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# A DISTRIBUTIONAL SOLUTION TO A HYPERBOLIC PROBLEM ARISING IN POPULATION DYNAMICS

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Abstract. We consider a generalization of the Lotka-McKendrick problem describing the dynamics of an age-structured population with time-dependent vital rates. The generalization consists in allowing the initial and the boundary conditions to be derivatives of the Dirac measure. We construct a unique $\mathcal{D}'$-solution in the framework of intrinsic multiplication of distributions. We also investigate the regularity of this solution.

## 1. Introduction

We consider a non-classical hyperbolic problem with integral boundary condition

$$(\partial_t + \partial_x)u = p(x,t)u + g(x,t), \quad (x,t) \in \Pi \tag{1.1}$$

$$u|_{t=0} = a(x), \quad x \in [0,L) \tag{1.2}$$

$$u|_{x=0} = c(t)\int_0^L b(x)u\,dx, \quad t \in [0,\infty)\,, \tag{1.3}$$

where $\Pi = \{(x,t) \in \mathbb{R}^2 : 0 < x < L, t > 0\}$. From the point of view of applications, (1.1)–(1.3) describes the dynamics of an age-structured population (see i.e. [1, 3, 15, 23, 28]). There $u$ denotes the distribution of individuals having age $x > 0$ at time $t > 0$, $a(x)$ is the initial distribution, $-p(x,t)$ denotes the mortality rate, $b(x)$ denotes the age-dependent fertility rate, $c(t)$ is the specific fertility rate of females, $g(x,t)$ is the distribution of migrants, $L$ is the maximum age attained by individuals. Furthermore, $b(x) = 0$ on $[0,L] \setminus [L_1, L_2]$, where $[L_1, L_2] \subset [0,L]$ is the fertility period of females. The evolution of $u$ without diffusion is governed by (1.1)–(1.3). The system (1.1)–(1.3) is a continuous model of a discrete structure. As in many problems of such a kind, it is natural to consider singular initial and boundary data. We focus on the case when these data have singular support in

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finitely many points, i.e.

$$\begin{aligned} a(x) &= a_r(x) + \sum_{i=1}^{m} d_{1i}\delta^{(m_i)}(x - x_i) \quad \text{for some } d_{1i} \in \mathbb{R},\ m_i \in \mathbb{N}_0, x_i \in (0, L),\\ b(x) &= b_r(x) + \sum_{k=1}^{s} d_{2i}\delta^{(n_k)}(x - x_k) \quad \text{for some } d_{2i} \in \mathbb{R},\ n_k \in \mathbb{N}_0, x_k \in (0, L),\\ c(t) &= c_r(t) + \sum_{j=1}^{q} d_{3i}\delta^{(l_j)}(t - t_j) \quad \text{for some } d_{3i} \in \mathbb{R},\ l_j \in \mathbb{N}_0, t_j \in (0, \infty). \end{aligned} \tag{1.4}$$

The data of the Dirac measure type enable us to model the point-concentration of various demographic parameters.

The problem under consideration is of interest from both biological and mathematical points of view.

**Biological motivation.** A basic model describing the evolution of an age-structured population is given by the Lotka-McKendrick system

$$\begin{aligned} (\partial_t + \partial_x)u &= -p(x)u\\ u|_{t=0} &= u_0(x)\\ u|_{x=0} &= \int_0^L b(x)u\,dx. \end{aligned} \tag{1.5}$$

This differential equation describes the aging of the population. While the integral $\int_{\alpha_1}^{\alpha_2} u(x, t)\,dx$ gives the number of individuals, at time $t$, having age $x$ in the range $\alpha_1 \le x \le \alpha_2$. Thus, the third equation is responsible for newborns, entering the population at age zero.

A biological generalization of (1.5) to (1.1)–(1.3) consists in allowing the fertility and mortality rates to depend on $t$ (see e.g. [9, 10, 14]). In reality the vital rates are never time-homogeneous and adapt to the changing social and technological environment. Introducing $\delta$-distributional data in (1.2) and (1.3) also has a biological meaning (see [15]).

In demography, $c(t)$ is the total fertility rate of the population at time $t$, in other words, the average number of childbirths per female during her reproductive period. On one side, the results presented in the paper could shed a new light on the so-called $c$-control problems when one wants to control the population only through changing $c(t)$. Chinese scientists used discrete models to provide mathematical background for the unicity child policy ($c$-control problem) in the People's Republic of China [25, 26, 29]. Continuous models in the context of the $c$–control problem were considered in [8]. In contrast to the aforementioned papers, the presence of strongly singular data in (1.2) and (1.3) allows one to combine the continuity of the model with the discreteness of the real evolutionary process. Occurrence of strong singularities in $c(x)$ can be motivated by synchronized and concentrated reproduction of the species. This also allows one to introduce statistical data in (1.1)–(1.3) and perhaps makes our model competitive with discrete-time and discrete-age models [2].

Introducing strong singularities in the model could have another interpretation: such singularities can be produced by a linearization of nonlinear problems with discontinuous data. Thus this opens a space for interesting nonlinear consequences.

**Mathematical motivation.** We consider our paper as a further step in the study of generalized solutions to initial-boundary hyperbolic problems in two variables.

Since the singularities given on $\partial\Pi$ expand inside $\Pi$ along characteristic curves of the equation (1.1), a solution preserves at least the same order of regularity as it has on $\partial\Pi$. This causes multiplication of distributions under the integral sign in (1.3). In spite of this complication, we find distributional solutions of (1.1)–(1.3). In parallel, we study propagation, interaction and creation of new singularities for the problem (1.1)–(1.3).

Semilinear hyperbolic initial-boundary value problems with distributional data were studied, among others, in [18, 11, 12]. There also appears a complication with multiplication of distributions that is caused by nonlinear right-hand sides of the differential equations and also by boundary conditions that are nonlinear (with bounded nonlinearity) in [18], nonseparable in [12], and integral in [11]. To overcome this complication, the authors use the framework of *delta waves* (see [20]). In other words, they find solutions by regularizing all singular data, solving the regularized system and then passing to a weak limit in the obtained sequential solution.

Boundary and initial-boundary value problems for a linear second order hyperbolic equation [22] and general strictly hyperbolic systems in the Leray-Volevich sense [21] are studied in a complete scale of *Sobolev type spaces* depending on parameters $s$ and $\tau$, where $s$ characterizes the smoothness of a solution in all variables and $\tau$ characterizes additional smoothness in the tangential variables. Sobolev-type a priori estimates are obtained and, based on them, existence and uniqueness results in Sobolev spaces are proved.

In contrast to the aforementioned papers we here treat *integral* boundary conditions and show that the problem (1.1)–(1.3) is solvable in the *distributional* sense. We construct a unique distributional solution by means of multiplication of distributions in the sense of Hörmander [7].

We show that the boundary condition (1.3) causes anomalous singularities at the time when singular characteristics and vertical singular lines arising from the data of (1.3) intersect. In the case that the singular part of $b(x)$ is a sum of derivatives of the Dirac measure, the solution becomes more singular. In the case that the initial and the boundary data are Dirac measures, the solution preserves the same order of regularity. A similar phenomenon was shown in [27] for a semilinear hyperbolic Cauchy problem with strongly singular initial data, where interaction of singularities was caused by nonlinearity of the equations. Anomalous singularities were considered also in [19] and [17], where propagation of singularities for, respectively, initial and initial-boundary semilinear hyperbolic problems were studied. There it was proved that, if the initial data have, at worst, jump discontinuities, then the singularities at the common point of singular characteristics of the differential equations are weaker. Furthermore, if the boundary data are regular enough, then reflected singularities cannot be stronger than the corresponding incoming singularities. It turns out [4, 13] that in some cases of nonseparable boundary conditions the solution becomes more regular in time, namely, for $C^1$-initial data it becomes $k$-times continuously differentiable for any desired $k \in \mathbb{N}_0$ in a finite time.

**Organization of the paper.** Section 2 contains some basic facts from the theory of distributions. In Section 3 we describe our problem in detail and state our results. Sections 4–9 present successive steps of construction of a distributional solution to the problem. In particular, the integral boundary condition is treated in Section 5. In parallel we analyze the regularity of the solution. The uniqueness is proved in Section 10.

## 2. Background

For convenience of the reader we here recall the relevant material from [5, 6, 7, 24] without proofs. Throughout the paper we will denote by $\langle\cdot,\cdot\rangle : \mathcal{D}' \times \mathcal{D} \to \mathbb{R}$ the dual pairing on the space $\mathcal{D}$ of $C^\infty$-functions having compact support.

**Definition 2.1** ([6, 2.5 ]). A distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ is *microlocally smooth* at $(x,t,\xi,\eta)$ ($(\xi,\eta) \neq 0$) if the following condition holds: If $u$ is localized about $(x,t)$ by $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\varphi \equiv 1$ in a neighborhood of $(x,t)$, then the Fourier transform of $\varphi u$ is rapidly decreasing in an open cone about $(\xi,\eta)$. The *wave front set* of $u$, $\mathrm{WF}(u)$, is the complement in $\mathbb{R}^4$ of the set of microlocally smooth points.

**Proposition 2.2** ([7, 8.1.5]). *Let $u \in \mathcal{D}'(\mathbb{R}^2)$ and $P(x,D)$ be a linear differential operator with smooth coefficients. Then*
$$\mathrm{WF}(Pu) \subset \mathrm{WF}(u).$$

**Definition 2.3** ([7, 6.1.2]). Let $X, Y \subset \mathbb{R}^2$ be open sets and $u \in \mathcal{D}'(Y)$. Let $f : X \to Y$ be a smooth invertible map such that its derivative is surjective. Then the *pullback of $u$ by $f$*, $f^*u$, is a unique continuous linear map: $\mathcal{D}'(Y) \to \mathcal{D}'(X)$ such that for all $\varphi \in \mathcal{D}(Y)$
$$\langle f^*u, \varphi\rangle = \langle u, |J(f^{-1})|(\varphi \circ f^{-1})\rangle,$$
where $J(f^{-1})$ is the Jacobian matrix of $f^{-1}$.

**Theorem 2.4** ([7, 8.2.7]). *Let $X$ be a manifold and $Y$ a submanifold with normal bundle denoted by $N(Y)$. For every distribution $u$ in $X$ with $\mathrm{WF}(u)$ disjoint from $N(Y)$, the restriction $u|_Y$ of $u$ to $Y$ is a well-defined distribution on $Y$ that is the pullback by the inclusion $Y \hookrightarrow X$.*

**Theorem 2.5** ([7, 5.1.1]). *For any distributions $u \in \mathcal{D}'(X_1)$ and $v \in \mathcal{D}'(X_2)$ there exists a unique distribution $w \in \mathcal{D}'(X_1 \times X_2)$ such that*
$$\langle w, \varphi_1 \otimes \varphi_2\rangle = \langle u, \varphi_1\rangle\langle v, \varphi_2\rangle, \quad \varphi_i \in \mathcal{D}(X_i),$$
$$\langle w, \varphi\rangle = \langle u, \langle v, \varphi(x_1,x_2)\rangle\rangle = \langle v, \langle u, \varphi(x_1,x_2)\rangle\rangle, \quad \varphi \in \mathcal{D}(X_1 \times X_2).$$
*Here $u$ acts on $\varphi(x_1,x_2)$ as on a function of $x_1$ and $v$ acts on $\varphi(x_1,x_2)$ as on a function of $x_2$.*

The distribution $w$ as in the above theorem is called the *tensor product* of $u$ and $v$, and denoted by $w = u \otimes v$.

**Theorem 2.6** ([5, 11.2.2]). *Let $X, Y$ be open sets in $\mathbb{R}^2$ and let $f : X \to Y$ be a diffeomorphism. If $u \in \mathcal{D}'(Y)$, then $f^*u$, the pull-back of $u$, is well defined, and we have*
$$\mathrm{WF}(f^*(u)) = \{(x, df_x^t\eta) : (f(x),\eta) \in \mathrm{WF}(u)\}.$$

**Theorem 2.7** ([7, 8.2.10]). *If $v, w \in \mathcal{D}'(X)$, then the product $v \cdot w$ is well defined as the pullback of the tensor product $v \otimes w$ by the diagonal map $\delta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ unless $(x,t,\xi,\eta) \in \mathrm{WF}(v)$ and $(x,t,-\xi,-\eta) \in \mathrm{WF}(w)$ for some $(x,t,\xi,\eta)$.*

**Theorem 2.8** ([24, 8.6]). *If a distribution $u$ is identically equal to 0 on each of the domains $G_i$, $i \geq 1$, then $u$ is identically equal to 0 on $G = \bigcup_{i\geq 1} G_i$.*

## 3. Statement of the results

For simplicity of technicalities we assume that both the initial and the boundary data have singular supports at a single point and are Dirac measures or derivatives of the Dirac measure. This causes no loss of generality for the problem if the singular parts of the initial and the boundary data are finite sums of the Dirac measures and derivatives thereof, i.e. they are of the form (1.4). Specifically, we consider the following system

$$(\partial_t + \partial_x)u = p(x,t)u + g(x,t), \quad (x,t) \in \Pi \tag{3.1}$$

$$u|_{t=0} = a_r(x) + \delta^{(m)}(x - x_1^*), \quad x \in [0, L) \tag{3.2}$$

$$u|_{x=0} = (c_r(t) + \delta^{(j)}(t - t_1)) \int_0^L (b_r(x) + \delta^{(n)}(x - x_1))u\,dx,\ t \in [0, \infty), \tag{3.3}$$

where $x_1 > 0, x_1^* > 0, t_1 > 0$, and $m, j, n \in \mathbb{N}_0$. Without loss of generality we can assume that $x_1^* < x_1$. We introduce the following assumptions:

- (A1) $a_r^{(i)}(0) = 0, c_r^{(i)}(0) = 0$ for all $i \in \mathbb{N}_0$.
- (A2) $b_r^{(i)}(L) = 0$ for all $i \in \mathbb{N}_0$ and there exists $\varepsilon > 0$ such that $b_r(x) = 0$ for $x \in [0, \varepsilon]$.
- (A3) The functions $p$ and $g$ are smooth in $\mathbb{R}^2$, $a_r$ is smooth on $[0, L)$, $b_r$ is smooth on $[0, L]$, and $c_r$ is smooth on $[0, \infty)$.

Note that (A1) ensures an arbitrary order compatibility between (3.2) and (3.3). (A2) is not particularly restrictive from the practical point of view, since $[0, L]$ covers the fertility period of females.

All characteristics of the differential equation (1.1) are solutions to the following initial value problem for ordinary differential equation

$$\frac{dx}{dt} = 1, \quad x(t_0) = x_0, \quad \text{where } (x_0, t_0) \in \mathbb{R}^2,$$

and therefore are given by the formula $x = t + x_0 - t_0$.

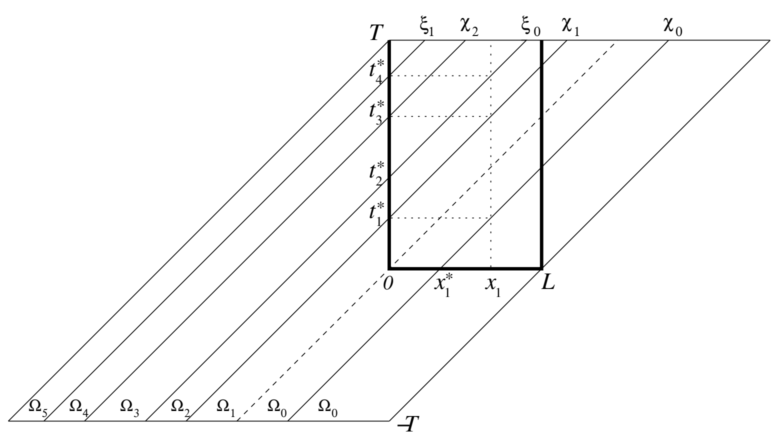


Figure 1. Singular characteristics $\chi_n$ and $\xi_n$ in the case $t_1 = t_2^*$.

**Definition 3.1.** Define $\chi_n$ and $\xi_n$, subsets of $\mathbb{R}^2$, inductively:

- $\chi_0$ is the characteristic passing through the point $(x_1^*, 0)$ and $\xi_0$ is the characteristic passing through the point $(0, t_1)$.
- Let $n \geq 1$. Then $\chi_n$ is the characteristic passing through the point $(0, t)$ such that $(x_1, t) \in \chi_{n-1}$. Furthermore, $\xi_n$ is the characteristic passing through the point $(0, t)$ such that $(x_1, t) \in \xi_{n-1}$.

Also, we set $I = \bigcup_{n \geq 0} (\chi_n \cup \xi_n)$.

For characteristics contributing into $I$ denote their intersection points with the positive semiaxis $x = 0$ by $t_1^*, t_2^*, \dots$. We assume that $t_j^* < t_{j+1}^*$ for $j \geq 1$ (see Figure 1, where we chose $t_1 = t_2^*$). The union of all singular characteristics of the initial problem, as it will be shown, is included in the set $I$. In fact, we will show that $\operatorname{sign\,supp} u \subset I$.

(A4) $(x_1, t_1) \notin \chi_n$ for all $n \geq 0$.

This assumption excludes the situation when three different singularities intersect at the same point. Without this assumption the distributional solution does not exist, because there appears a multiplication of two Dirac measures at the same point.

Our goal is, using distributional multiplication, to obtain a distributional solution to (3.1)–(3.3). We use the notion of the so-called "WF favorable" product which is due to L. Hörmander [7] and is in the second level of M. Oberguggenberger's hierarchy of intrinsic distributional products [16, p. 69].

We will actually obtain a distributional solution in the domain

$$\Omega = \{(x, t) \in \mathbb{R}^2 : x < t + L\}.$$

This is the domain of influence of the data on the part of the boundary of $\Pi$ where the conditions (3.2) and (3.3) are given.

**Definition 3.2.** A distribution $u$ is called a $\mathcal{D}'(\Omega)$-solution to the problem (3.1)–(3.3) if the following conditions are met.

(1) The equation (3.1) is satisfied in $\mathcal{D}'(\Omega)$: for every $\varphi \in \mathcal{D}(\Omega)$

$$\langle (\partial_t + \partial_x - p(x, t))u, \varphi \rangle = \langle g(x, t), \varphi \rangle.$$

(2) $u$ is restrictable to $[0, L) \times \{0\}$ in the sense of Hörmander (see Theorem 2.4) and $u|_{t=0} = a_r(x) + \delta^{(m)}(x - x_1^*), \quad x \in [0, L)$.

(3) The product of $(b_r(x) + \delta^{(n)}(x - x_1)) \otimes 1(t)$ and $u(x, t)$ exists in $\mathcal{D}'(\Pi)$ in the sense of Hörmander (see Theorem 2.7).

(4) $\int_0^L \left[ \left( b_r(x) + \delta^{(n)}(x - x_1) \right) \otimes 1(t) \right] u \, dx$ is a distribution $v \in \mathcal{D}'(\mathbb{R}_+)$ defined by

$$\langle v, \psi(t) \rangle = \langle [(b_r(x) + \delta^{(n)}(x - x_1)) \otimes 1(t)]u, 1(x) \otimes \psi(t) \rangle, \quad \psi(t) \in \mathcal{D}(\mathbb{R}_+),$$

where $b_r(x) = 0$, $x \notin [0, L]$.

(5) $v$ is a smooth function in $t_1$.

(6) $u$ is restrictable to $\{0\} \times [0, \infty)$ in the sense of Hörmander (see Theorem 2.4) and $u|_{x=0} = (c_r(t) + \delta^{(j)}(t - t_1))v, \quad t \in [0, \infty)$.

(7) $\operatorname{sign\,supp} u \subset \Omega \setminus \{(x, t) : x = t\}$.

Our next objective is to define the solution concept for (3.1)–(3.3) on $\Pi$. It is not so obvious how we should define the restriction of $u \in \mathcal{D}'(\Pi)$ to the boundary of $\Pi$ so that the initial and the boundary conditions are meaningful. In this respect let us make the following observation.

Note that $\overline{\Pi}\setminus\{(L,0)\}\subset\Omega$. Let $\Omega_0\subset\Omega$ be a domain such that $\overline{\Pi}\setminus\{(L,0)\}\subset\Omega_0$ and $u$ be a $\mathcal{D}'(\Omega)$-solution to the problem (3.1)–(3.3) in the sense of Definition 3.2. Then $u$ restricted to $\Omega_0$ is a $\mathcal{D}'(\Omega_0)$-solution to the problem (3.1)–(3.3) in the sense of the same definition. This suggests the following definition.

**Definition 3.3.** Let $u$ be a $\mathcal{D}'(\Omega)$-solution to the problem (3.1)–(3.3) in the sense of Definition 3.2. Then $u$ restricted to $\Pi$ is called a $\mathcal{D}'(\Pi)$-solution to the problem (3.1)–(3.3).

Set

$$\Omega_+=\{(x,t)\in\Omega : x>0, t>0\}.$$

We are now prepared to state the existence result.

**Theorem 3.4.** (1) *Let* (A1)–(A4) *hold. Then there exists a $\mathcal{D}'(\Omega)$-solution $u$ to the problem* (3.1)–(3.3) *in the sense of Definition 3.2 such that*

*the restriction of $u$ to any domain $\Omega'_+\supset\Omega_+$ such that any characteristic of* (3.1) *intersects $\partial\Omega'_+$ at a single point does not depend on the values of the functions $p$ and $g$ on $\Omega\setminus\Omega'_+$.* (3.4)

(2) *Let* (A1)–(A4) *hold. Then there exists a $\mathcal{D}'(\Pi)$-solution to the problem* (3.1)–(3.3) *in the sense of Definition 3.3.*

Given a domain $G$, set

$$\mathcal{D}'_+(G)=\{u\in\mathcal{D}'(G) : u=0 \text{ whenever } x<0 \text{ or } t<0\}.$$

**Definition 3.5.** $u\in\mathcal{D}'_+(\Omega)$ is called a $\mathcal{D}'_+(\Omega)$-solution to the problem (3.1)–(3.3) if the following conditions are met.

(1) Items 3–5 of Definition 3.2 hold.
(2) Equation (3.1) is satisfied in $\mathcal{D}'_+(\Omega)$: for every $\varphi\in\mathcal{D}(\Omega)$

$$\begin{aligned}&\langle(\partial_t+\partial_x-p(x,t))u,\varphi\rangle\\&=\langle g(x,t),\varphi\rangle+\big\langle(a_r(x)+\delta^{(m)}(x-x_1^*))\otimes\delta(t)\\&\quad+\delta(x)\otimes[(c_r(t)+\delta^{(j)}(t-t_1))v],\varphi\big\rangle,\end{aligned}$$

where $a_r(x)=0$ if $x<0$ and $v(t)=0$ if $t<0$.
(3) $\operatorname{sign}\operatorname{supp}u\setminus\partial\Omega_+\subset\Omega_+\setminus\{(x,t):x=t\}$.

**Proposition 3.6.** *Let $u$ be a $\mathcal{D}'(\Omega)$-solution to the problem* (3.1)–(3.3) *in the sense of Definition 3.2 that satisfies (3.4). Then there exists a $\mathcal{D}'_+(\Omega)$-solution $\tilde u$ to the problem* (3.1)–(3.3) *in the sense of Definition 3.5 such that*

$$u=\tilde u \quad in\ \mathcal{D}'(\Omega_+).$$

This proposition is a straightforward consequence of Definitions 3.2 and 3.5. Since $\Pi\subset\Omega_+$, it makes sense to state the uniqueness result in $\mathcal{D}'_+(\Omega)$. Write

$$S(x,t)=\exp\Big\{\int_{\theta(x,t)}^{t}p(\tau+x-t,\tau)\,d\tau\Big\},\tag{3.5}$$

where $\theta(x,t)=(t-x)H(t-x)$ with $H(z)$ denoting the Heaviside function. We write $\hat S$ for the function $S$ given by (3.5), where $p$ is replaced by $-p$.

**Theorem 3.7.** (1) *Let* (A1)–(A4) *hold. Then a $\mathcal{D}'_+(\Omega)$-solution to the problem* (3.1)–(3.3) *is unique.*

(2) *Let* (A1)–(A4) *hold. Then a $\mathcal{D}'(\Pi)$-solution to the problem* (3.1)–(3.3) *is unique.*

From the construction of a $\mathcal{D}'(\Omega)$-solution presented in the proof of Theorem 3.4 we will see that in general there appear new singularities stronger than the initial singularities. In other words, the singular order (cf. [24, §13]) of the distributional solution grows in time. We state this result in the following theorem.

**Theorem 3.8.** (1) *Let $u$ be the $\mathcal{D}'(\Pi)$-solution to the problem* (3.1)–(3.3), *where $n \geq 1$. Then for each $i \geq 1$ there exist $k > i$ and $n' \geq 1$ such that the singular order of $u$ is equal to $n'$ in a neighborhood of $x = t - t_i^*$ and the singular order of $u$ is equal to $n' + n$ in a neighborhood of $x = t - t_k^*$.*

(2) *If $n = j = m = 0$, then the singular order of $u$ on $\Pi$ is equal to 1.*

We now start with the proof of Theorem 3.4 which will take Sections 4–9. By our construction of the set $I$, we have $t_1 \in \{t_1^*, t_2^*, \dots\}$. Let, say, $t_1 = t_2^*$ (for any other $t_1 = t_i^*$ the proof is virtually the same, see Footnotes 1 and 2). It is sufficient to solve the problem in the domain

$$\Omega^T = \{(x,t) \in \Omega : t - T < x, -T < t < T\}$$

for an arbitrary fixed $T > 0$. Observe that $\Omega^T$ is the intersection of the strip $\mathbb{R} \times (-T, T)$ and the domain of determinacy of (3.1) with respect to the set $([0, L) \times \{0\}) \cup (\{0\} \times [0, T))$. Fix $T > 0$ and start with a subdomain

$$\Omega_0 = \{(x,t) \in \Omega^T : t < x < t + L\}$$

(see Figure 1). To abuse notation, we do not indicate the dependence of $\Omega_0$ on $T$.

## 4. The solution on $\Omega_0$

Observe that $\Omega_0$ is the intersection of the strip $\mathbb{R} \times (-T, T)$ with the domain of determinacy of the problem (3.1)–(3.2). In the case that the initial data are functions, a unique solution to the problem (3.1)–(3.2) on $\Omega_0$ can be written in the form

$$u(x,t) = S_1(x,t) + S(x,t)a_r(x-t) + S(x,t)\delta^{(m)}(x - t - x_1^*) \tag{4.1}$$

with the functions $S(x,t)$ given by (3.5) and

$$\begin{aligned} S_1(x,t) = \exp\Big\{ \int_{\theta(x,t)}^t p(\tau + x - t, \tau)\, d\tau \Big\} \\ \times \int_{\theta(x,t)}^t \exp\Big\{ - \int_{\theta(x,t)}^\tau p(\tau_1 + x - t, \tau_1)\, d\tau_1 \Big\} g(\tau + x - t, \tau)\, d\tau. \end{aligned} \tag{4.2}$$

Let $A_i(x,t) = \delta^{(i)}(x) \otimes 1(t)$ and $B_i(x,t) = 1(x) \otimes \delta^{(i)}(t)$ be the distributions in $\mathbb{R}^2$ that are derivatives of the Dirac measure $\delta^{(i)}(x)$ and $\delta^{(i)}(t)$ supported along the $t$-axis and the $x$-axis, respectively. They are defined by the equalities

$$\langle A_i(x,t), \varphi(x,t) \rangle = (-1)^i \int \varphi_x^{(i)}(0,t)\, dt,$$

$$\langle B_i(x,t), \varphi(x,t) \rangle = (-1)^i \int \varphi_t^{(i)}(x,0)\, dx$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$. When $i = 0$, then we have the Dirac measure supported along the respective axes.

Let $f$ be the smooth map

$$f : (x,t) \to (x, x - t - x_1^*).$$

Then its inverse

$$f^{-1} : (x,t) \to (x, x - t - x_1^*)$$

is unique and maps the $x$-axis to the line $t = x - x_1^*$ and the $t$-axis onto itself. Moreover,

$$f'(x,t) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

For the Jacobian of $f$ we hence have $J(f) = |f'| = -1 \neq 0$ and $f^* B_m = \delta^{(m)}(x - t - x_1^*)$, the pullback of $B_m$ by $f$ (see Definition 2.3), is well defined. Therefore the distribution $\delta^{(m)}(x - t - x_1^*)$ acts on test functions $\varphi \in \mathcal{D}(\mathbb{R}^2)$ in the following way:

$$\begin{aligned} \langle \delta^{(m)}(x - t - x_1^*), \varphi(x,t) \rangle &= \langle f^* B_m, \varphi(x,t) \rangle \\ &= -\langle B_m, \varphi(x,t) \circ f^{-1}(x,t) \rangle \\ &= (-1)^{m+1} \int \partial_t^m \varphi(x, x - t - x_1^*)\big|_{t=0}\, dx \\ &= -\int \partial_t^m \varphi(x,t)|_{t = x - x_1^*}\, dx. \end{aligned}$$

Hence, similarly to $B_m$, $f^* B_m$ is the $m$-th derivative of the Dirac measure supported along the line $t = x - x_1^*$.

**Definition 4.1.** A distribution $u$ is called a $\mathcal{D}'(\Omega_0)$-solution to the problem (3.1), (3.2) if Items 1 and 2 of Definition 3.2 with $\Omega$ replaced by $\Omega_0$ hold.

**Lemma 4.2.** *The function $u(x,t)$ given by the formula (4.1) is a $\mathcal{D}'(\Omega_0)$-solution to the problem* (3.1)*–(3.2).*

*Proof.* A straightforward verification shows that the sum of the first two summands in (4.1) is a smooth (and, therefore, distributional) solution to the problem (3.1)–(3.2) with the singular part of the initial condition (3.2) identically equal to 0. Our goal is now to prove that the third summand in (4.1) is a distributional solution to the homogeneous equation (3.1) with singular initial condition $\delta^{(m)}(x - x_1^*)$. Indeed, for all $\varphi \in \mathcal{D}(\Omega_0)$, we have

$$\begin{aligned} &\langle (\partial_t + \partial_x)(S\delta^{(m)}(x - t - x_1^*)), \varphi \rangle \\ &= -\langle S\delta^{(m)}(x - t - x_1^*), \partial_t \varphi + \partial_x \varphi \rangle \\ &= -\langle \delta^{(m)}(x - t - x_1^*), S\partial_t \varphi + S\partial_x \varphi \rangle \\ &= -\langle \delta^{(m)}(x - t - x_1^*), \partial_t(S\varphi) + \partial_x(S\varphi) - \partial_t S\varphi - \partial_x S\varphi \rangle. \end{aligned}$$

Since $w = \delta^{(m)}(x - t - x_1^*)$ is a distribution in $x - t$, this is a weak solution to the equation $(\partial_t + \partial_x)w = 0$. Note that $S\varphi \in \mathcal{D}(\Omega_0)$. Therefore,

$$\langle \delta^{(m)}(x - t - x_1^*), \partial_t(S\varphi) + \partial_x(S\varphi) \rangle = 0.$$

By (3.5), we have $\partial_t S + \partial_x S = pS$. The desired assertion is therewith proved.

It remains to prove that $S(x,t)\delta^{(m)}(x - t - x_1^*)$ can be restricted to the initial interval $X = [0, L) \times \{0\}$. For this purpose we use Theorems 2.4 and 2.6. Observe that $f$ restricted to $\Omega_0$ is a diffeomorphism. We check the condition

$$\mathrm{WF}(Sf^* B_m) \cap N(X) = \emptyset, \tag{4.3}$$

where the normal bundle $N(X)$ to $X$ is defined by the formula

$$N(X) = \{(x, t, \xi, \eta) : (x, t) \in X, \langle T_{(x,t)}(X), (\xi, \eta)\rangle = 0\}$$

and $T_{(x,t)}(X)$ is the space of all tangent vectors to $X$ at $(x, t)$. It is clear that in our case

$$N(X) = \{(x, 0, 0, \eta), \eta \neq 0\}.$$

Let us now look at $\mathrm{WF}(Sf^*B_m)$. By Proposition 2.2, we have

$$\mathrm{WF}(Sf^*B_m) \subset \mathrm{WF}(f^*B_m).$$

By definition,

$$\mathrm{WF}(f^*B_m) = \left\{(x, t, df_x^t \cdot (\xi, \eta)) : (f(x, t), \xi, \eta) \in \mathrm{WF}(B_m)\right\}. \tag{4.4}$$

We also have

$$\mathrm{WF}(B_m) \subset \mathrm{WF}(B_0) = \{(x, 0, 0, \eta), \eta \neq 0\}.$$

It follows that $f(x, t) = (x, 0)$ in (4.4) and therefore $(x, t) = (x, x - x_1^*)$. Furthermore,

$$df_x^t = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad df_x^t \cdot (0, \eta) = \begin{pmatrix} \eta \\ -\eta \end{pmatrix}.$$

As a consequence,

$$\mathrm{WF}(Sf^*B_m) \subset \{(x, x - x_1^*, \eta, -\eta), \eta \neq 0\}.$$

This implies that $S(x, t)\delta^{(m)}(x - t - x_1^*)$ is restrictable to $X$. Consider the distribution $\delta^{(m)}(x - t - x_1^*)$ to be smooth in $t$ with distributional values in $x$. Then initial condition (4.3) follows from (4.1), completing the proof. $\square$

We have proved that $u$ defined by (4.1) satisfies Items 1 and 2 of Definition 3.2 with $\Omega$ replaced by $\Omega_0$. Items 4–7 on $\Omega_0$ do not need any proof. Item 3 will be given by Lemma 5.2 in the next section.

## 5. Multiplication of distributions under the integral in (1.3)

In the further sections we will extend the solution to $\Omega^T \setminus \overline{\Omega_0}$. We use the fact that any $\mathcal{D}'(\Omega)$-solution $u$ to our problem is representable as

$$u(x, t) = u_0(x, t) + u_1(x, t), \tag{5.1}$$

where $u_0 = u$ in $\mathcal{D}'(\Omega_0)$, $u_0$ is identically equal to 0 on $\overline{\Omega^T}\setminus\Omega_0$, $u_1 = u$ in $\mathcal{D}'(\Omega^T\setminus\overline{\Omega_0})$, and $u_1$ is identically equal to 0 on $\overline{\Omega_0}$. Indeed, if $u$ is a solution, then it is a smooth function in a neighborhood of the line $x = t$ (see Item 7 of Definition 3.2). Given an arbitrary $\varphi \in \mathcal{D}(\Omega^T)$, consider a representation $\varphi(x, t) = \varphi_1(x, t) + \varphi_2(x, t) + \varphi_3(x, t)$ such that $\varphi_i(x, t) \in \mathcal{D}(\Omega^T)$, $\operatorname{supp}\varphi_1 \subset \Omega_0$, $\operatorname{supp}\varphi_2 \cap \operatorname{sign\,supp} u = \emptyset$, and $\operatorname{supp}\varphi_3 \subset \Omega^T \setminus \overline{\Omega_0}$. We have

$$\begin{aligned}
\langle u_0 + u_1, \varphi\rangle &= \langle u_0, \varphi_1 + \varphi_2\rangle + \langle u_1, \varphi_2 + \varphi_3\rangle \\
&= \langle u, \varphi_1\rangle + \langle u_0, \varphi_2\rangle + \langle u_1, \varphi_2\rangle + \langle u, \varphi_3\rangle \\
&= \langle u, \varphi_1 + \varphi_2 + \varphi_3\rangle = \langle u, \varphi\rangle.
\end{aligned}$$

Using (5.1), we rewrite $v(t)$ (see Item 4 of Definition 3.2) in the form

$$v(t) = \int_0^L b(x)u_0(x, t)\,dx + \int_0^L b(x)u_1(x, t)\,dx.$$

In this section we compute the integral

$$J_0(t) = \int_0^L b(x)u_0(x,t)\,dx, \quad 0 < t < T, \tag{5.2}$$

that will be used in the construction. We have to tackle the multiplication of distributions involved in the integrand. For technical reasons we extend $a_r(x)$ and $b_r(x)$ over all $\mathbb{R}$ defining them to be 0 outside $[0, L]$. By (4.1), we rewrite (5.2) as follows

$$\begin{aligned}
J_0(t) = &\int_t^L b_r(x)\left(S_1(x,t) + S(x,t)a_r(x-t)\right)\,dx \\
&+ \int_0^L \delta^{(n)}(x-x_1)\left(S_1(x,t) + S(x,t)a_r(x-t)\right)\,dx \\
&+ \int_0^L b_r(x)S(x,t)\delta^{(m)}(x-t-x_1^*)\,dx \\
&+ \int_0^L \delta^{(n)}(x-x_1)S(x,t)\delta^{(m)}(x-t-x_1^*)\,dx.
\end{aligned}$$

To evaluate the second and the third integrals, we take a test function $\psi(t) \in \mathcal{D}(0,T)$ and compute the actions (see Definition 3.2, Item 4),

$$\begin{aligned}
&\langle \delta^{(n)}(x-x_1)\left(S_1(x,t) + S(x,t)a_r(x-t)\right), 1(x)\otimes\psi(t)\rangle \\
&= (-1)^n \langle \partial_x^n \left(S_1(x,t) + S(x,t)a_r(x-t)\right)\big|_{x=x_1}, \psi(t)\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\langle S(x,t)b_r(x)\delta^{(m)}(x-t-x_1^*), 1(x)\otimes\psi(t)\rangle \\
&= (-1)^m \langle \partial_x^m \left(S(x+t+x_1^*,t)b_r(x+t+x_1^*)\right)\big|_{x=0}, \psi(t)\rangle.
\end{aligned}$$

To evaluate the last integral in the expression for $J_0(t)$ we need the following fact.

**Lemma 5.1.** *The product of two distributions $v = \delta^{(n)}(x-x_1)\otimes 1(t)$ and $w = \delta^{(m)}(x-t-x_1^*)$ exists in the sense of Hörmander (see Theorem 2.7).*

*Proof.* Recall that

$$\begin{gathered}
\mathrm{WF}(v) = \{(x_1, t, \xi_1, 0), \xi_1 \neq 0\}, \\
\mathrm{WF}(w) \subset \{(x, x-x_1^*, \xi_2, -\xi_2), \xi_2 \neq 0\}.
\end{gathered}$$

Thus all the conditions of Theorem 2.7 are true and the lemma follows. □

We have proved the following lemma.

**Lemma 5.2.** *The distribution $u$ defined by (4.1) satisfies Item 3 of Definition 3.2 with $\Pi$ replaced by $\Pi\cap\Omega_0$.*

Turning back to computing the last integral in $J_0(t)$, consider the map

$$H : (x,t) \to (x-x_1, x-t-x_1^*)$$

and the inverse map

$$H^{-1} : (x,t) \to (x+x_1, x-t+x_1-x_1^*).$$

Define $H^*A_n = \delta^{(n)}(x-x_1)\otimes 1(t)$ and $H^*B_m = \delta^{(m)}(x-t-x_1^*)$. Let us compute the actions of $H^*A_n$ and $H^*B_m$ on a test function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ explicitly,

$$
\begin{aligned}
\langle H^*A_n, \varphi(x,t)\rangle &= \langle A_n, \varphi(x+x_1, x-t+x_1-x_1^*)\rangle\\
&= \langle \delta^{(n)}(x), \int \varphi(x+x_1, x-t+x_1-x_1^*)\,dt\rangle\\
&= \langle \delta^{(n)}(x), \int \varphi(x+x_1, \tau)\,d\tau\rangle\\
&= (-1)^n \int \varphi_x^{(n)}(x_1,\tau)\,d\tau
\end{aligned}
$$

and similarly with $H^*B_m$.

We are now in a position to compute the product of two distributions $\delta^{(n)}(x-x_1)$ and $\delta^{(m)}(x-t-x_1^*)$: For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have

$$
\begin{aligned}
&\langle S(x,t)\delta^{(n)}(x-x_1)\delta^{(m)}(x-t-x_1^*), \varphi(x,t)\rangle\\
&= \langle H^*A_nH^*B_m, S(x,t)\varphi(x,t)\rangle\\
&= \langle H^*(A_nB_m), S(x,t)\varphi(x,t)\rangle\\
&= \langle A_nB_m, (S\varphi)(x+x_1, x-t+x_1-x_1^*)\rangle\\
&= \langle \delta^{(n)}(x)\otimes\delta^{(m)}(t), (S\varphi)(x+x_1, x-t+x_1-x_1^*)\rangle\\
&= (-1)^{n+m}\partial_x^n\partial_t^m (S\varphi)(x+x_1, x-t+x_1-x_1^*)\big|_{x=0,t=0}\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n+m} F_{ji}(x,t)\partial_x^j\partial_t^i\varphi(x+x_1, t+x_1-x_1^*)\big|_{x=0,t=0}\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n+m} F_{ji}(0,0)\partial_x^j\partial_t^i\varphi(x_1,t_1^*)\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n+m} (-1)^{j+i}F_{ji}(0,0)\langle\delta^{(j)}(x-x_1)\otimes\delta^{(i)}(t-t_1^*), \varphi(x,t)\rangle.
\end{aligned}
$$

Here $F_{ji}(x,t)$ are known smooth functions of $S$ and of all its derivatives up to the order $n+m$. Hence, for all $\psi(t) \in \mathcal{D}(0,T)$ we get

$$
\begin{aligned}
&\Big\langle \int_0^L \delta^{(n)}(x-x_1)S(x,t)\delta^{(m)}(x-t-x_1^*)\,dx, \psi(t)\Big\rangle\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n+m} (-1)^{j+i}F_{ji}(0,0)\Big\langle \int_0^L \delta^{(j)}(x-x_1)\otimes\delta^{(i)}(t-t_1^*)\,dx, \psi(t)\Big\rangle\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n+m} (-1)^{j+i}F_{ji}(0,0)\langle\delta^{(j)}(x-x_1)\otimes\delta^{(i)}(t-t_1^*), 1(x)\otimes\psi(t)\rangle\\
&= \sum_{i=0}^{n+m}(-1)^iF_{0i}(0,0)\langle\delta^{(i)}(t-t_1^*), \psi(t)\rangle.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
J_0(t) = \int_t^L & b_r(x)(S(x,t)a_r(x-t) + S_1(x,t))\,dx \\
&+ (-1)^n \partial_x^m \left(S(x,t)a_r(x-t) + S_1(x,t)\right)\big|_{x=x_1} \\
&+ (-1)^m \partial_x^m \left(S(x+t+x_1^*,t)b_r(x+t+x_1^*)\right)\big|_{x=0} \\
&+ \sum_{i=0}^{n+m} (-1)^i F_{0i}(0,0)\delta^{(i)}(t-t_1^*).
\end{aligned} \tag{5.3}
$$

Observe that the first three summands in (5.3) are smooth for $t > 0$. Indeed, the second summand is smooth due to $a_r^{(i)}(0) = 0$ for $0 \le i \le n$ (see (A1)). The third summand is smooth due to $b_r^{(i)}(L) = 0$ for $0 \le i \le m$ (see (A2)).

**Further plan of the solution construction.** We split $\Omega^T \setminus \overline{\Omega_0}$ into subdomains

$$
\Omega_i = \left\{(x,t) \in \Omega^T \setminus \overline{\Omega_0} : t - t_i^* < x < t - t_{i-1}^*\right\}
$$

(see Figure 1) and construct the solution separately in each $\Omega_i$ and in a neighborhood of each border between $\Omega_i$ and $\Omega_{i+1}$. Here $t_0^* = 0$, $1 \le i \le k(T)$, where $k(T)$ is defined by inequalities $t_{k(T)}^* < T$ and $t_{k(T)+1}^* \ge T$. The finiteness of $k(T)$ is obvious.

## 6. Existence of the smooth solution on $\Omega_1$

**Lemma 6.1.** *There exists a smooth solution to the problem* (3.1)–(3.3) *on* $\Omega_1$.

*Proof.* Under the assumption that $x_1^* < x_1$, we have $t_1^* < L$. Hence $(x_1, t_1^*) \in \Omega_0$. Therefore any solution which is given by (4.1) on $\Omega_0$, is smooth on $\Omega_1$, and has the property given by Item 7 of Definition 3.2, satisfies the Volterra integral equation of the second kind

$$
u(x,t) = S_3(x,t) + S_2(x,t)\int_0^{t-x} b_r(\xi)u(\xi, t-x)\,d\xi, \tag{6.1}
$$

where $S_2(x,t) = S(x,t)c_r(t-x)$ and

$$
S_3(x,t) = S_2(x,t)J_0(t-x) + S_1(x,t)
$$

are known by (5.3). The smoothness of $J_0(t-x)$ at every point $(x,t) \in \Omega_1$ follows from the facts that $t - x < t_1^*$ and that $J_0(t)$ restricted to the interval $(0, t_1^*)$ is smooth. Therefore $S_2$ and $S_3$ are smooth.

The lemma will follow from two claims. Given $s > 0$, set

$$
\Omega_1^s = \{(x,t) \in \Omega_1 : 0 < t < s\}.
$$

**Claim 1:** Given $m \in \mathbb{N}_0$, there exists a unique solution $u \in C^m(\overline{\Omega_1^{s_m}})$ to the problem (3.1)–(3.3) for some $s_m > 0$. We apply the contraction principle to (6.1). Comparing the difference of two continuous functions $u$ and $\tilde{u}$ satisfying (6.1), we have

$$
|u - \tilde{u}| \le s_0 q \max_{(x,t)\in\overline{\Omega_1^{s_0}}} |u - \tilde{u}|,
$$

where

$$
q = \max_{(x,t)\in\overline{\Omega_1}} |S| \max_{t\in[0,t_1^*]} |c_r| \max_{x\in[0,L]} |b_r|.
$$

Choosing $s_0 < 1/q$, we obtain the contraction property for the operator defined by the right-hand side of (6.1). The claim for $m = 0$ follows.

Our next concern is the existence and uniqueness of a $C^1(\overline{\Omega_1^{s_1}})$-solution for some $s_1$. Let us consider the problem

$$\begin{aligned} \partial_x u(x,t) = \partial_x S_3(x,t) + \partial_x S_2(x,t) \int_0^{t-x} b_r(\xi) u(\xi, \theta(x,t))\, d\xi \\ - b_r(t-x) u(t-x, t-x) - S_2(x,t) \int_0^{t-x} b_r(\xi)(\partial_t u)(\xi, t-x)\, d\xi. \end{aligned} \tag{6.2}$$

From (3.1) we have $\partial_t u = p(x,t)u + g(x,t) - \partial_x u$. We choose an arbitrary $s_1 \le s_0$. Since $u$ is a known $C(\Omega_1^{s_1})$-function, (6.2) on $\overline{\Omega_1^{s_1}}$ is a Volterra integral equation of the second kind with respect to $\partial_x u$. Assuming in addition to the condition $s_1 \le s_0$ that $s_1 < q$, we obtain the contraction property for (6.2). On the account of (3.1), the claim for $m = 1$ follows.

Proceeding further by induction and using in parallel (3.1), (6.1), and their suitable differentiations, we complete the proof of the claim.

**Claim 2:** In the domain $\Omega_1^{t_1^*}$ there exists a unique smooth solution to the problem (3.1)–(3.3). Given $m \in \mathbb{N}_0$, we prove that there exists a unique $u \in C^m(\Omega_1^{t_1^*})$ in at most $\lceil t_1^*/s_m \rceil$ steps by iterating the local existence and uniqueness result in domains

$$\Omega_1^{ks_m} \setminus \overline{\Omega_1^{(k-1)s_m}}, \quad 1 \le k \le \lceil T/s_m \rceil.$$

In particular, for $m = 0$ in the $k$-th step of the proof we have

$$\begin{aligned} u(x,t) = S_3(x,t) + S_2(x,t) \int_0^{t-x-(k-1)s_m} b_r(\xi) u(\xi, t-x)\, d\xi \\ + S_2(x,t) \int_{t-x-(k-1)s_m}^{t-x} b_r(\xi) u(\xi, t-x)\, d\xi \end{aligned} \tag{6.3}$$

on $\{(x,t) \in \Omega_1^{ks_m} : x \le t - (k-1)s_m\}$, and

$$u(x,t) = S(x,t)u(0,t-x) + S_1(x,t) \text{ on } \{(x,t) \in \Omega_1 : x \ge t - (k-1)s_m\}. \tag{6.4}$$

As in the latter formula $t - x \le (k-1)s_m$, the function $u$ defined by (6.4) is smooth and known from the previous steps. This implies that the last summand in (6.3) is known and smooth. Hence (6.3) is a Volterra integral equation of the second kind. Applying now the argument used to prove Claim 1, we obtain the existence and uniqueness of a continuous solution $u$ to (6.3) on $\Omega_1^{ks_m} \setminus \overline{\Omega_1^{(k-1)s_m}}$. Since $k$ is an arbitrary integer in the range $1 \le k \le \lceil T/s_m \rceil$, we have $u \in C(\Omega_1^{t_1^*})$. Further we similarly proceed with all derivatives of $u$. Claim 2 is therewith proved.

The solution on the whole $\Omega_1$ is now uniquely determined by the formula

$$u(x,t) = S(x,t)u(0,t-x) + S_1(x,t),$$

where $u(0, t-x)$ is a known smooth function. The latter is true due to $0 < t-x < t_1^*$ and Claim 2. The proof of the lemma is complete. $\square$

From formulas (4.1) and (6.1), Lemma 6.1, and (A1) it follows that $u$ is smooth in a neighborhood of the characteristic line $x = t$. This ensures that $u$ we construct satisfies Item 7 of Definition 3.2.

Under the assumption that $\Omega_2$ is nonempty, in the next section we give the formula of the solution on

$$\Omega_{1,\varepsilon} = \Omega_1 \cup \big\{(x,t) \in \overline{\Omega_2} : x > t - t_1^* - \varepsilon\big\}$$

for a fixed $\varepsilon > 0$ such that $t_1^* - \varepsilon > 0$, $t_1^* + \varepsilon < t_2^*$, and

$$b_r(x) = 0, \quad x \in [0, 2\varepsilon]. \tag{6.5}$$

Such $\varepsilon$ exists by (A2).

## 7. The solution on $\Omega_{1,\varepsilon}$

Write now

$$v(t) = \int_0^L \Big(b_r(x) + \delta^{(n)}(x - x_1)\Big)\, u\, dx = v_r(t) + v_s(t), \tag{7.1}$$

where $v_r(t)$ and $v_s(t)$ are, respectively, the regular (smooth) and singular parts of $v(t)$. On the account of (5.1), (5.3), (6.5), and the fact that $x_1^* < x_1$, we have on $[0, t_1^* + \varepsilon]$:

$$\begin{aligned} v_r(t) = {} & \int_{2\varepsilon}^t b_r(x)u(x,t)\,dx + \int_t^L b_r(x)\left(S(x,t)a_r(x-t) + S_1(x,t)\right)dx \\ & + (-1)^n \partial_x^n (S(x,t)a_r(x-t) + S_1(x,t))\big|_{x=x_1} \\ & + (-1)^m \partial_x^m \left(S(x+t+x_1^*, t) b_r(x+t+x_1^*)\right)\big|_{x=0} \end{aligned} \tag{7.2}$$

and

$$v_s(t) = \sum_{i=0}^{n+m} (-1)^i F_{0i}(0,0)\delta^{(i)}(t - t_1^*). \tag{7.3}$$

Note that the first summand in (7.2) is a known smooth function. This follows from the inclusion $[t - t_1^* + \varepsilon, t] \times \{t\} \subset \Omega_1 \cup \{(x,t) : x = t\}$, Lemma 6.1, and (A1).

On the account of (7.1)–(7.3) and the fact that $x_1 - x_1^* = t_1^*$, we derive the following formula for $u(0,t)$ on $(0, t_1^* + \varepsilon)$:

$$\begin{aligned} u(0,t) &= c_r(t) \sum_{i=0}^{n+m} (-1)^i F_{0i}(0,0)\delta^{(i)}(t - t_1^*) + c_r(t)v_r(t) \\ &= \sum_{i=0}^{n+m} E_i \delta^{(i)}(t - t_1^*) + c_r(t)v_r(t), \end{aligned} \tag{7.4}$$

where $E_i$ are constants depending on $F_{0k}(0,0)$ and $c_r^{(k)}(t_1^*)$ for $0 \le k \le i$. Note that if $t_1^* = t_1$, then $x_1 - x_1^* > t_1^*$ by (A4). This implies $v(t) = v_r(t)$ on $[0, t_1^* + \varepsilon]$. Thus, Item 6 of Definition 3.2 for $u$ we construct is fulfilled. Furthermore, we have an expression for $u(0,t)$ on $(0, t_1^* + \varepsilon)$ similar to (7.4), namely, $u(0,t) = (\delta^{(j)}(t - t_1^*) + c_r(t))v_r(t) = v_r^{(j)}(t_1^*)\delta^{(j)}(t - t_1^*) + c_r(t)v_r(t)$.

Set

$$Q(t) = \sum_{i=0}^{n+m} E_i \delta^{(i)}(t - t_1^*).$$

**Lemma 7.1.** *$u(x,t)$ given by the formula*
$$u(x,t)=S(x,t)c_r(t-x)v_r(t-x)+S_1(x,t)+S(x,t)Q(t-x), \tag{7.5}$$
*where $v_r(t)$ is determined by (7.2), is a $\mathcal{D}'(\Omega)$-solution to the problem* (3.1)–(3.3) *restricted to $\Omega_{1,\varepsilon}$.*

*Proof.* On the account of (7.4) and the construction of the solution on $\Omega_1$ done in Section 6, it is enough to prove that the restriction of $S(x,t)Q(t-x)$ to $Y=\{0\}\times(0,t_1^*+\varepsilon)$ is well defined and that $S(x,t)Q(t-x)$ satisfies (3.1) with $g(x,t)\equiv 0$ on $\Omega_{1,\varepsilon}$ in a distributional sense. The proof of the latter uses the argument as in the proof of Lemma 4.2. To prove the former claim, consider the smooth bijective map
$$\Phi:(x,t)\to(x,t-x-t_1^*)$$
and its inverse
$$\Phi^{-1}:(x,t)\to(x,x+t+t_1^*).$$
Applying Theorem 2.6, we have
$$\mathrm{WF}(\Phi^*B_i)\subset\{(0,t+t_1^*,-\eta,\eta),\eta\neq 0\}.$$
Furthermore, $N(Y)=\{(0,t,\xi,0)\}$ and therefore
$$\mathrm{WF}(\Phi^*B_i)\cap N(Y)=\emptyset\quad\text{for all } 0\le i\le n+m.$$
By Theorem 2.4, the restriction of $S(x,t)Q(\theta(x,t))$ to $Y$ is well defined. The lemma is therewith proved. ☐

## 8. Construction of the smooth solution on $\Omega_2$

To shorten notation, without loss of generality we assume that $t_2^*\le T$.

**Lemma 8.1.** *There exists a smooth solution to the problem* (3.1)–(3.3) *on $\Omega_2$.*

*Proof.* We start from the general formula of a smooth solution on $\Omega_2$:
$$u(x,t)=S(x,t)u(0,t-x)+S_1(x,t). \tag{8.1}$$
Since $S$ and $S_1$ are smooth, our task is to prove that there exists a smooth function identically equal to $u(0,t-x)$ on $\Omega_2$. Since $t_1^*<t-x<t_2^*$ if $(x,t)\in\Omega_2$ and $c(t)=c_r(t)$ if $t\in(t_1^*,t_2^*)$, it suffices to show the existence of a smooth function $v_r(t)$ identically equal to $v(t)$ on $(t_1^*,t_2^*)$. From the formula (7.3) for $v_s(t)$ on $(0,t_1^*+\varepsilon)$ it follows that $v(t)=v_r(t)$ if $t\in(t_1^*,t_1^*+\varepsilon)$, where $\varepsilon$ is as in Section 7 and $v_r(t)$ is known and determined by (7.2). To prove the lemma, it is sufficient to show that there exists a smooth extension of $v_r(t)$ from $(0,t_1^*+\varepsilon)$ to $[t_1^*+\varepsilon,t_2^*)$ such that $v_r(t)=v(t)$ if $t\in[t_1^*+\varepsilon,t_2^*)$. By (7.5), such an extension must satisfy the following integral equation on $[t_1^*+\varepsilon,t_2^*)$:
$$v_r(t)=\int_0^{t-t_1^*-\varepsilon}b_r(x)S(x,t)c_r(t-x)v_r(t-x)\,dx+R(t), \tag{8.2}$$
where
$$\begin{aligned}R(t)=&\int_{t-t_1^*-\varepsilon}^{P(t)}b_r(x)S(x,t)c_r(t-x)v_r(t-x)\,dx+\int_0^{P(t)}b_r(x)S_1(x,t)\,dx\\&+J_0(t)+\int_0^L b_r(x)S(x,t)Q(t-x)\,dx\end{aligned} \tag{8.3}$$

and

$$P(t) = \begin{cases} t & \text{if } L \le t, \\ L & \text{if } L \ge t. \end{cases}$$

Here $b_r(x)$ is defined to be 0 outside $[0, L]$, and $v_r$ in the formula (8.3) is known and defined by (7.2). One can easily see that the first three summands in (8.3) are smooth functions on $[t_1^* + \varepsilon, t_2^*)$. We now show that the last summand is a $C^\infty[t_1^* + \varepsilon, t_2^*)$-function as well. Indeed, take $\psi(t) \in \mathcal{D}(t_1^* + \varepsilon/2, t_1^*)$ and compute

$$\begin{aligned}
&\Big\langle \int_0^L b_r(x)S(x,t)\delta^{(j)}(t - x - t_1^*)\,dx, \psi(t)\Big\rangle \\
&= \langle \delta^{(j)}(t - x - t_1^*), b_r(x)S(x,t)\psi(t)\rangle \\
&= -\langle \delta^{(j)}(x) \otimes 1(t), b_r(t - x - t_1^*)S(t - x - t_1^*, t)\psi(t)\rangle \\
&= (-1)^{j+1}\langle \partial_x^j \left(b_r(t - x - t_1^*)S(t - x - t_1^*, t)\right)\big|_{x=0}, \psi(t)\rangle.
\end{aligned}$$

The desired assertion follows. As follows from (6.5), the functions $v_r(t)$ defined by (7.2) and (8.2) coincide at $t = t_1^* + \varepsilon$. The same is true with respect to all the derivatives of $v_r$.

Our task is therefore reduced to show that there exists a $C^\infty[t_1^* + \varepsilon, t_2^*)$-function $v_r(t)$ satisfying (8.2). This follows from the fact that (8.2) is a Volterra integral equation of the second kind with respect to $v_r(t)$ (for details see the proof of Lemma 6.1). The proof is complete. □

## 9. Completion of the construction

Continuing our construction in this fashion, we extend $u$ over a neighborhood of each subsequent border between $\Omega_{i-1}$ and $\Omega_i$ and over $\Omega_i$ for all $3 \le i \le k(T)$. Eventually we construct $u$ on $\Omega^T$ for any $T > 0$ in the sense of Definition 3.2 with $\Omega$ replaced by $\Omega^T$ and $\Pi$ replaced by $\Pi^T = \{(x,t) \in \Pi : t < T\}$. As easily seen from our construction, the condition (3.4) is fulfilled with $\Omega_+$ and $\Omega'_+$ replaced by $\Omega^T \cap \Omega_+$ and $\Omega^T \cap \Omega'_+$, respectively. Since $T$ is arbitrary, the proof of Item 1 of Theorem 3.4 is complete. On the account of Definition 3.3 and the definition of the restriction $u \in \mathcal{D}'(\Omega)$ to a subset of $\Omega$ (see [7, Section 5]), Item 2 of Theorem 3.4 is a straightforward consequence of Item 1. Theorem 3.4 is therewith proved.

By (7.5) it follows from the construction, that if the singular part of $b(x)$ is the derivative of the Dirac measure of order $n$, then for each $i \ge 1$ there exist $k > i$ and $n' \ge 1$ such that $u$ is the derivative of the Dirac measure of order $n'$ along the characteristic line $t - t_i^*$ and $u$ is the derivative of the Dirac measure of order $n' + n$ along the characteristic line $t - t_k^*$. In contrast, this is not so if singular parts of the initial and the boundary data are Dirac measures. In the latter case the solution preserves the same order of regularity in time. Furthermore, the assumption $b_r^{(i)}(L) = 0$ for all $i \in \mathbb{N}_0$ can be weakened to $b_r(L) = 0$. Since $u$ restricted to $\Pi \setminus I$ is smooth, Theorem 3.8 follows from Item 2 of Theorem 3.7.

## 10. Uniqueness of the solution (Proof of Theorem 3.7)

In this section we reuse notation $\Omega_i$, $i \ge 0$, by setting

$$\Omega_0 = \{(x,t) \in \Omega : t < x < t + L\},$$
$$\Omega_i = \{(x,t) \in \Omega : t - t_i^* < x < t - t_{i-1}^*\}, \quad i \ge 1.$$

Recall that $t_0^* = 0$.

Without loss of generality, we make the same assumption as in the proof of Theorem 3.4, namely, that $t_1 = t_2^*$. The proof of Theorem 3.7 is based on five lemmas.

**Lemma 10.1.** *A $\mathcal{D}'_+(\Omega)$-solution $u$ to the problem* (3.1)–(3.3) *is unique on $\Omega_0$.*

*Proof.* Note that any $\mathcal{D}'_+(\Omega)$-solution $u$ to the problem (3.1)–(3.3) on $\Omega_0$ is a $\mathcal{D}'_+(\Omega_0)$-solution to the problem (3.1)–(3.2). Let $u$ and $\tilde{u}$ be two $\mathcal{D}'_+(\Omega_0)$-solutions to the problem (3.1)–(3.2). Then

$$\langle L(u-\tilde{u}), \varphi\rangle = \langle u-\tilde{u}, L^*\varphi\rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega_0), \tag{10.1}$$

where

$$L = \partial_t + \partial_x - p, \quad L^* = -(\partial_t + \partial_x + p). \tag{10.2}$$

Our goal is to show that

$$\langle u-\tilde{u}, \psi\rangle = 0 \quad \text{for all } \psi \in \mathcal{D}(\Omega_0). \tag{10.3}$$

Using the definition of $\mathcal{D}'_+(\Omega_0)$ and (10.1), it is sufficient to prove that for every $\psi \in \mathcal{D}(\Omega_0)$ there exists $\varphi \in \mathcal{D}(\Omega_0)$ such that

$$L^*\varphi = \psi \quad \text{on} \quad \{(x,t) \in \Omega_0 : t \ge 0\}. \tag{10.4}$$

Fix $\psi \in \mathcal{D}(\Omega_0)$. If $\operatorname{supp}\psi \cap \{(x,t) : t > 0\} = \emptyset$, then (10.3) follows immediately from the definition of $\mathcal{D}'_+(\Omega_0)$. We therefore assume that $\operatorname{supp}\psi \cap \{(x,t) : t > 0\} \neq \emptyset$. Consider the problem

$$\begin{gathered} \varphi_t + \varphi_x = -p\varphi - \psi, \quad (x,t) \in \{(x,t) \in \Omega_0 : t > 0\}, \\ \varphi|_{t=0} = \varphi_0(x), \quad x \in (0,L), \end{gathered}$$

where $\varphi_0(x) \in \mathcal{D}(0,L)$ will be specified below. This problem has a unique smooth solution given by the formula

$$\varphi(x,t) = \hat{S}(x,t)\varphi_0(x-t) + \hat{S}_1(x,t),$$

where $\hat{S}_1$ is given by (4.2) with $p$ and $g$ replaced by $-p$ and $-\psi$, respectively.

Fix $T(\psi) > 0$ so that $\operatorname{supp}\psi \cap \{(x,t) : t \ge T(\psi)\} = \emptyset$ for all $x$ with $(x, T(\psi)) \in \Omega_0$. Set

$$\varphi_0(x - T(\psi)) = -\frac{\hat{S}_1(x, T(\psi))}{\hat{S}(x, T(\psi))}$$

for $x$ such that $(x, T(\psi)) \in \Omega_0$. Changing coordinates $x \to \xi = x - T(\psi)$, we obtain

$$\varphi_0(\xi) = -\frac{\hat{S}_1(\xi + T(\psi), T(\psi))}{\hat{S}(\xi + T(\psi), T(\psi))}. \tag{10.5}$$

We construct the desired function $\varphi(x,t)$ by the formula

$$\varphi(x,t) = \begin{cases} 0 & \text{if } (x,t) \in \Omega_0 \text{ and } t \ge T(\psi), \\ \hat{S}(x,t)\varphi_0(x-t) + \hat{S}_1(x,t) & \text{if } (x,t) \in \Omega_0 \text{ and } 0 \le t \le T(\psi), \\ \tilde{\varphi}(x,t) & \text{if } (x,t) \in \Omega_0 \text{ and } t \le 0, \end{cases}$$

where $\tilde{\varphi}(x,t)$ is chosen so that $\varphi \in \mathcal{D}(\Omega_0)$. The proof is complete. □

**Lemma 10.2.** *A $\mathcal{D}'_+(\Omega)$-solution to the problem* (3.1)–(3.3) *is unique on $\Omega_1$.*

*Proof.* Assume that there exist two $\mathcal{D}'_+(\Omega)$-solutions $u$ and $\tilde{u}$. We will show that

$$\langle v(t)-\tilde{v}(t),\psi(t)\rangle = 0 \quad \text{for all } \psi(t)\in\mathcal{D}(0,t_1^*), \tag{10.6}$$

where $v(t)$ is defined by Item 5 of Definition 3.2 and $\tilde{v}(t)$ is defined similarly with $u$ replaced by $\tilde{u}$. Postponing the proof, assume that (10.6) is true. Taking into account Item 2 of Definition 3.5 and the fact that $c(t)=c_r(t)$ if $0<t<t_1^*$, we have

$$\langle L(u-\tilde{u}),\varphi\rangle = \langle u-\tilde{u},L^*\varphi\rangle = 0 \quad \text{for all } \varphi\in\mathcal{D}(\Omega_1).$$

Let us prove that

$$\langle u-\tilde{u},\psi\rangle = 0 \quad \text{for all } \psi\in\mathcal{D}(\Omega_1). \tag{10.7}$$

Following the argument used in the proof of Lemma 10.1, it is sufficient to show that, given $\psi\in\mathcal{D}(\Omega_1)$, there exists $\varphi\in\mathcal{D}(\Omega_1)$ such that

$$L^*\varphi = \psi \text{ on } \{(x,t)\in\Omega_1 : x\ge 0\}.$$

We concentrate on the case that $\operatorname{supp}\psi\cap\{(x,t):x>0\}\neq\emptyset$. Otherwise (10.7) is immediate because $u-\tilde{u}\in\mathcal{D}'_+(\Omega_1)$. Consider the problem

$$\begin{gathered}\varphi_t+\varphi_x = -p\varphi-\psi, \quad (x,t)\in\{(x,t)\in\Omega_1 : x>0\},\\ \varphi|_{x=0} = \varphi_1(t), \quad t\in(0,t_1^*),\end{gathered}$$

where $\varphi_1(t)\in\mathcal{D}(0,t_1^*)$ is a fixed function. Let $T(\psi)>0$ be the same as in the proof of Lemma 10.1. We specify $\varphi_1(\xi)$ by

$$\varphi_1(\xi) = -\frac{\hat{S}_1(T(\psi)-\xi,T(\psi))}{\hat{S}(T(\psi)-\xi,T(\psi))} \tag{10.8}$$

and construct the desired $\varphi$ similarly to the construction of $\varphi$ in the proof of Lemma 10.1. To finish the proof of the lemma, it remains to show that

$$\langle v-\tilde{v},\psi(t)\rangle = 0 \quad \text{for all } \psi(t)\in\mathcal{D}(\varepsilon i,\varepsilon i+2\varepsilon), \tag{10.9}$$

for each $0\le i\le t_1^*/\varepsilon-2$, where $\varepsilon>0$ is chosen so that $t_1^*/\varepsilon$ is an integer and

$$b_r(x) = 0 \quad \text{for } x\in[0,2\varepsilon]. \tag{10.10}$$

Such $\varepsilon$ exists by (A2). We prove (10.9) by induction on $i$.
*Base case:* (10.9) is true for $i=0$. We will use the following representations for $u$ and $\tilde{u}$ on $\Omega_+$ which are possible owing to Item 3 of Definition 3.5:

$$\begin{aligned} u &= u_0+u_1 \quad \text{in } \mathcal{D}'(\Omega_+),\\ \tilde{u} &= \tilde{u}_0+\tilde{u}_1 \quad \text{in } \mathcal{D}'(\Omega_+), \end{aligned} \tag{10.11}$$

where $u_0=u$ and $\tilde{u}_0=\tilde{u}$ in $\mathcal{D}'(\Omega_0\cap\Omega_+)$, $u_0=\tilde{u}_0\equiv 0$ on $(\Omega\setminus\Omega_0)\cap\Omega_+$, $u_1=u$ and $\tilde{u}_1=\tilde{u}$ in $\mathcal{D}'\big((\Omega\setminus\overline{\Omega_0})\cap\Omega_+\big)$, $u_1=\tilde{u}_1\equiv 0$ on $\overline{\Omega_0}\cap\Omega_+$.

We first prove that

$$\langle v-\tilde{v},\psi(t)\rangle = \langle u_1-\tilde{u}_1, b_r(x)\psi(t)\rangle \quad \text{for all } \psi(t)\in\mathcal{D}(0,4\varepsilon). \tag{10.12}$$

According to Item 1 of Definition 3.5,

$$\begin{aligned} \langle v-\tilde{v},\psi(t)\rangle &= \langle (u-\tilde{u})\,b(x), 1(x)\otimes\psi(t)\rangle\\ &= \langle (u_0-\tilde{u}_0)b(x),1(x)\otimes\psi(t)\rangle + \langle (u_1-\tilde{u}_1)b(x),1(x)\otimes\psi(t)\rangle, \end{aligned} \tag{10.13}$$

where $b_r(x)=0$, $x\notin[0,L]$. By Lemma 10.1, $u_0=\tilde{u}_0$ in $\mathcal{D}'(\Omega_0\cap\Omega_+)$. Applying in addition Item 1 of Theorem 3.4 and Proposition 3.6, we have

$$\langle (u_0-\tilde{u}_0)(x,t)b(x),1(x)\otimes\psi(t)\rangle = \langle J_0(t)-\tilde{J}_0(t),\psi(t)\rangle, \tag{10.14}$$

where $J_0(t)$ is defined by (5.2) and $\tilde{J}_0(t)$ is defined by (5.2) with $u_0$ replaced by $\tilde{u}_0$. From (5.3) we have $J_0(t) = \tilde{J}_0(t)$ for $0 < t < 4\varepsilon$. Hence the right-hand side of (10.14) is equal to 0. On the account of the inclusions $\operatorname{supp}(u_1 - \tilde{u}_1) \subset \Omega \setminus \Omega_0$ and $\operatorname{supp} \psi(t) \subset [0, 2\varepsilon]$, (10.13) does not depend on $b(x)$ outside $[0, 2\varepsilon]$. Since $x_1^* < x_1$, $b(x) = b_r(x)$ on $[0, 2\varepsilon]$. Therefore (10.13) implies (10.12). The base case now follows from (10.10).

Assume that (10.9) is true for $i = k - 1$, where $k \geq 1$, and prove that it is true for $i = k$.

*Induction step:* (10.9) is true for $i = k$, $k \geq 1$. The proof is similar to the proof of the base case. Based on the induction assumption and applying the argument used in the proof of (10.7), we obtain

$$u = \tilde{u} \quad \text{in } \mathcal{D}'_+(G_{k-1}), \tag{10.15}$$

where

$$G_k = \Omega_1 \cap \{(x, t) : x > t - \varepsilon k - 2\varepsilon\}.$$

Applying in addition Item 1 of Theorem 3.4, Proposition 3.6, and Lemma 6.1, we conclude that $u$ is smooth on $G_{k-1} \cap \Omega_+$. Owing to (10.15) and the latter fact, the following representations for $u$ and $\tilde{u}$ on $\Omega_+$ are possible:

$$\begin{gathered} u = u_0 + u_{k-1} + u_k \quad \text{in } \mathcal{D}'(\Omega_+), \\ \tilde{u} = u_0 + u_{k-1} + \tilde{u}_k \quad \text{in } \mathcal{D}'(\Omega_+), \end{gathered}$$

where $u_0$ is the same as in (10.11), $u_{k-1} = u$ in $\mathcal{D}'(G_{k-1} \cap \Omega_+)$, $u_{k-1} \equiv 0$ on $\Omega_+ \setminus G_{k-1}$, $u_k = u$ and $\tilde{u}_k = \tilde{u}$ in $\mathcal{D}'\big(\Omega_+ \setminus (\overline{G_{k-1}} \cup \overline{\Omega_0})\big)$, $u_k = \tilde{u}_k \equiv 0$ on $\Omega_+ \cap (\overline{G_{k-1}} \cup \overline{\Omega_0})$. Similarly to (10.12), we derive the equality

$$\langle v - \tilde{v}, \psi(t) \rangle = \langle u_k - \tilde{u}_k, b_r(x)\psi(t) \rangle \quad \text{for all } \psi(t) \in \mathcal{D}(\varepsilon k, \varepsilon k + 2\varepsilon).$$

The induction step follows from the support properties of $u_k - \tilde{u}_k$, $\psi(t)$, and $b_r$ given by (10.10). The proof is complete. $\square$

Set

$$\Omega_{0,1}^{\varepsilon} = \{(x, t) \in \Omega : x - \varepsilon < t < x + \varepsilon)\}.$$

**Lemma 10.3.** *A $\mathcal{D}'_+(\Omega)$-solution to* (3.1)–(3.3) *is unique on $\Omega_{0,1}^{\varepsilon}$ provided $\varepsilon$ is small enough.*

*Proof.* Let $u$ and $\tilde{u}$ be two $\mathcal{D}'_+(\Omega)$-solutions to the problem (3.1)–(3.3). Fix $\varepsilon > 0$ so that the condition (10.10) is fulfilled. By Base case in the proof of Lemma 10.2, (10.9) is true for $i = 0$. Therefore

$$\langle L(u - \tilde{u}), \varphi \rangle = \langle u - \tilde{u}, L^* \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega_{0,1}^{\varepsilon}).$$

Our task is to prove (10.7) with $\Omega_1$ replaced by $\Omega_{0,1}^{\varepsilon}$. In fact, we prove that, given $\psi \in \mathcal{D}(\Omega_{0,1}^{\varepsilon})$ with $\operatorname{supp} \psi \cap \{(x, t) : x > 0\} \neq \emptyset$, there exists $\varphi \in \mathcal{D}(\Omega_{0,1}^{\varepsilon})$ satisfying the initial boundary problem

$$\begin{gathered} \varphi_t + \varphi_x = -p\varphi - \psi, \quad (x, t) \in \Omega_{0,1}^{\varepsilon} \cap \Omega_+, \\ \varphi|_{t=0} = \varphi_0(x), \quad x \in [0, \varepsilon), \\ \varphi|_{x=0} = \varphi_1(t), \quad t \in [0, \varepsilon). \end{gathered}$$

Here $\varphi_0(x) \in C^\infty[0, \varepsilon)$ is a fixed function identically equal to 0 in a neighborhood of $\varepsilon$, $\varphi_1(t) \in C^\infty[0, \varepsilon)$ is a fixed function identically equal to 0 in a neighborhood of $\varepsilon$, and $\varphi_0^{(i)}(0) = \varphi_1^{(i)}(0)$ for all $i \in \mathbb{N}_0$. We construct $\varphi(x, t)$, combining the constructions of $\varphi(x,t)$ in the proofs of Lemmas 10.1 and 10.2. Thus we fix $T(\psi)>0$ to be the same as in the proof of Lemma 10.1 and specify $\varphi_0(x)$ and $\varphi_1(t)$ by (10.5) and (10.8), respectively. Let

$$\varphi(x,t)=\begin{cases}0 & \text{if } (x,t)\in\Omega_{0,1}^{\varepsilon} \text{ and } t\ge T(\psi),\\ \hat S(x,t)\varphi_0(x-t)+\hat S_1(x,t) & \text{if } (x,t)\in\overline{\Omega_0}\cap\Omega_{0,1}^{\varepsilon} \text{ and } 0\le t\le T(\psi),\\ \hat S(x,t)\varphi_1(t-x)+\hat S_1(x,t) & \text{if } (x,t)\in\overline{\Omega_1}\cap\Omega_{0,1}^{\varepsilon} \text{ and } 0\le t\le T(\psi),\\ \tilde\varphi(x,t) & \text{if } (x,t)\in\Omega_{0,1}^{\varepsilon} \text{ and } (x\le 0 \,\text{or}\, t\le 0),\end{cases}$$

where $\tilde\varphi(x,t)$ is chosen so that $\varphi\in\mathcal{D}(\Omega_{0,1}^{\varepsilon})$. The proof is complete. □

For every $i\ge 1$ fix $\varepsilon_i$ such that $t_i^*-\varepsilon_i>t_{i-1}^*$, $t_i^*+\varepsilon_i<t_{i+1}^*$, and

$$b_r(x)=0 \quad \text{for } x\in[0,4\varepsilon_i]. \tag{10.16}$$

Set

$$Q_i=\{(x,t): t-t_i^*-\varepsilon_i<x<t-t_i^*+\varepsilon_i\}.$$

**Lemma 10.4.** *A $\mathcal{D}'_+(\Omega)$-solution to problem* (3.1)–(3.3) *is unique on $Q_1$.*

*Proof.* Assume that there exist two $\mathcal{D}'_+(\Omega)$-solutions $u$ and $\tilde u$ and show that

$$\langle v-\tilde v,\psi(t)\rangle=0 \quad \text{for all } \psi(t)\in\mathcal{D}(t_1^*-\varepsilon_1,t_1^*+\varepsilon_1). \tag{10.17}$$

By Lemmas 6.1 and 10.2, Item 1 of Theorem 3.4, and Proposition 3.6, any solution to (3.1)–(3.3) restricted to $\Omega_1$ is smooth. Based on this fact and on Lemmas 10.1–10.3, similarly to (10.12), we derive the equality

$$\langle v-\tilde v,\psi(t)\rangle=\langle u_1-\tilde u_1,b_r(x)\psi(t)\rangle \quad \text{for all } \psi(t)\in\mathcal{D}(t_1^*-\varepsilon_1,t_1^*+\varepsilon_1),$$

where $u_1=u$ and $\tilde u_1=\tilde u$ in $\mathcal{D}'(G)$, $u_1$ and $\tilde u_1$ are identically equal to zero on $\Omega_+\setminus G$. Here

$$G=\{(x,t)\in\Omega_+: x<t-t_1^*+\varepsilon_1\}.$$

The equality (10.17) now follows from the support properties of $u_1-\tilde u_1$, $\psi$, and $b_r$ given by (10.16) for $i=1$.

Note that $c(t)=c_r(t)$ for $t$ in the range $t_1^*-\varepsilon_1<t<t_1^*+\varepsilon_1$. Applying (10.17) and Item 2 of Definition 3.5, we have

$$L(u-\tilde u)=0 \quad \text{in } \mathcal{D}'(Q_1). \tag{10.18}$$

Note that if $t_1^*=t_1$, then $c(t)=\delta^{(j)}(t-t_1)+c_r(t)$. By Item 5 of Definition 3.2, $v-\tilde v$ is smooth in a neighborhood of $t_1^*$. Combining the latter with (10.17), we get (10.18).

For the rest of the proof we proceed as in the proof of Lemma 10.2. □

**Lemma 10.5.** *A $\mathcal{D}'_+(\Omega)$-solution to problem* (3.1)–(3.3) *is unique on $\Omega_2$.*

*Proof.* We follow the proof of Lemma 10.2 with $\Omega_1$ replaced by $\Omega_2$ and with minor changes caused by the fact that due to Lemmas 8.1 and 10.4, $u$ and $\tilde u$ are smooth on $\Omega_2\cap\Omega_+\cap\{(x,t): x>t-t_1^*-\varepsilon_1\}$. Hence (10.6) is true with $\mathcal{D}(0,t_1^*)$ replaced by $\mathcal{D}(t_1^*+\varepsilon_1/2,t_2^*)$. □

Continuing in this fashion, we eventually prove the uniqueness over subsequent $\Omega_i$ and $Q_i$ for any desired $i\in\mathbb{N}$. Combining it with Lemmas 10.1 and 10.3 and Theorem 2.8, we obtain Item 1 of Theorem 3.7.

Item 2 of Theorem 3.7 is a straightforward consequence of Item 1 of Theorem 3.7, Item 2 of Theorem 3.4, and Proposition 3.6.

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