our main result. So, we make the following assumption

(G1) $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing differentiable functions satisfying

$$1 - \int_0^\infty g(s)ds = l > 0, \quad g'(t) \le 0, \quad t \ge 0.$$
$$1 - \int_0^\infty h(s)ds = k > 0, \quad h'(t) \le 0, \quad t \ge 0.$$

(G2) There exists a function $I(u,v) \ge 0$ such that

$$\frac{\partial I}{\partial u} = f_1(u,v), \quad \frac{\partial I}{\partial v} = f_2(u,v).$$

(G3) There exists a constant $\rho > 2$ such that

$$\int_{\mathbb{R}^n} [u f_1(u,v) + v f_2(u,v) - \rho I(u,v)]dx \ge 0.$$

(G4) There exists a constant $d > 0$ such that

$$|f_1(\xi,\varsigma)| \le d(|\xi|^{\beta_1} + |\varsigma|^{\beta_2}), \quad \forall (\xi,\varsigma) \in \mathbb{R}^2,$$
$$|f_2(\xi,\varsigma)| \le d(|\xi|^{\beta_3} + |\varsigma|^{\beta_4}), \quad \forall (\xi,\varsigma) \in \mathbb{R}^2,$$

where

$$\beta_i \ge 1, \quad (n-2)\beta_i \le n, \quad i = 1,2,3,4.$$

Note that (G1) is necessary to guarantee the hyperbolicity of the system (1.3). As an example of functions satisfying (G2)-(G4), we have

$$I(u,v) = \frac{a}{\rho}|u-v|^\rho, \quad \rho > 2, \quad (n-2)\rho \le 2(n-1).$$

Condition (G4) is necessary for the existence of a local solution to (1.3).

We introduce the "modified" energy functional

$$
\begin{aligned}
E(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{1}{2}(1 - \int_0^t h(s)ds)\|\nabla v\|_2^2 \\
+ \frac{1}{2}(g \circ \nabla u) + \frac{1}{2}(h \circ \nabla v) - \int_{\mathbb{R}^n} I(u,v)dx,
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
(g \circ \nabla u)(t) &= \int_0^t g(t-\tau)||u(t) - \nabla u(\tau)||_2^2 d\tau. \\
(h \circ \nabla v)(t) &= \int_0^t h(t-\tau)||\nabla v(t) - \nabla v(\tau)||_2^2 d\tau.
\end{aligned}
\tag{2.2}
$$

3. Blow up results

In this section we state and prove our main result.

Theorem 3.1. *Assume that* (G1)–(G4) *hold and that*

$$
\max\Big\{\int_0^{+\infty} g(s)ds, \int_0^{+\infty} h(s)ds\Big\} \le \frac{\rho(\rho-2)}{1+\rho(\rho-2)}. \tag{3.1}
$$

Then for initial data $(u_0,v_0),(u_1,v_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, *with compact support, satisfying*

$$
E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\|\nabla v_0\|_2^2 - \int_{\mathbb{R}^n} I(u_0,v_0)dx < 0, \tag{3.2}
$$

the corresponding solution (of (1.3))blows up in finite time.

Proof. Multiplying (1.3) by u_t and v_t respectively, integrating over $\mathbb{R}^n$, using integration by parts, and repeating the same computations as in [7], we obtain

$$
E'(t) = \frac{1}{2}(g' \circ \nabla u) + \frac{1}{2}(h' \circ \nabla v) - \frac{1}{2}g(s)\|\nabla u\|_2^2 - \frac{1}{2}h(s)\|\nabla v\|_2^2 \le 0. \tag{3.3}
$$

Hence,

$$
E(t) \le E(0) < 0. \tag{3.4}
$$

We then define

$$
F(t) = \frac{1}{2}\int_{\mathbb{R}^n} [|u(x,t)|^2 + |v(x,t)|^2]dx + \frac{1}{2}\beta(t+t_0)^2, \tag{3.5}
$$

for $t_0 > 0$ and $\beta > 0$ to be chosen later. By differentiating F twice we get

$$
F'(t) = \int_{\mathbb{R}^n} (u_t u + v_t v)dx + \beta(t+t_0), \tag{3.6}
$$

$$
F''(t) = \int_{\mathbb{R}^n} (u_{tt}u + v_{tt}v)dx + \int_{\mathbb{R}^n} (|u_t|^2 + |v_t|^2)dx + \beta. \tag{3.7}
$$

To estimate the term $\int_{\mathbb{R}^n}(u_{tt}u+v_{tt}v)dx$ in (3.7), we multiply the equations in (1.3) by u and v respectively and integrate by parts over $\mathbb{R}^n$ to get

$$\begin{aligned}\int_{\mathbb{R}^n}(uu_{tt}+vv_{tt})dx &= -\int_{\mathbb{R}^n}(|\nabla u|^2+|\nabla v|^2)dx+\int_{\mathbb{R}^n}[uf_1(u,v)+vf_2(u,v)]dx\\ &\quad+\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(x,t).\nabla u(x,s)\,dx\,ds\\ &\quad+\int_0^t h(t-s)\int_{\mathbb{R}^n}\nabla v(x,t).\nabla v(x,s)\,dx\,ds.\end{aligned}$$

Using Young's inequality and (G3) we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n}(uu_{tt}+vv_{tt})dx &\geq \Big[-1-\delta+\int_0^t g(s)ds\Big]\|\nabla u\|_2^2+\rho\int_{\mathbb{R}^n}I(u,v)dx\\ &\quad-\frac{1}{4\delta}\Big(\int_0^t g(s)ds\Big)(g\circ\nabla u)+\Big[-1-\delta+\int_0^t h(s)ds\Big]\|\nabla v\|_2^2\\ &\quad-\frac{1}{4\delta}\Big(\int_0^t h(s)ds\Big)(h\circ\nabla v)+\int_{\mathbb{R}^n}\big(|u_t|^2+|v_t|^2\big)dx,\end{aligned}\tag{3.8}$$

we then insert (3.8) in (3.7) to obtain

$$\begin{aligned}F''(t) &\geq (-1-\delta+\int_0^t g(s)ds)\|\nabla u\|_2^2-\frac{1}{4\delta}(\int_0^t g(s)ds)(g\circ\nabla u)\\ &\quad+(-1-\delta+\int_0^t h(s)ds)\|\nabla v\|_2^2-\frac{1}{4\delta}(\int_0^t h(s)ds)(h\circ\nabla v)\\ &\quad+\rho\int_{\mathbb{R}^n}I(u,v)dx+2\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)dx+\beta.\end{aligned}\tag{3.9}$$

Now, we exploit (2.1) to substitute for $\int_{\mathbb{R}^n}I(u,v)dx$, thus (3.9) takes the form

$$\begin{aligned}F''(t) &\geq -\rho E(t)+\beta+[(-1-\delta+\int_0^t g(s)ds)+\frac{\rho}{2}\Big(1-\int_0^t g(s)ds\Big)]\|\nabla u\|_2^2\\ &\quad+[(-1-\delta+\int_0^t h(s)ds)+\frac{\rho}{2}(1-\int_0^t h(s)ds)]\|\nabla v\|_2^2\\ &\quad+\Big[\frac{\rho}{2}-\frac{1}{4\delta}(\int_0^t g(s)ds)\Big](g\circ\nabla u)\\ &\quad+\Big[\frac{\rho}{2}-\frac{1}{4\delta}(\int_0^t h(s)ds)\Big](h\circ\nabla v)+(\frac{\rho}{2}+2)[\|u_t\|_2^2+\|v_t\|_2^2].\end{aligned}\tag{3.10}$$

At this point, we introduce

$$G(t):=F^{-\gamma}(t),$$

for $\gamma>0$ to be chosen properly. By differentiating G twice we arrive at

$$G'(t)=-\gamma F^{-(\gamma+1)}(t)F'(t),\quad G''(t)=-\gamma F^{-(\gamma+2)}(t)Q(t),$$

where

$$
\begin{aligned}
Q(t) &= F(t)F''(t) - (\gamma+1)(F')^2(t) \\
&\geq F(t)\Big\{ -\rho E(t) + \beta + \Big[(-1-\delta+\frac{\rho}{2}) - (\frac{\rho}{2}-1)\int_0^t g(s)ds)\Big]\|\nabla u\|_2^2 \\
&\quad + \Big[(-1-\delta+\frac{\rho}{2}) - (\frac{\rho}{2}-1)\int_0^t h(s)ds)\Big]\|\nabla v\|_2^2 \\
&\quad + \big[\frac{\rho}{2} - \frac{1}{4\delta}(\int_0^t g(s)ds)\big](g\circ\nabla u) + \big[\frac{\rho}{2} - \frac{1}{4\delta}(\int_0^t h(s)ds)\big](h\circ\nabla v) \\
&\quad + (\frac{\rho}{2}+2)[\|u_t\|_2^2 + \|v_t\|_2^2]\Big\} - (\gamma+1)\Big[\int_{\mathbb{R}^n}(u_tu+v_tv)dx + \beta(t+t_0)\Big]^2.
\end{aligned}
\tag{3.11}
$$

Using Young's and Cauchy-Schwartz inequalities, we estimate the last term in (3.11) as follows:

$$
\begin{aligned}
&\Big[\int_{\mathbb{R}^n}(u_tu+v_tv)dx + \beta(t+t_0)\Big]^2 \\
&\leq \Big(\int_{\mathbb{R}^n}(u_tu+v_tv)dx\Big)^2 + 2\Big[\frac{\varepsilon}{2}\Big(\int_{\mathbb{R}^n}(u_tu+v_tv)dx\Big)^2 \\
&\quad + \frac{1}{2\varepsilon}\beta^2(t+t_0)^2\Big] + \beta^2(t+t_0)^2 \\
&\leq (1+\varepsilon)(\int_{\mathbb{R}^n}(u_tu+v_tv)dx)^2 + (1+\frac{1}{\varepsilon})\beta^2(t+t_0)^2 \\
&\leq (1+\varepsilon)\Big[\int_{\mathbb{R}^n}u^2dx + \int_{\mathbb{R}^n}v^2dx\Big]\Big[\int_{\mathbb{R}^n}u_t^2dx + \int_{\mathbb{R}^n}v_t^2dx\Big] \\
&\quad + (1+\frac{1}{\varepsilon})\beta^2(t+t_0)^2 \\
&\leq 2F(x)\Big[(1+\varepsilon)\Big(\int_{\mathbb{R}^n}u_t^2dx + \int_{\mathbb{R}^n}v_t^2dx\Big) + (1+\frac{1}{\varepsilon})\beta\Big].
\end{aligned}
$$

Hence, (3.11) becomes

$$
\begin{aligned}
Q(t) &\geq F(t)\Big\{\Big[(-1-\delta+\frac{\rho}{2}) - (\frac{\rho}{2}-1)\int_0^t g(s)ds)\Big]\|\nabla u\|_2^2 \\
&\quad + \Big[(-1-\delta+\frac{\rho}{2}) - (\frac{\rho}{2}-1)\int_0^t h(s)ds)\Big]\|\nabla v\|_2^2 \\
&\quad + [\frac{\rho}{2} - \frac{1}{4\delta}\int_0^t g(s)ds](g\circ\nabla u) + [\frac{\rho}{2} - \frac{1}{4\delta}\int_0^t h(s)ds](h\circ\nabla v) \\
&\quad + \big[\frac{\rho}{2} + 2 - 2(\gamma+1)(1+\varepsilon\big]\big[\|u_t\|_2^2 + \|v_t\|_2^2\big] \\
&\quad - \rho E_0 - 2(\gamma+1)(1+\frac{1}{\varepsilon})\beta\Big\}, \quad \forall \varepsilon > 0.
\end{aligned}
\tag{3.12}
$$

We choose $\varepsilon < \rho/4$, $0 < \gamma < (\rho - 4\varepsilon)/(4(1+\varepsilon))$, and β small so that

$$
-\rho E_0 - [2 + \frac{2}{\varepsilon} + \gamma(2 + \frac{2}{\varepsilon})]\beta \geq 0.
$$

Next, we choose $\delta > 0$ so that

$$-1-\delta+\int_0^t g(s)ds+\frac{\rho}{2}(1-\int_0^t g(s)ds)\geq 0, \quad \frac{\rho}{2}-\frac{1}{4\delta}\int_0^t g(s)ds\geq 0,$$

and

$$-1-\delta+\int_0^t h(s)ds+\frac{\rho}{2}\big(1-\int_0^t h(s)ds\big)\geq 0, \quad \frac{\rho}{2}-\frac{1}{4\delta}\int_0^t h(s)ds\geq 0.$$

This is, of course, possible by (3.1), we then conclude, from (3.12), that $Q(t)\geq 0$, for all $t\geq 0$. Therefore $G''(t)\leq 0$ for all $t\geq 0$; which implies that G' is decreasing. By choosing t_0 large enough we get

$$F'(0)=\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)dx+\beta t_0>0,$$

hence $G'(0)<0$. Finally Taylor expansion of G yields

$$G(t)\leq G(0)+tG'(0), \quad \forall t\geq 0,$$

which shows that $G(t)$ vanishes at a time $t_m\leq -\frac{G(0)}{G'(0)}$. Consequently $F(t)$ must become infinite at time t_m. $\square$

Acknowledgments. The authors would like to express their sincere thanks to King Fahd University of Petroleum and Minerals for its support.

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