Electronic Journal of Differential Equations, Vol. 2007(2007), No. 130, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

VARIATION OF CONSTANTS FORMULA FOR FUNCTIONAL PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This paper presents a variation of constants formula for the system of functional parabolic partial differential equations

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= D\Delta u + Lu_t + f(t,x), \quad t > 0,\ u \in \mathbb{R}^n \\ \frac{\partial u(t,x)}{\partial \eta} &= 0, \quad t > 0,\ x \in \partial\Omega \\ u(0,x) &= \phi(x) \\ u(s,x) &= \phi(s,x), \quad s \in [-\tau, 0),\ x \in \Omega\,. \end{aligned}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, the $n \times n$ matrix D is block diagonal with semi-simple eigenvalues having non negative real part, the operator L is bounded and linear, the delay in time is bounded, and the standard notation $u_t(x)(s) = u(t+s,x)$ is used.

1. Introduction

In this paper we find a variation of constants formula for the system of functional parabolic partial differential equations

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= D\Delta u + Lu_t + f(t,x), \quad t > 0,\ u \in \mathbb{R}^n \\ \frac{\partial u(t,x)}{\partial \eta} &= 0, \quad t > 0,\ x \in \partial\Omega \\ u(0,x) &= \phi(x) \\ u(s,x) &= \phi(s,x), \quad s \in [-\tau, 0),\ x \in \Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, the $n \times n$ matrix D is non diagonal with semi-simple eigenvalues having non negative real part, and $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is an smooth function. The standard notation $u_t(x)$ defines a function from $[-\tau, 0]$ to $\mathbb{R}^n$ by $u_t(x)(s) = u(t+s,x)$, $-\tau \le s \le 0$ (with x fixed). Here $\tau \ge 0$ is the maximum delay, which is suppose to be finite. We assume the operator $L : L^2([-\tau, 0]; Z) \to Z$ is linear and bounded with $Z = L^2(\Omega)$ and $\phi_0 \in Z$, $\phi \in L^2([-\tau, 0]; Z)$.

2000 *Mathematics Subject Classification.* 34G10, 35B40.
Key words and phrases. Functional partial parabolic equations; variation of constants formula; strongly continuous semigroups.

Submitted July 2, 2007. Published October 5, 2007.

The variational constant formula plays an important role in the study of the stability, existence of bounded solutions and the asymptotic behavior of non linear ordinary differential equations. The variation of constants formula is well known for the finite dimensional semi-linear ordinary differential equation

$$\begin{gathered} x'(t) = A(t) + f(t,x), \quad x \in \mathbb{R}^n \\ x(0) = x_0, \end{gathered} \tag{1.2}$$

and it gives the solution

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)f(s,x(s))ds$$

where $\Phi(\cdot)$ is the fundamental matrix of the system

$$x'(t) = A(t)x. \tag{1.3}$$

Due to the importance of this formula, for semi linear ordinary differential equations, in 1961 the Russian mathematician Alekseev [1] found a formula for the nonlinear ordinary differential equation

$$y'(t) = f(t,y) + g(t,y), \quad y(t_0) = y_0, \tag{1.4}$$

which is given by

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^t \Phi(t,s,y(s))g(s,y(s))ds,$$

where $x(t,t_0,y_0)$ is the solution of the initial value problem

$$x'(t) = f(t,x), \quad x(t_0) = y_0, \tag{1.5}$$

and

$$\Phi(t,s,\xi) = \frac{\partial x(t,t_0,y_0)}{\partial y_0}.$$

This formula is used to compare the solutions of (1.4) with the solutions of (1.5). In fact, it was used in [9].

In infinite dimensional Banach spaces Z, we have the following general situation. If A is the infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in Z and $f : [0,\beta] \to Z$ is a suitable function, then the solution of the initial value problem

$$\begin{gathered} z'(t) = Az(t) + f(t), \quad t > 0,\ z \in Z \\ z(0) = z_0, \end{gathered} \tag{1.6}$$

is given by the variation constant formula

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0,\infty). \tag{1.7}$$

Therefore, any solution of the problem (1.6) is also solution of the integral equation (1.7). However, the converse may not be true, since a solution of (1.7) is not necessarily differentiable. We shall refer to a continuous solution of (1.7) as a mild solution of problem (1.6); a mild solution is thus a kind of generalized solution. However, if $\{T(t)\}_{t\geq 0}$ is an analytic semigroup and the function f satisfies the following Hölder condition

$$\|f(s) - f(t)\| \leq L|s-t|^\theta, \quad s,t \in [0,\beta],$$

with $L > 0$, $\theta \geq 1$, then the mild solution (1.7) is also solution of the initial value problem (1.6).

Our work and many others are motivated by the legendary paper by Borisovic and Turbabin [3]; there they found a variational constants formula for the system of nonhomogeneous differential equation with delay

$$\begin{gathered} z'(t) = Lz_t + f(t), \quad t > 0,\ z \in \mathbb{R}^n \\ z(0) = z_0, \\ z(s) = \phi(s), \quad s \in [-\tau, 0), \end{gathered} \tag{1.8}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^n$ is a suitable function. The standard notation z_t defines a function from $[-\tau, 0]$ to $\mathbb{R}^n$ by $z_t(s) = z(t+s), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^p([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is linear and bounded, and $z_0 \in \mathbb{R}^n$, $\phi \in L^p([-\tau, 0]; \mathbb{R}^n)$. Under some conditions they prove the existence and the uniqueness of solutions for this system and associate to it a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in the Banach space $\mathbb{M}_p([-\tau, 0]; \mathbb{R}^n) = \mathbb{R}^n \oplus L_p([-\tau, 0]; \mathbb{R}^n)$.

Therefore, system (1.8) is equivalent to the following system of ordinary differential equations, in $\mathbb{M}_p$,

$$\begin{gathered} \frac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0, \\ W(0) = W_0 = (z_0, \phi(\cdot)) \end{gathered} \tag{1.9}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ and $\Phi(t) = (f(t), 0)$.

Hence, the solution of system (1.8) is given by the variational constant formula or mild solution

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\Phi(s)ds. \tag{1.10}$$

Finally, the formula we found here is valid for those system of PDEs that can be rewritten in the form $\frac{\partial}{\partial t}u = D\Delta u$, like damped nonlinear vibration of a string or a beam, thermoplastic plate equation, etc. For more information about this, see the paper by Oliveira [12].

To the best of our knowledge, there are variational constant formulas for reaction diffusion equations, functional equations and neutral equations [6], but for functional partial parabolic equations we are not aware of results similar to the one presented here. At the same time, if we change the Neumann boundary condition by Dirichlet boundary condition, the result follows trivially.

2. Abstract Formulation of the Problem

In this section we choose a Hilbert Space where system (1.1) can be written as an abstract functional differential equation. To this end, we consider the following hypothesis.

(H1) The matrix D is semi simple (block diagonal) and the eigenvalues $d_i \in \mathbb{C}$ of D satisfy $\mathrm{Re}(d_i) \geq 0$. Consequently, if $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ are the eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, then there exists a constant $M \geq 1$ such that :

$\|e^{-\lambda_n Dt}\| \leq M, \quad t \geq 0, \quad n = 1, 2, 3, \dots$

H2). For all $I > 0$ and $z \in L^2_{\rm loc}([-\tau, 0); Z)$ we have the following inequality

$$\int_0^t |Lz_s| ds \leq M_0(t)|z|_{L^2([-\tau,t),Z)}, \quad \forall t \in [0, I],$$

where $M_0(\cdot)$ is a positive continuous function on $[0, \infty)$.

Consider $H = L^2(\Omega, \mathbb{R})$ and $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n \to \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Then

(i) There exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$ we have

$$-\Delta\xi = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle \xi, \phi_{n,k}\rangle \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in H and

$$E_n x = \sum_{k=1}^{\gamma_n} \langle \xi, \phi_{n,k}\rangle \phi_{n,k}. \tag{2.2}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in H and $\xi = \sum_{n=1}^{\infty} E_n \xi$, $\xi \in H$.

(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)\xi = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n \xi. \tag{2.3}$$

Now, we denote by Z the Hilbert space $L^2(\Omega, \mathbb{R}^n)$ and define the following operator

$$A : D(A) \subset Z \to Z, \quad A\psi = -D\Delta\psi$$

with $D(A) = H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n)$.
Therefore, for all $z \in D(A)$ we obtain

$$Az = \sum_{n=1}^{\infty} \lambda_n D P_n z, \quad z = \sum_{n=1}^{\infty} P_n z, \quad \|z\|^2 = \sum_{n=1}^{\infty} \|P_n z\|^2, \quad z \in Z$$

where $P_n = \operatorname{diag}(E_n, E_n, \dots, E_n)$ is a family of complete orthogonal proyections in Z. Consequently, system (1.1) can be written as an abstract functional differential equation in Z:

$$\begin{gathered} \frac{dz(t)}{dt} = -Az(t) + Lz_t + f^e(t), \quad t > 0 \\ z(0) = \phi_0 \\ z(s) = \phi(s), \quad s \in [-\tau, 0) \end{gathered} \tag{2.4}$$

Here $f^e : (0, \infty) \to Z$ is a function defined as follows:

$$f^e(t)(x) = f(t, x), \quad t > 0, \ x \in \Omega.$$

3. Preliminaries Results

For the rest of this article, we will use the following generalization of lemma 2.1 from [8].

Lemma 3.1. *Let Z be a separable Hilbert space, $\{S_n(t)\}_{n\geq 1}$ a family of strongly continuous semigroups and $\{P_n\}_{n\geq 1}$ a family of complete orthogonal projection in Z such that*
$$\Lambda_n P_n = P_n \Lambda_n, \quad n \geq 1, 2, \dots$$
where Λ_n is the infinitesimal generator of S_n. Define the family of linear operators
$$S(t)z = \sum_{n=1}^{\infty} S_n(t) P_n z, \quad t \geq 0.$$
Then:

(a) *$S(t)$ is a linear and bounded operator if $\|S_n(t)\| \leq g(t)$, $n = 1, 2, \dots$, with $g(t) \geq 0$, continuous for $t \geq 0$.*

(b) *$\{S(t)\}_{t\geq 0}$ is an strongly continuous semigroup in the Hilbert space Z whose infinitesimal generator Λ is given by*
$$\Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z, \quad z \in D(\Lambda)$$
with
$$D(\Lambda) = \big\{ z \in Z \,/\, \sum_{n=1}^{\infty} \|\Lambda_n P_n z\|^2 < \infty \big\}$$

(c) *the spectrum $\sigma(\Lambda)$ of Λ is given by*
$$\sigma(\Lambda) = \overline{\cup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}, \tag{3.1}$$
where $\bar{\Lambda}_n = \Lambda_n P_n : \mathcal{R}(P_n) \to \mathcal{R}(P_n)$.

Proof. First, from Hille-Yosida Theorem, $S_n(t)P_n = P_n S_n(t)$ since $\Lambda_n P_n = P_n \Lambda_n$. So that $\{S_n(t)P_n z\}_{n\geq 1}$ is a family of orthogonal vectors in Z. Then
$$\begin{aligned}
\|S(t)z\|^2 &= \langle S(t)z, S(t)z \rangle \\
&= \Big\langle \sum_{n=1}^{\infty} S_n(t) P_n z, \sum_{m=1}^{\infty} S_m(t) P_m z \Big\rangle \\
&= \sum_{n=1}^{\infty} \|S_n(t) P_n z\|^2 \\
&\leq (g(t))^2 \sum_{n=1}^{\infty} \|P_n z\|^2 \\
&= (g(t)\|z\|)^2
\end{aligned}$$
Therefore, $S(t)$ is a bounded linear operator.

Second, we have the following relations: (i)
$$S(t)S(s)z = \sum_{n=1}^{\infty} S_n(t) P_n S(s) z$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} S_n(t)P_n\Big(\sum_{m=1}^{\infty} S_m(s)P_m z\Big) \\
&= \sum_{n=1}^{\infty} S_n(t+s)P_n z \\
&= S(t+s)z
\end{aligned}
$$

(ii)

$$
S(0)z = \sum_{n=1}^{\infty} S_n(0)P_n z = \sum_{n=1}^{\infty} P_n z = z
$$

(iii)

$$
\begin{aligned}
\|S(t)z - z\|^2 &= \|\sum_{n=1}^{\infty} S_n(t)P_n z - \sum_{n=1}^{\infty} P_n z\|^2 \\
&= \sum_{n=1}^{\infty} \|(S_n(t) - I)P_n z\|^2 \\
&= \sum_{n=1}^{N} \|(S_n(t) - I)P_n z)\|^2 + \sum_{n=N+1}^{\infty} \|(S_n(t) - I)P_n z\|^2 \\
&\le \sup_{1\le n\le N} \|(S_n(t) - I)P_n z\|^2 \sum_{n=1}^{N} + K \sum_{n=N+1}^{\infty} \|P_n z\|^2,
\end{aligned}
$$

where $K = \sup_{0\le t\le 1;\, n\ge 1} \|(S_n(t) - I)\|^2 \le (g(t) + 1)^2$. Since $\{S_n(t)\}_{t\ge 0}$ $(n = 1, 2, \dots)$ is an strongly continuous semigroup and $\{P_n\}_{n\ge 1}$ is a complete orthogonal projections, given an arbitrary $\epsilon > 0$ we have, for some natural number N and $0 < t < 1$, the following estimates:

$$
\sum_{n=N+1}^{\infty} \|P_n z\|^2 < \frac{\epsilon}{2K}, \quad \sup_{1\le n\le N} \|(S_n(t) - I)P_n z\|^2 \le \frac{\epsilon}{2N},
$$

$$
\|S(t)z - z\|^2 < \frac{\epsilon}{2N} \sum_{n=1}^{N} + K \frac{\epsilon}{2K} < \epsilon
$$

Hence, $S(t)$ is an strongly continuous semigroup.

Let Λ be the infinitesimal generator of this semigroup. By definition, for all $z \in D(\Lambda)$, we have

$$
\Lambda z = \lim_{t\to 0^+} \frac{S(t)z - z}{t} = \lim_{t\to 0^+} \sum_{n=1}^{\infty} \frac{(S_n(t) - I)}{t} P_n z.
$$

Next,

$$
P_m \Lambda z = P_m\Big(\lim_{t\to 0^+} \sum_{n=1}^{\infty} \frac{(S_n(t) - I)}{t} P_n z\Big) = \lim_{t\to 0^+} \frac{S_m(t) - I}{t} P_m z = \Lambda_m P_m z
$$

So,

$$
\Lambda z = \sum_{n=1}^{\infty} P_n \Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z
$$

and

$$D(\Lambda) \subset \big\{z \in Z / \sum_{n=1}^{\infty} \|\Lambda_n P_n z\|^2 < \infty\big\}$$

On the other hand, if we assume that $z \in \big\{z \in Z / \sum_{n=1}^{\infty} \|\Lambda_n P_n z\|^2 < \infty\big\}$, then

$$\sum_{n=1}^{\infty} \Lambda_n P_n z = y \in Z$$

Next, making $z_n = \sum_{k=1}^{n} P_k z$, we obtain

$$\lim_{t\to 0^+} \frac{S(t)z_n - z_n}{t} = \sum_{k=1}^{n} P_k \Lambda_k z < \infty.$$

Therefore, $z_n \in D(\Lambda)$ and $\Lambda z_n = \sum_{k=1}^{n} P_k \Lambda_k z$. Finally, if $z_n \to z$ when $n \to \infty$ and $\lim_{t\to 0^+} \Lambda z_n = y$, then, since Λ is closed, we obtain that $z \in D(\Lambda)$ and $\Lambda z = y$.

To complete the proof of the lemma, we shall prove part (c). It is equivalent to prove that

$$\cup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n) \subset \sigma(\Lambda) \quad \text{and} \quad \sigma(\Lambda) \subset \overline{\cup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}.$$

To prove the first part, We shall show that $\rho(\Lambda) \subset \bigcap_{n=1}^{\infty} \rho(\bar{\Lambda}_n)$. In fact, let λ be in $\rho(\Lambda)$. Then $(\lambda - \Lambda)^{-1} : Z \to D(\Lambda)$ is a bounded linear operator. We need to prove that

$$(\lambda - \bar{\Lambda}_m)^{-1} : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$$

exists and is bounded for $m \geq 1$. Suppose that $(\lambda - \bar{\Lambda}_m)^{-1} P_m z = 0$. Then

$$(\lambda - \Lambda) P_m z = \sum_{n=1}^{\infty} (\lambda - \Lambda_n) P_n P_m z = (\lambda - \Lambda_m) P_m z = (\lambda - \bar{\Lambda}_m) P_m z = 0.$$

Which implies that, $P_m z = 0$. So, $(\lambda - \bar{\Lambda}_m)$ is one to one.

Now, given y in $\mathcal{R}(P_m)$ we want to solve the equation $(\lambda - \bar{\Lambda}_m) w = y$. In fact, since $\lambda \in \rho(\Lambda)$ there exists $z \in Z$ such that

$$(\lambda - \Lambda) z = \sum_{n=1}^{\infty} (\lambda - \Lambda_n) P_n z = y.$$

Then, applying P_m to the both side of this equation we obtain

$$P_m(\lambda - \Lambda) z = (\lambda - \Lambda_m) P_m z = (\lambda - \bar{\Lambda}_m) P_m z = P_m y = y.$$

Therefore, $(\lambda - \bar{\Lambda}_m) : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$ is a bijection. Since $\bar{\Lambda}_m$ is close, then, by the closed-graph theorem, we get

$$\lambda \in \rho(\bar{\Lambda}_m) = \{\lambda \in \mathbb{C} : (\bar{\Lambda}_m - \lambda I) \text{ is bijective }\} = \{\lambda \in \mathbb{C} : (\bar{\Lambda}_m - \lambda I)^{-1} \text{is bounded }\}$$

for all $m \geq 1$. We have proved that

$$\rho(\Lambda) \subset \bigcap_{n=1}^{\infty} \rho(\bar{\Lambda}_n) \iff \bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n) \subset \sigma(\Lambda).$$

Now, we shall prove the other part of (c), that is to say:

$$\sigma(\Lambda) \subset \overline{\cup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}.$$

In fact, if $\lambda \in \sigma(\Lambda)$, then

(1) $\lambda \in \sigma_p(\Lambda) = \{\lambda \in \mathbb{C} : (\Lambda - \lambda I) \text{ is not injective }\}$
(2) $\lambda \in \sigma_r(V) = \{\lambda \in \mathbb{C} : (\Lambda - \lambda I) \text{ is injective , but } \overline{R(\Lambda - \lambda I)} \neq Z\}$

(3) $\lambda \in \sigma_c(\Lambda) = \{\lambda \in \mathbb{C} : (\Lambda - \lambda I)$ is injective, $\overline{R(\Lambda - \lambda I)} = Z$, but $R(\Lambda - \lambda I) \neq Z\}$.

(1) If $(A\Lambda - \lambda I)$ is not injective, then there exists $z \in Z$ non zero such that: $(\Lambda - \lambda I)z = 0$. This implies that for some n_0 we have

$$(\overline{\Lambda_{n_0}} - \lambda I)P_{n_0}z = 0, \quad P_{n_0}z \neq 0.$$

¿From here we obtain that $\lambda \in \sigma(\overline{\Lambda_{n_0}})$, and therefore $\lambda \in \overline{\cup_{n=1}^{\infty}\sigma(\overline{\Lambda_n})}$.

(2) If $\overline{R(\Lambda - \lambda I)} \neq Z$, then there exists $z_0 \in Z$ non zero such that

$$\langle z_0, (\Lambda - \lambda I)z\rangle = 0, \quad \forall z \in D(A).$$

But, $z = \sum_{n=1}^{\infty} P_n z$, so

$$\langle z_0, \sum_{n=1}^{\infty}(\overline{\Lambda_n} - \lambda I)P_n z\rangle = 0.$$

Now, if $z_0 \neq 0$, then there is $n_0 \in \mathbf{N}$ such that $P_{n_0}z_0 \neq 0$. Hence,

$$0 = \langle z_0, \sum_{n=1}^{\infty}(\overline{\Lambda_n} - \lambda I)P_n z\rangle = \langle z_0, (\overline{\Lambda_{n_0}} - \lambda I)P_{n_0}z\rangle = \langle P_{n_0}z_0, (\overline{\Lambda_{n_0}} - \lambda I)P_{n_0}z\rangle$$

So, $R(\overline{\Lambda}_{n_0} - \lambda I) \neq P_{n_0}Z$. Therefore, $\lambda \in \sigma(\overline{\Lambda_{n_0}}) \subset \overline{\cup_{n=1}^{\infty}\sigma(\overline{\Lambda_n})}$.

(3) Assume that $(\Lambda - \lambda I)$ is injective, $\overline{R(\Lambda - \lambda I)} = Z$ and $R(\Lambda - \lambda I) \subsetneq Z$. For the purpose of getting a contradiction, we suppose that $\lambda \in \left(\overline{\cup_{n=1}^{\infty}\sigma(\overline{\Lambda_n})}\right)^C$.
However,

$$\left(\overline{\cup_{n=1}^{\infty}\sigma(\overline{\Lambda_n})}\right)^C \subset \Big(\bigcup_{n=1}^{\infty}\sigma(\overline{\Lambda_n})\Big)^C = \bigcap_{n\geq 1}\big(\sigma(\overline{\Lambda_n})\big)^C = \bigcap_{n\geq 1}\rho(\overline{\Lambda_n}),$$

which implies that, $\lambda \in \rho(\overline{\Lambda_n})$, for all $n \geq 1$. Then we get that

$$(\overline{\Lambda_n} - \lambda I) : R(P_n) \to R(P_n)$$

is invertible, with $(\overline{\Lambda_n} - \lambda I)^{-1}$ bounded. Hence, for all $z \in D(\Lambda)$ we obtain

$$P_j(\Lambda - \lambda I)z = (\overline{\Lambda_j} - \lambda I)P_j z, \quad j = 1, 2, \dots;$$

i.e.,

$$(\overline{\Lambda_j} - \lambda I)^{-1}P_j(\Lambda - \lambda I)z = P_j z, \quad j = 1, 2, \dots$$

Now, since $D(A)$ is dense in Z, we may extend the operator $(\overline{\Lambda_j} - \lambda I)^{-1}P_j(\Lambda - \lambda I)$ to a bounded operator T_j defined on Z. Therefore, it follows that

$$T_j z = P_j z, \quad \forall z \in Z,\ j = 1, 2, \dots,$$

and

$$\|T_j\| = \|P_j\| \leq 1, \quad j = 1, 2, \dots.$$

Since $\overline{R(\Lambda - \lambda I)} = Z$, we get

$$\|(\overline{\Lambda_j} - \lambda I)^{-1}\| \leq 1, \quad j = 1, 2, \dots. \tag{3.2}$$

Now we shall see that $R(\Lambda - \lambda I) = Z$. In fact, given $z \in Z$ we define y as

$$y = \sum_{j=1}^{\infty}(\overline{\Lambda_j} - \lambda I)^{-1}P_j z.$$

¿From (3.2) we get that y is well defined. We shall see now that $y \in D(\Lambda)$ and $(\Lambda - \lambda I)y = z$. In fact, we know that

$$y \in D(\Lambda) \iff \sum_{j=1}^{\infty} \|\Lambda_j P_j y\|^2 < \infty.$$

On the other hand, we have

$$\sum_{j=1}^{\infty} \|\overline{\Lambda}_j P_j y\|^2 = \sum_{j=1}^{\infty} \|\Lambda_j (\overline{\Lambda_j} - \lambda I)^{-1} P_j z\|^2 = \sum_{j=1}^{\infty} \|\{I + \lambda(\overline{\Lambda_j} - \lambda I)^{-1}\} P_j z\|^2.$$

So,

$$\sum_{j=1}^{\infty} \|\Lambda_j P_j y\|^2 \le \sum_{j=1}^{\infty} \|(1+|\lambda|)^2 \|P_j z\|^2 = (1+|\lambda|)^2 \|z\|^2 < \infty.$$

Then, $y \in D(\Lambda)$ and $(\Lambda - \lambda I) = z$. Therefore $R(\Lambda - \lambda I) = Z$, which is a contradiction that came from the assumption: $\lambda \in \overline{\left(\cup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})\right)}^C$. $\square$

Lemma 3.2. *Let Z be a separable Hilbert space, $\{S_n(t)\}_{t \ge 0}$ a family of strongly continuous semigroups with generators Λ_n and $\{P_n\}_{n \ge 1}$ a family of complete orthogonal projections such that*

$$\Lambda_n P_m = P_m \Lambda_n, \quad n, m = 1, 2, \dots \tag{3.3}$$

If the operator

$$\Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z, \quad z \in D(\Lambda)$$

with

$$D(\Lambda) = \{z \in Z : \sum_{n=1}^{\infty} \|\Lambda_n P_n z\|^2 < \infty\}$$

generates a strongly continuous semigroup $\{S(t)\}_{t \ge 0}$, then

$$S(t)z = \sum_{n=1}^{\infty} S_n(t) P_n z, \quad z \in Z.$$

Proof. If $z_0 \in Z$, then $P_n z_0 \in D(\Lambda)$ and the mild solution of the problem

$$\begin{aligned} z'(t) &= \Lambda z(t) \\ z(0) &= P_n z_0 \end{aligned} \tag{3.4}$$

is given by $z_n(t) = S(t) P_n z_0$ and it is a classic solution. Using (3.3) and the Hille-Yosida Theorem, we get $P_n S(t) = S(t) P_n$, which implies

$$S(t) z_0 = \sum_{n=1}^{\infty} P_n S(t) z_0 = \sum_{n=1}^{\infty} S(t) P_n z_0. \tag{3.5}$$

On the other hand, since $z_n(t)$ is a classic solution of (3.4), we obtain

$$\begin{aligned} z_n'(t) &= \Lambda z_n(t) \\ &= \Lambda S(t) P_n z_0 \\ &= \sum_{m=1}^{\infty} \Lambda_m P_m S(t) P_n z_0 \\ &= \Lambda_n P_n S(t) P_n z_0 \end{aligned}$$

$$= \Lambda_n S(t) P_n z_0 = \Lambda_n z_n(t)$$

So that, $z_n(t) = S_n(t)P_n z_0 = S(t)P_n z_0$ and from (3.5) we get

$$S_n(t)z_0 = \sum_{n=1}^{\infty} S_n(t) P_n z_0.$$

□

Now, applying Lemma 3.1 we can prove the following result.

Theorem 3.3. *The operator $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{T_A(t)\}_{t\geq 0}$ in the space Z, given by*

$$T_A(t)z = \sum_{n=1}^{\infty} e^{-\lambda_n D t} P_n z, \quad z \in Z, \ t \geq 0. \tag{3.6}$$

3.1. **Existence and Uniqueness of Solutions.** In this part we study the existence and the uniqueness of the solutions for system (2.4) in case $f^e \equiv 0$. That is, we analyze the homogeneous system

$$\begin{gathered} \frac{dz(t)}{dt} = -Az(t) + Lz_t, \quad t > 0 \\ z(0) = \phi_0 = z_0 \\ z(s) = \phi(s), \quad s \in [-\tau, 0) \end{gathered} . \tag{3.7}$$

Definition 3.4. A function $z(\cdot)$ define on $[-\tau, \alpha)$ is called a Mild Solution of (3.7) if

$$z(t) = \begin{cases} \phi(t) & -\tau \leq t < 0, \\ T_A(t)z_0 + \int_0^t T_A(t-s) L z_s ds, & t \in [0, \alpha) \end{cases}$$

Theorem 3.5. *Problem (3.7) admits only one mild solution defined on $[-\tau, \infty)$.*

Proof. Consider the initial function

$$\varphi(s) = \begin{cases} \phi(s), & -\tau \leq s < 0 \\ T_A(s)z_0 & s \geq 0 \end{cases}$$

which belongs to $L^2_{\text{loc}}([-\tau, \infty), Z)$. For a moment we shall set the problem on $[-\tau, I]$, $I > 0$ and denote by G the set

$$G = \{\psi : \psi \in L^2[[-\tau, \alpha], Z] \quad \text{and} \quad |\psi - \varphi|_{L^2} \leq \rho, \quad \rho > 0\},$$

where $\alpha > 0$ is a number to be determine. It is clear that G endowed with the norm of $L^2([-\tau, \alpha]; Z)$ is a complete metric space.

Now, we consider the application $S : G \to Z$, for $z \in G$, given by

$$(Sz)(t) = Sz(t) = \begin{cases} \phi(t), & -\tau \leq t < 0 \\ T_A(t)z_0 + \int_0^t T_A(t-s) L z_s ds, & t \in [0, \alpha] \end{cases}$$

Claim 1. There exists $\alpha > 0$ such that

- (i) $Sz \in G$, for all $z \in G$.
- (ii) S is a contraction mapping.

In fact, we prove (i) as follows:

$$|Sz(t) - \varphi(t)| \le \int_0^t |T_A(t-s)Lz_s|ds \le \int_0^\alpha M|Lz_s|ds \le MM_0(\alpha)|z|_{L^2([-\tau,\alpha),Z)}.$$

Integrating, we have

$$|Sz - \varphi|_{L^2} \le K\alpha^{1/2}|z|_{L^2}$$

where $K = max\{MM_0(\alpha)/\alpha \in [0, I]\}$. ¿From here we get

$$|Sz - \varphi|_{L^2} \le K\alpha^{1/2}(|\varphi|_{L^2} + \rho), \quad z \in G.$$

Taking

$$\alpha < \Big(\frac{\rho}{K(|\varphi|_{L^2} + \rho)}\Big)^2$$

we obtain that $Sz \in G$, for all $z \in G$.

To prove (ii), we use the linearity of L to obtain:

$$|Sz - Sw|_{L^2} \le K\alpha^{1/2}|z - w|_{L^2}, \quad \forall z, w \in G.$$

Next, to prove that S it is a contraction and $S(G) \subset G$ it is sufficient to choose α so that

$$\alpha < \min\Big\{\Big(\frac{1}{K}\Big)^2, \Big(\frac{\rho}{K(|\varphi|_{L^2} + \rho)}\Big)^2\Big\}$$

Therefore, S is a contraction mapping. So, if we apply the contraction mapping Theorem, there exists a unique point $z \in G$ such that $Sz = z$. i.e.,

$$z(t) = Sz(t) = \begin{cases} \phi(t), & -\tau \le t < 0 \\ T_A(t)z_0 + \int_0^t T_A(t-s)Lz_s ds, & t \in [0, \alpha], \end{cases}$$

which proves the existence and the uniqueness of the mild solution of the initial value problem (3.7) on $[-\tau, \alpha]$.

Claim 2. α could be equal to ∞. In fact, let z be the unique mild solution define in a maximal interval $[-\tau, \delta)(\delta \ge \alpha)$.

By contradiction, let us suppose that $\delta < \infty$. Since z is a mild solution of (3.7), we have that

$$z(t) = T_A(t)z_0 + \int_0^t T_A(t-s)Lz_s ds, \quad t \in [0, \delta).$$

Consider the sequence $\{t_n\}$ such that $t_n \to \delta^-$. Let us prove that $\{z(t_n)\}$ is a Cauchy sequence. In fact,

$$\begin{aligned}
&|z(t_n) - z(t_m)| \\
&= |T_A(t_n)z_0 - T_A(t_m)z_0 + \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds| \\
&\le |(T_A(t_n) - T_A(t_m))z_0| + |\int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds|
\end{aligned}$$

But,

$$\begin{aligned}
&|\int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds| \\
&\le |\int_0^{t_m} (T_A(t_n - s) - T_A(t_m - s))Lz_s ds| + |\int_{t_n}^{t_m} T_A(t_n - s)Lz_s ds|
\end{aligned}$$

Now, for $z \in L^2([-\tau, \delta])$ we obtain

$$\int_0^{t_m} |(T_A(t_n - s) - T_A(t_m - s))Lz_s|ds \le \int_0^{\delta} |(T_A(t_n - s) - T_A(t_m - s))Lz_s|ds$$

We know that

$$\lim_{n,m\to\infty} |(T_A(t_n - s) - T_A(t_m - s))Lz_s| = 0,$$
$$|(T_A(t_n - s) - T_A(t_m - s))Lz_s| \le 2M|Lz_s|$$

But, from the hypothesis (H1), we obtain

$$\int_0^{\delta} 2M|Lz_s|ds \le 2MM_0(\delta)|z|_{L^2([-\tau,\delta);Z)}$$

Therefore, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n,m\to\infty} \int_0^{\delta} |(T_A(t_n - s) - T_A(t_m - s))Lz_s|ds = 0$$

Then, since the family $\{T_A(t)\}_{t\ge 0}$ is strongly continuous and $t_n, t_m \to \delta^-$ when $n, m \to \infty$, the sequence $\{z(t_n)\}$ is a Cauchy sequence and therefore there exists $B \in Z$ such that

$$\lim_{n\to\infty} z(t_n) = B.$$

Now, for $t \in [0, \delta)$ we obtain that

$$\begin{aligned}
|z(t) - B| &\le |z(t) - z(t_n)| + |z(t_n) - B| \\
&\le |(T_A(t) - T_A(t_n))z_0| \ + \ |z(t_n) - B| \\
&\quad + |\int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^t T_A(t - s)Lz_s ds|
\end{aligned}$$

However,

$$\begin{aligned}
&|\int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^t T_A(t - s)Lz_s ds| \\
&\le \int_0^{t_n} |(T_A(t - s) - T_A(t_n - s))Lz_s|ds + \int_t^{t_n} |T_A(t - s)Lz_s|ds.
\end{aligned}$$

On the other hand, for $z \in L^2([-\tau, \delta])$ we get the estimate

$$\int_0^{t_n} |(T_A(t - s) - T_A(t_n - s))Lz_s|ds \le \int_0^{\delta} |(T_A(t - s) - T_A(t_n - s))Lz_s|ds$$

Therefore, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_0^{\delta} |(T_A(t - s) - T_A(t_n - s))Lz_s| = 0$$

Then, since the family $\{T_A(t)\}_{t\ge 0}$ is strongly continuous and $t_n \to \delta^-$ when $n \to \infty$, it follows that $z(t) \to B$ as $t \to \delta^-$. The function

$$\varphi(s) = \begin{cases} z(s), & \delta - \tau \le s < \delta \\ T_A(s)B, & s \ge \delta \end{cases}$$

belongs to $L^2_{\rm loc}([\delta-\tau,\infty),Z)$. So, if we apply again the contraction mapping Theorem to the Cauchy problem

$$\begin{gathered}\frac{dy(t)}{dt} = -Ay(t) + Ly_t, \quad t>\delta\\ y(\delta)=B\\ y(s)=z(s), \quad s\in[\delta-\tau,\delta)\end{gathered} \tag{3.8}$$

where $z(\cdot)$ is the unique solution of the system (3.7), then we get that (3.8) admits only one solution $y(\cdot)$ on the interval $[\delta-\tau,\delta+\epsilon]$ with $\epsilon>0$. Therefore, the function

$$\widetilde{z}(s)=\begin{cases} z(s) & -\tau\le s<\delta\\ y(s), & \delta\le s<\delta+\epsilon\end{cases}$$

is also a mild solution of (3.7) which is a contradiction. So, $\delta=\infty$. □

4. The Variation Of Constants Formula

Now we are ready to find the formula announced in the title of this paper for the system (2.4), but first we need to write this system as an abstract ordinary differential equation in an appropriate Hilbert space. In fact, we consider the Hilbert space $\mathbb{M}_2([-\tau,0];Z)=Z\oplus L_2([-\tau,0];Z)$ with the usual innerproduct given by

$$\Big\langle \begin{pmatrix}\phi_{01}\\ \phi_1\end{pmatrix}, \begin{pmatrix}\phi_{02}\\ \phi_2\end{pmatrix}\Big\rangle = \langle\phi_{01},\phi_{02}\rangle_Z + \langle\phi_1,\phi_2\rangle_{L_2}.$$

Define the operators $T(t)$ in the space $\mathbb{M}_2$ for $t\ge 0$ by

$$T(t)\begin{pmatrix}\phi_0\\ \phi(.)\end{pmatrix} = \begin{pmatrix} z(t)\\ z_t\end{pmatrix} \tag{4.1}$$

where $z(\cdot)$ is the only mild solution of the system (3.7).

Theorem 4.1. *The family of operators $\{T(t)\}_{t\ge0}$ defined by (4.1) is an strongly continuous semigroup on $\mathbb{M}_2$ such that*

$$T(t)W=\sum_{n=1}^{\infty} T_n(t)Q_nW, \quad W\in\mathbb{M}_2,\ t\ge 0, \tag{4.2}$$

where

$$Q_n=\begin{pmatrix} P_n & 0\\ 0 & \widetilde{P}_n\end{pmatrix},$$

with $(\widetilde{P}_n\phi)(s)=P_n\phi(s)$, $\phi\in L^2([-\tau,0];Z)$, $s\in[-\tau,0]$, and $\{\{T_n(t)\}_{t\ge0},\ n=1,2.3,\dots\}$ is a family of strongly continuous semigroups on $\mathbb{M}_2^n=Q_n\mathbb{M}_2$ given in the same way as in [5, Theorem 2.4.4] *and defined by*

$$T_n(t)\begin{pmatrix} w_n^0\\ w_n\end{pmatrix} = \begin{pmatrix} W^n(t)\\ W^n(t+\cdot)\end{pmatrix}, \quad \begin{pmatrix} w_n^0\\ w_n\end{pmatrix}\in\mathbb{M}_2^n,$$

where $W^n(\cdot)$ is the unique solution of the initial value problem

$$\begin{gathered}\frac{dw(t)}{dt} = -\lambda_n Dw(t) + L_n w_t, \quad t>0\\ w(0)=w_n^0\\ w(s)=w_n(s), \quad s\in[-\tau,0)\end{gathered} \tag{4.3}$$

and $L_n = L\widetilde{P}_n = P_n L$, as it is in most the case practical problems.

Proof of Theorem 4.1. First, we shall prove that

$$T(t)W = \sum_{n=1}^{\infty} T_n(t) Q_n W, \quad W \in \mathbb{M}_2,\ t \geq 0.$$

In fact, let $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{M}_2$.

$$\begin{aligned}
&\sum_{n=1}^{\infty} T_n(t) Q_n W \\
&= \sum_{n=1}^{\infty} T_n(t) \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} T_n(t) \begin{pmatrix} P_n w_1 \\ \widetilde{P}_n w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} z^n(t) \\ z^n(t+\cdot) \end{pmatrix} \quad z^n(\cdot) \text{ is the only mild solution of (4.3)} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} e^{\mathcal{A}_n t} P_n w_1 + \int_0^t e^{\mathcal{A}_n (t-s)} L_n (\widetilde{P}_n z^n(s+\cdot)) ds \\ (\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=1}^{\infty} e^{\mathcal{A}_n t} P_n w_1 + \int_0^t \sum_{n=1}^{\infty} e^{\mathcal{A}_n (t-s)} P_n \Big(L \sum_{m=1}^{\infty} (\widetilde{P}_m z(s+\cdot)) \Big) ds \\ \sum_{n=1}^{\infty} (\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} T_{\mathcal{A}}(t) w_1 + \int_0^t T_{\mathcal{A}}(t-s) L z(s+\cdot) ds \\ z(t+\cdot) \end{pmatrix} \\
&= \begin{pmatrix} z(t) \\ z_t(\cdot) \end{pmatrix}, \quad z(\cdot) \text{ is the only mild solution of (3.7)} \\
&= T(t)W.
\end{aligned}$$

In the same way as in [5, Theorem 2.4.4] we can prove that the infinitesimal generator of $\{T_n(t)\}_{t\geq 0}$ is given by

$$\Lambda_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\Lambda_n D w_n^0 + L_n w_n(\cdot) \\ \frac{\partial w_n(\cdot)}{\partial s} \end{pmatrix}$$

with

$$D(\Lambda_n) = \{ \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} \in \mathbb{M}_2^n : w_n \text{ is a.c.}, \frac{\partial w_n(\cdot)}{\partial s} \in L_2([-\tau, 0]; Q_n Z),\ w_n(0) = w_n^0 \}.$$

Furthermore, the spectrum of Λ_n is discrete and given by

$$\sigma(\Lambda_n) = \sigma_p(\Lambda_n) = \{\lambda \in \mathbb{C} : \det(A_n(\lambda)) = 0\}, \tag{4.4}$$

where $A_n(\lambda)$ is given by

$$\Lambda_n(\lambda) z = \lambda z + \lambda_n D z - L_n e^{\lambda(\cdot)} z, \quad z \in Z_n = P_n Z,$$

which can be considered a matrix since $\dim(Z_n) < \infty$.

On the other hand, $\{Q_n\}_{n\geq 1}$ is a family of complete orthogonal projection on $\mathbb{M}_2$ and

$$\Lambda_n Q_n = Q_n \Lambda_n, \quad n = 1, 2, 3, \dots$$

In fact,

$$\begin{aligned}
\Lambda_n Q_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} &= \Lambda_n \begin{pmatrix} Pnw_n^0 \\ \widetilde{P_n} w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\Lambda_n DP_n w_n^0 + L_n \widetilde{P_n} w_n(\cdot) \\ \frac{\partial \widetilde{P_n} w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= \begin{pmatrix} -\Lambda_n DP_n w_n^0 + L\widetilde{P_n}\widetilde{P_n} w_n(\cdot) \\ \widetilde{P_n} \frac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= \begin{pmatrix} -\Lambda_n DP_n w_n^0 + P_n L_n w_n(\cdot) \\ \widetilde{P_n} \frac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} -\Lambda_n Dw_n^0 + L_n w_n(\cdot) \\ \frac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= Q_n \Lambda_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix}
\end{aligned}$$

Now, we shall check condition (a) of Lemma 3.1. To this end we need to prove the following claim.

Claim. If $W^n(t)$ is the solution of (4.3), then the following inequalities hold

$$\|W^n(t)\|_Z \le c_2 e^{c_1 t} \|w_n^0\|, \quad t \ge 0, \tag{4.5}$$

$$\int_0^t \|W^n(u)\|_Z du \le k e^{c_2 t} \|w_n^0\|, \quad t \ge 0. \tag{4.6}$$

In fact, if we put $M_1 = \max\{M, \|L\|\}$, then we get

$$\|W^n(t+\theta)\|_Z \ \le \ M_1 \|w_n^0\| + M_1^2 \int_0^t \|W_s^n\|_{L^2} ds; \quad \theta \in [-\tau, 0],$$

this implies

$$\|W^n(t+\theta)\|_Z^2 \le \Big(M_1 \|w_n^0\| + M_1^2 \int_0^t \|W_s^n\|_{L^2} ds \Big)^2.$$

Next,

$$\begin{aligned}
\int_{-\tau}^0 \|W^n(t+\theta)\|_Z^2 d\theta &\le \int_{-\tau}^0 \Big(M_1 \|w_n^0\| + M_1^2 \int_0^t \|W_s^n\|_{L^2} ds \Big)^2 d\theta \\
&\le \int_{-\tau}^0 2^2 \Big(M_1^2 \|w_n^0\|^2 + M_1^4 \Big(\int_0^t \|W_s^n\|_{L^2} ds \Big)^2 \Big) d\theta \\
&= 2^2 \tau M_1^2 \|w_n^0\|^2 + M_1^4 \Big(\int_0^t \|W_s^n\|_{L^2} ds \Big)^2 \int_{-\tau}^0 d\theta \\
&= c_2^2 \|w_n^0\|^2 + c_1^2 \Big(\int_0^t \|W_s^n\|_{L^2} ds \Big)^2 \\
&\le \Big(c_2 \|w_n^0\| + c_1 \Big(\int_0^t \|W_s^n\|_{L^2} ds \Big) \Big)^2
\end{aligned}$$

So that

$$\|W_t^n\|_{L^2} \le c_2 \|w_n^0\| + c_1 \Big(\int_0^t \|W_s^n\|_{L^2} ds \Big)$$

Therefore, applying Gronwall's lemma we obtain

$$\|W_t^n\|_{L^2} \le c_2 e^{c_1 t} \|w_n^0\|, \quad t \ge 0.$$

On the other hand, we obtain the estimate

$$\begin{aligned}
\|W^n(t)\|_Z &\le \|T_{A_n}(t)w_n^0\| + \|\int_0^t T_{A_n}(t-s)L_nW^n(s+\cdot)ds\| \\
&\le M_1\|w_n^0\| + M_1^2\int_0^t \|W^n(s+\cdot)ds\| \\
&\le M_1\|w_n^0\| + M_1^2\int_0^t c_1e^{c_2t}\|w_n^0\|ds \\
&= \big(M_1 + \frac{M_1^2c_1}{c_2}e^{c_2t}\big)\|w_n^0\| \\
&\le ce^{c_2t}\|w_n^0\|,
\end{aligned}$$

where $c = M_1 + \frac{M_1^2c_1}{c_2}$, $t \ge 0$. Finally, we get

$$\int_0^t \|W^n(u)\|_Z du \le ke^{c_2t}\|w_n^0\|, \quad k = \frac{c}{c_2},\ t \ge 0.$$

This completes the proof of the claim.

Now, we use the above inequalities:

$$\begin{aligned}
\big\|T_n(t)\begin{pmatrix} w_n^0 \\ w_n\end{pmatrix}\big\|^2 &= \|W^n(t)\|_Z^2 + \int_{-\tau}^0 \|W^n(t+\tau)\|_Z^2 d\tau \\
&= \|W^n(t)\|_Z^2 + \int_{t-\tau}^t \|W^n(u)\|_Z^2 du \\
&\le \|W^n(t)\|_Z^2 + \int_0^t \|W^n(u)\|_Z^2 du + \|w_n\|_{L^2}^2 \\
&\le \big(c_2^2e^{2c_2t} + k^2e^{2c_2t}\big)\|w_n^0\|^2 + \|w_n\|_{L^2}^2 \\
&\le g(t)2\big(\|w_n^0\|^2 + \|w_n\|_{L^2}^2\big), \quad n \ge 1, 2, \dots .
\end{aligned}$$

Hence,

$$\|T_n(t)\| \le g(t), \quad n \ge 1, 2, \dots .$$

Therefore, applying Lemma 3.1, we obtain that $T(t)$ is bounded and $\{T(t)\}_{t\ge0}$ is a strongly continuous semigroup on the Hilbert space $\mathbb{M}_2$, whose generator Λ is given by

$$\Lambda W = \sum_{n=1}^{\infty} \Lambda_nQ_nW, \quad W \in D(\Lambda),$$

with

$$D(\Lambda) = \big\{W \in \mathbb{M}_2 / \sum_{n=1}^{\infty} \|\Lambda_nQ_nW\|^2 < \infty\big\}$$

and the spectrum $\sigma(\Lambda)$ of Λ is given by

$$\sigma(\Lambda) = \overline{\cup_{n=1}^{\infty}\sigma(\bar\Lambda_n)}, \tag{4.7}$$

where $\bar\Lambda_n = \Lambda_nQ_n : \mathcal{R}(Q_n) \to \mathcal{R}(Q_n)$. □

Lemma 4.2. *Let Λ be the infinitesimal generator of the semi-group $\{T(t)\}_{t\ge0}$. Then*

$$\Lambda\tilde\varphi(s) = \begin{pmatrix} -A\varphi(0) + L\phi(s) \\ \frac{\partial\phi(s)}{\partial s}\end{pmatrix}, \quad -\tau \le s \le 0,$$

$$\begin{aligned} D(\Lambda) = \Big\{ \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} \in \mathbb{M}_2 : \phi_0 \in D(A), \phi \textit{ is a.c.}, \frac{\partial \phi(s)}{\partial s} \in L^2([-\tau, 0]; Z) \\ \textit{and } \phi(0) = \phi_0 \Big\}, \end{aligned}$$

and

$$\sigma(\Lambda) = \overline{\cup_{n=1}^{\infty} \{\lambda \in \mathbb{C} : det(\Lambda_n(\lambda)) = 0\}}$$

Proof. Consider $\begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix}$ in $\mathbb{M}_2$. Then

$$\begin{aligned} \Lambda W = \Lambda \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} &= \sum_{n=1}^{\infty} \Lambda_n Q_n W \\ &= \sum_{n=1}^{\infty} \Lambda_n \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} = \sum_{n=1}^{\infty} \Lambda_n \begin{pmatrix} P_n \phi_0 \\ \widetilde{P_n} \phi(\cdot) \end{pmatrix} \\ &= \sum_{n=1}^{\infty} \begin{pmatrix} -\Lambda_n D \widetilde{P_n} \phi(0) + L_n \widetilde{P_n} \phi \\ \frac{\partial \widetilde{P}_n \phi(\cdot)}{\partial (s)} \end{pmatrix} \\ &= \begin{pmatrix} -\sum_{n=1}^{\infty} \Lambda_n D P_n \phi(0) + L \sum_{n=1}^{\infty} \widetilde{P}_n \phi \\ \frac{\partial}{\partial s} \Big(\sum_{n=1}^{\infty} \widetilde{P}_n \phi(\cdot) \Big) \end{pmatrix} \\ &= \begin{pmatrix} -A\phi(0) + L\phi(\cdot) \\ \frac{\partial \phi(\cdot)}{\partial s} \end{pmatrix}. \end{aligned}$$

The other part of the lemma follows from (4.7) □

Therefore, the systems (3.7) and (2.4) are equivalent to the following two systems of ordinary di-fferential equations in $\mathbb{M}_2$ respectively:

$$\begin{gathered} \frac{dW(t)}{dt} = \Lambda W(t), \quad t > 0 \\ W(0) = W_0 = (\phi_0, \phi(\cdot)) \end{gathered} \tag{4.8}$$

and

$$\begin{gathered} \frac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0 \\ W(0) = W_0 = (\phi_0, \phi(\cdot)), \end{gathered} \tag{4.9}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$ and $\Phi(t) = (f^e(t), 0)$.

The steps we have taken to arrive here allow us to conclude the proof of the main result of this work: The Variation of Constants Formula for Functional Partial Parabolic Equations. This result is presented as the final Theorem of the this work.

Theorem 4.3. *The abstract Cauchy problem in the Hilbert space* $\mathbb{M}_2$*,*

$$\begin{gathered} \frac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0 \\ W(0) = W_0 \end{gathered}$$

where Λ *is the infinitesimal generator of the semigroup* $\{T(t)\}_{t \geq 0}$ *and* $\Phi(t) = (f^e(t), 0)$ *is a function taking values in* $\mathbb{M}_2$*, admits one and only one mild solution given by*

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\Phi(s)ds \tag{4.10}$$

Corollary 4.4. *If $z(t)$ is a solution of* (2.4)*, then the function $W(t) := (z(t), z_t)$ is solution of the equation* (4.9)

5. Conclusion

As one can see, this work can be generalized to a broad class of functional reaction diffusion equation in a Hilbert space Z of the form

$$\begin{gathered} \frac{dz(t)}{dt} = \mathcal{A}z(t) + Lz_t + F(t), \quad t > 0 \\ z(0) = \phi_0 \\ z(s) = \phi(s), \quad s \in [-\tau, 0), \end{gathered} \tag{5.1}$$

where

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \quad z \in D(\mathcal{A}), \tag{5.2}$$

where $L : L^2([-\tau, 0]; Z) \to Z$ is linear and bounded $F : [-\tau, \infty) \to Z$ is a suitable function. Some examples of this class are the following well known systems of partial differential equations with delay:

The equation modelling a damped flexible beam:

$$\begin{gathered} \frac{\partial^2 z}{\partial t^2} = -\frac{\partial 3z}{\partial 3x} + 2\alpha \frac{\partial 3z}{\partial t \partial 2x} + z(t-\tau, x) + f(t, x) \quad t \geq 0,\ 0 \leq x \leq 1 \\ z(t, 1) = z(t, 0) = \frac{\partial 2z}{\partial 2x}(0, t) = \frac{\partial 2z}{\partial 2x}(1, t) = 0, \\ z(0, x) = \phi_0(x), \quad \frac{\partial z}{\partial t}(0, x) = \psi_0(x), \quad 0 \leq x \leq 1 \\ z(s, x) = \phi(s, x), \quad \frac{\partial z}{\partial t}(s, x) = \psi(s, x), \quad s \in [-\tau, 0),\ 0 \leq x \leq 1 \end{gathered} \tag{5.3}$$

where $\alpha > 0$, $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is a smooth function, $\phi_0, \psi_0 \in L^2[0, 1]$ and $\phi, \psi \in L^2([-\tau, 0]; L^2[0, 1])$.

The strongly damped wave equation with Dirichlet boundary conditions

$$\begin{gathered} \frac{\partial^2 w}{\partial t^2} + \eta(-\Delta)^{1/2} \frac{\partial w}{\partial t} + \gamma(-\Delta)w = Lw_t + f(t, x), \quad t \geq 0,\ x \in \Omega, \\ w(t, x) = 0, \quad t \geq 0,\ x \in \partial\Omega. \\ w(0, x) = \phi_0(x), \quad \frac{\partial z}{\partial t}(0, x) = \psi_0(x), \quad x \in \Omega, \\ w(s, x) = \phi(s, x), \quad \frac{\partial z}{\partial t}(s, x) = \psi(s, x), \quad s \in [-\tau, 0),\ x \in \Omega, \end{gathered} \tag{5.4}$$

where Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is a smooth function, $\phi_0, \psi_0 \in L^2(\Omega)$ and $\phi, \psi \in L^2([-\tau, 0]; L^2(\Omega))$ and $\tau \geq 0$ is the maximum delay, which is supposed to be finite. We assume that the operators $L : L^2([-\tau, 0]; Z) \to Z$ is linear and bounded and $Z = L^2(\Omega)$.

The thermoelastic plate equation with Dirichlet boundary conditions

$$
\begin{gathered}
\frac{\partial 2w}{\partial 2t} + \Delta^2 w + \alpha\Delta\theta = L_1 w_t + f_1(t,x) \quad t \geq 0,\ x \in \Omega, \\
\frac{\partial \theta}{\partial t} - \beta\Delta\theta - \alpha\Delta\frac{\partial w}{\partial t} = L_2\theta_t + f_2(t,x) \quad t \geq 0,\ x \in \Omega, \\
\theta = w = \Delta w = 0, \quad t \geq 0,\ x \in \partial\Omega, \\
w(0,x) = \phi_0(x), \quad \frac{\partial w}{\partial t}(0,x) = \psi_0(x), \quad \theta(0,x) = \xi_0(x) \quad x \in \Omega, \\
w(s,x) = \phi(s,x), \quad \frac{\partial w}{\partial t}(s,x) = \psi(s,x), \quad \theta(0,x) = \xi(s,x), \quad s \in [-\tau, 0),\ x \in \Omega,
\end{gathered}
\tag{5.5}
$$

where Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $f_1, f_2 : \mathbb{R} \times \Omega \to \mathbb{R}$ are smooth functions, $\phi_0, \psi_0, \xi_0 \in L^2(\Omega)$ and $\phi, \psi, \xi \in L^2([-\tau, 0]; L^2(\Omega))$ and $\tau \geq 0$ is the maximum delay, which is supposed to be finite. We assume that the operators $L_1, L_2 : L^2([-\tau, 0]; Z) \to Z$ are linear and bounded and $Z = L^2(\Omega)$.

References

[1] V. M. Alekseev, "An estimate for perturbations of the solutions of ordinary diffe rential equations" Westnik Moskov Unn. Ser. 1. Math. Merk. 2 (1961).28-36.

[2] H. T. Banks, "The representation of solutions of linear functional di-fferential equations" J. Diff. Eqns., 5 (1969).

[3] J. G. Borisovich and A. S. Turbabin, "On the Cauchy problem for linear non-homegeneous di-fferential equations with retarded arguments" Soviet Math. Dokl., 10 (1969), pp. 401-405

[4] R. F. Curtain and A. J. Pritchard, "Infinite Dimensional Linear Systems", Lecture Notes in Control and Information Sciences, Vol. 8. Springer Verlag, Berlin (1978).

[5] R. F. Curtain and H. J. Zwart, "An Introduction to Infinite Dimensional Linear Systems Theory", Tex in Applied Mathematics, Vol. 21. Springer Verlag, New York (1995).

[6] A. E. Ize and A. Ventura, "Asymptotic behavior of a perturbed neutral funcional differential equation related to the solution of the unperturbed linear system", Pacific J. Math., Vol. 1. pp. 57-91 (1984)

[7] D. Henry, "Geometric theory of semilinear parabolic equations" , Springer, New York (1981).

[8] H. Leiva, "A Lemma on C_0-Semigroups and Appications PDEs Systems", Quaestions Mathematicae 26(2003), 247-265.

[9] J. Lopez G. and R. Pardo San Gil, "Coexistence in a Simple Food Chain with Diffusion" J. Math. Biol. (1992) 30: 655-668.

[10] R. H. Marlin and H. L. Struble, "Asymptotic Equivalence of Nonlinear Systems", Journal of Differential Equations, Vol. 6, 578-596(1969).

[11] R. H. Martin and H. L. Smith, "Reaction-diffusion systems with time delays: monotonicity, invariance, comparison and convergence", J. reine angew. Math. 413 (1991),1-35.

[12] Luiz A. F. de Oliveira, "On Reaction-Diffusion Systems" E. Journal of Differential Equations, Vol. 1998(1998), N0. 24, pp. 1-10.

[13] G. A. Shanholt, "A nonlinear variations of constants formula for functional differential equations", Math Systems Theory 6, No. 5 (1973).

[14] C. C. Travis and G. F. Webb (1974), "Existence and Stability for Partial Functional Differential Equations", Transation of the A.M.S, Vol 200.

[15] J. Wu (1996), "Theory and Applications of Partial Functional Differential Equatios", Springer Verlag, Applied Math. Science Vol 119.

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