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POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, we present the Green's functions for a second-order linear differential equation with three-point boundary conditions. We give exact expressions of the solutions for the linear three-point boundary problems by the Green's functions. As applications, we study uniqueness and iteration of the solutions for a nonlinear singular second-order three-point boundary value problem.

1. Introduction

The Green's function plays an important role in solving boundary-value problems of differential equations. The exact expressions of the solutions for some linear ordinary differential equations boundary value problems can be denoted by Green's functions of the problems (see [3, 12, 20]). The Green's function method might be used to obtain an initial estimate for shooting method. the Green's function method for solving the boundary value problem is an effective tool in numerical experiments [6]. Some boundary value problems for nonlinear differential equations can be transformed into the nonlinear integral equations the kernel of which are the Green's functions of corresponding linear differential equations. The integral equations can be solved by to investigate the property of the Green's functions (see [2, 4, 7, 8, 14]). The concept, the significance and the development of Green's functions can be seen in [15]. The other study of second-order three-point boundary value problems can be seen in [5, 9, 10, 16, 18, 19] and its references. In above literatures, the three-point boundary values are all same conditions $u(0) = 0, u(1) = ku(\eta)$, the investigation on the boundary condition $u'(0) = 0$, $u(1) = ku(\eta)$ can be seen [1, 11, 13, 17], the investigation for other three-point boundary conditions is few, since people may be not familiar with their Green's functions. The solutions of the Green's functions diffuse in the literature, there is a lack of uniform method. The undetermined parametric method we use in this paper is a universal method, the Green's functions of many boundary value problems for ordinary differential

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equations can obtained by similar the method. In addition, our Green's functions have orderly expressions.

We consider the Green's function for the following second-order linear differential equation three-point boundary value problems

$$u'' + f(t) = 0, \quad t \in [a, b], \tag{1.1}$$

subject to the boundary value conditions, respectively,

$$u(a) = 0, \quad u'(b) = ku(\eta); \tag{1.2}$$
$$u(a) = ku(\eta), \quad u'(b) = 0; \tag{1.3}$$
$$u(a) = 0, \quad u'(b) = ku'(\eta); \tag{1.4}$$
$$u(a) = ku'(\eta), \quad u'(b) = 0; \tag{1.5}$$
$$u'(a) = 0, \quad u(b) = ku(\eta); \tag{1.6}$$
$$u'(a) = ku(\eta), \quad u(b) = 0; \tag{1.7}$$
$$u'(a) = 0, \quad u(b) = ku'(\eta); \tag{1.8}$$
$$u'(a) = ku'(\eta), \quad u(b) = 0; \tag{1.9}$$

where $a < \eta < b$ and k is a constant.

This paper is organized as follows. In §2, we study the Green's function for the equation (1.1) satisfying the three-point boundary conditions (1.2) and give the expression of the unique solution by the Green's function, that incarnate the general method of deriving the Green's functions for a class of boundary problems. In §3, for some interrelated boundary conditions, we give the Green's functions of the problems directly, omitting the particular of derivation. The correctness of the Green's functions only need direct verification. As applications, in §4, we study the uniqueness of the solutions, the iteration and the rate of convergence by the iteration for a nonlinear singular second-order three-point boundary value problem.

2. The Green's Function of Equations (1.1) with the Boundary Condition (1.2)

About the boundary value problem (1.1)-(1.2), we have the following conclusions.

theorem 2.1. *If $k(\eta - a) \neq 1$, then the second-order linear three-point boundary value problem* (1.1)(1.2) *has a unique solution $u(t)$, which is given via*

$$u(t) = \int_0^1 G_1(t, s) f(s) ds, \tag{2.1}$$

where

$$G_1(t, s) = K(t, s) + \frac{k(t - a)}{1 - k(\eta - a)} K(\eta, s), \tag{2.2}$$

$$K(t, s) = \begin{cases} s - a, & a \leq s \leq t \leq b \\ t - a, & a \leq t < s \leq b. \end{cases} \tag{2.3}$$

Remark 2.2. We call $G_1(t, s)$ the Green's function of the boundary value problem (1.1)-(1.2).

Proof. It is well known that the second-order two-point linear boundary value problem

$$u'' + f(t) = 0, \quad t \in [a, b],$$
$$u(a) = 0, \quad u'(b) = 0$$

has a unique solution

$$w(t) = \int_a^b K(t,s)f(s)ds. \tag{2.4}$$

where $K(t,s)$ is as described in (2.3). From (2.4) we obtain

$$w(a) = 0, \quad w'(b) = 0, \quad w(\eta) = \int_a^b K(\eta,s)f(s)ds. \tag{2.5}$$

Assume that $u(t)$ is a solution of problem (1.1)-(1.2). Then $u''(t) = w''(t) = -f(t)$, thus, we can be assume that

$$u(t) = w(t) + c + dt, \tag{2.6}$$

where c, d are constants that will be determined. From (2.6) we know that

$$u'(t) = w'(t) + d. \tag{2.7}$$

Equations (2.5), (2.6) and (2.7) imply

$$u(a) = c + da,$$
$$u'(b) = d,$$
$$u(\eta) = c + d\eta + w(\eta).$$

Putting these into (1.2) yields

$$c + da = 0,$$
$$d = k(c + d\eta + w(\eta)).$$

Since $k(\eta - a) \neq 1$, solving the system of linear equations on the unknown numbers c, d, we obtain

$$c = \frac{-akw(\eta)}{1 - k(\eta - a)},$$
$$d = \frac{kw(\eta)}{1 - k(\eta - a)},$$

hence, $c + dt = \frac{k(t-a)}{1-k(\eta-a)}w(\eta)$. Putting this into (2.6), we obtain

$$u(t) = w(t) + \frac{k(t-a)}{1 - k(\eta - a)}w(\eta),$$

which a solution of (1.1)-(1.2). This together with (2.4) imply

$$u(t) = \int_a^b K(t,s)f(s)ds + \frac{k(t-a)}{1 - k(\eta - a)} \int_a^b K(\eta,s)f(s)ds. \tag{2.8}$$

Consequently, (2.1) holds.

The uniqueness of a solutions (1.1) (1.2) follows from the fact that the corresponding homogeneous problem has only the trivial solution. □

From Theorem 2.1 we obtain the following corollary.

Corollary 2.3. *Suppose the nonlinear function $g(t,u)$ is continuous on $[a,b]\times R$, then if $k(\eta-a)\neq 1$, the nonlinear three-point boundary-value problem*

$$u''+g(t,u)=0,\quad t\in[a,b],$$
$$u(a)=0,\quad u'(b)=ku(\eta)$$

is equivalent to the nonlinear integral equation

$$u(t)=\int_a^b G_1(t,s)g(s,u(s))ds$$

with $G_1(t,s)$ as in (2.2).

Example 2.4. The second-order three-point linear boundary value problem

$$u''(t)+\cos(t)=0,\quad t\in[0,1],$$
$$u(0)=0,\quad u'(1)=-\frac{3}{2}u(\frac{1}{3})$$

has an unique solution

$$u_1(t)=\frac{2}{3}t\sin(1)-t\cos\left(\frac{1}{3}\right)+t+\cos(t)-1,\quad 0\le t\le 1. \tag{2.9}$$

It can be obtained by letting $a=0$, $b=1$, $\eta=\frac{1}{3}$, $k=-\frac{3}{2}$, $f(t)=\cos(t)$ in Theorem 2.1 that

$$u_1(t)=\int_0^1 B(t,s)\cos(s)ds-t\int_0^1 B(\frac{1}{3},s)\cos(s)ds,$$

where $B(t,s)=\min\{t,s\}$, $t,s\in[0,1]$. Therefore, (2.9) is obtained by direct computation. Some properties of $u_1(t)$ are shown in the image Figure 1.

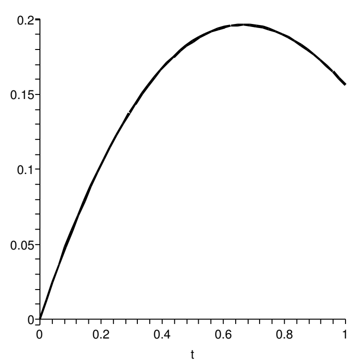

FIGURE 1. Graph of $u_1(t)$

3. The Relate Results for Other Boundary Conditions

In this section, we give the Green's functions for some boundary value problems directly via the following table, omitting the particular of derivation. The proof is similar to that of Theorem 2.1. Similarly, the unique solutions of the linear problems can be denoted by its Green's functions. Some nonlinear boundary value problems can be transformed into the nonlinear integral equations the kernel of which are the Green's functions of the corresponding linear problems.

Equation: $u''(t)+f(t)=0$, $t\in[a,b]$, $a<\eta<b$, k is a constant.			
no.	assume	boundary	Green's function
1	$k(\eta-a)\neq 1$	(1.2)	$G_1(t,s)=K(t,s)+\frac{k(t-a)}{1-k(\eta-a)}K(\eta,s)$
2	$k\neq 1$	(1.3)	$G_2(t,s)=K(t,s)+\frac{k}{1-k}K(\eta,s)$,
3	$k\neq 1$	(1.4)	$G_3(t,s)=K(t,s)+\frac{k(t-a)}{1-k}K_t(\eta,s)$,
4		(1.5)	$G_4(t,s)=K(t,s)+kK_t(\eta,s)$,
5	$k\neq 1$	(1.6)	$G_5(t,s)=H(t,s)+\frac{k}{1-k}H(\eta,s)$,
6	$k(b-\eta)\neq -1$	(1.7)	$G_6(t,s)=H(t,s)-\frac{k(b-t)}{1+k(b-\eta)}H(\eta,s)$,
7		(1.8)	$G_7(t,s)=H(t,s)+kH_t(\eta,s)$,
8	$k\neq 1$	(1.9)	$G_8(t,s)=H(t,s)-\frac{k(b-t)}{1-k}H_t(\eta,s)$,

where

$$K(t,s)=\begin{cases} s-a, & a\le s\le t\le b\\ t-a, & a\le t<s\le b;\end{cases}\qquad K_t(\eta,s)=\begin{cases}0, & a\le s<\eta,\\ 1, & \eta<s\le b;\end{cases}$$

$$H(t,s)=\begin{cases} b-t, & a\le s\le t\le b\\ b-s, & a\le t<s\le b;\end{cases}\qquad H_t(\eta,s)=\begin{cases}-1, & a\le s<\eta,\\ 0, & \eta<s\le b.\end{cases}$$

4. Applications in Nonlinear Singular Boundary Value Problems

In this section,we study the iteration process for the following nonlinear three-point boundary value problem

$$\begin{gathered} u''+f(t,u)=0,\quad t\in(0,1),\\ u(0)=0,\quad u'(1)=ku(\eta),\end{gathered}\tag{4.1}$$

with $\eta\in(0,1)$, $k\eta<1$, $f(t,u)$ may be singular at $t=0$ and/or $t=1$.

Concerning the function f we impose the following hypotheses:

$$\begin{gathered} f(t,u)\text{ is nonnegative continuous on }(0,1)\times[0,+\infty),\\ f(t,u)\text{ is monotone increasing on }u,\text{ for fixed }t\in(0,1),\\ \text{there exist }q\in(0,1)\text{ such that}\\ f(t,ru)\ge r^q f(t,u),\quad\forall\ 0<r<1,\ (t,u)\in(0,1)\times[0,+\infty).\end{gathered}\tag{4.2}$$

Obviously, from (4.2) we obtain

$$f(t,\lambda u)\le\lambda^q f(t,u),\quad\forall\ \lambda>1,\ (t,u)\in(0,1)\times[0,+\infty).\tag{4.3}$$

It is easy to see that if $0<\alpha_i<1$, $a_i(t)$ are nonnegative continuous on $(0,1)$, for $i=0,1,2,\dots,m$, then $f(t,u)=\sum_{i=1}^m a_i(t)u^{\alpha_i}$ satisfy the condition (4.2).

Concerning the boundary value problem (4.1), we have following conclusions.

theorem 4.1. *Suppose the function $f(t,u)$ satisfy the condition (4.2), and*

$$0<\int_0^1 f(t,t)dt<\infty. \tag{4.4}$$

Then the problem (4.1) *has an unique solution $w(t)$ in $D\bigcap C^2(0,1)$, here*

$$D=\{x\in C[0,1]\mid \exists M_x\geq m_x>0, \text{such that}\, m_x t\leq x(t)\leq M_x t,\ t\in I\}.$$

Constructing successively the sequence of functions

$$h_n(t)=\int_0^1 G(t,s)f(s,h_{n-1}(s))ds,\quad n=1,2,\dots, \tag{4.5}$$

for any initial function $h_0(t)\in D$, then $\{h_n(t)\}$ must converge to $w(t)$ uniformly on $[0,1]$ and the rate of convergence is

$$\max_{t\in[0,1]}|h_n(t)-w(t)|=O\big(1-N^{q^n}\big), \tag{4.6}$$

where $0<N<1$, which depends on initial function $h_0(t)$,

$$G(t,s)=B\left(t,s\right)+\frac{ktB\left(\eta,s\right)}{1-k\eta},\quad B(t,s)=\min\{t,s\},\quad t,s\in[0,1]. \tag{4.7}$$

Proof. Let $J=(0,1)$, $I=[0,1]$, $R^+=[0,+\infty)$,

$$P=\{x(t)\big|\ x(t)\in C(I),x(t)\geq 0\},$$

$$Fx(t)=\int_0^1 G(t,s)f(s,x(s))ds,\quad \forall x(t)\in D. \tag{4.8}$$

It is easy to see that the operator $F\colon D\to P$ is increasing. By direct verifications we know that if $u\in D$ satisfies

$$u(t)=Fu(t),\quad t\in I, \tag{4.9}$$

then $u\in C^1(I)\bigcap C^2(J)$ is a solution of (4.1).

For any $x\in D$, there exist positive numbers $0<m_x<1<M_x$ such that

$$\begin{gathered} m_x\ s\leq x(s)\leq M_x\ s,\quad s\in I,\\ (m_x)^q f(s,s)\leq f(s,x(s))\leq (M_x)^q f(s,s),\quad s\in J. \end{gathered} \tag{4.10}$$

By (4.7) we have

$$G(t,s)=B(t,s)+\frac{kt}{1-k\eta}B(\eta,s)\geq t\frac{k}{1-k\eta}B(\eta,s), \tag{4.11}$$

$$G(t,s)\leq t+\frac{kt}{1-k\eta}B(\eta,s)\leq t\left(1+\frac{k}{1-k\eta}B(\eta,s)\right). \tag{4.12}$$

Using (4.8), (4.3), (4.10), (4.11), (4.12) and the conditions (4.2), we obtain

$$Fx(t)\geq t(m_x)^q\frac{k}{1-k\eta}\int_0^1 B(\eta,s)f(s,s)ds,\quad t\in I, \tag{4.13}$$

$$\begin{aligned} Fx(t)&=\int_0^1 G(t,s)f(s,x(s))ds\\ &\leq t(M_x)^q\int_0^1\left(1+\frac{k}{1-k\eta}B(\eta,s)\right)f(s,s)ds,\quad t\in I. \end{aligned} \tag{4.14}$$

Equations (4.4), (4.13) and (4.14) imply that $F\colon D\to D$.

For any $h_0 \in D$, we let

$$\begin{aligned} l_{h_0} &= \sup\left\{l > 0 : lh_0(t) \le (Fh_0)(t),\ t \in I\right\}, \\ L_{h_0} &= \inf\left\{L > 0 : (Fh_0)(t)) \le Lh_0(t),\ t \in I\right\}, \\ m = \min\{1, (l_{h_0})^{\frac{1}{1-q}}\}, &\quad M = \max\{1, (L_{h_0})^{\frac{1}{1-q}}\}, \end{aligned} \tag{4.15}$$

$$\begin{gathered} u_0(t) = mh_0(t), \quad v_0(t) = Mh_0(t), \\ u_n(t) = Fu_{n-1}(t), \quad v_n(t) = Fv_{n-1}(t), \quad n = 0, 1, 2, \dots. \end{gathered} \tag{4.16}$$

Since the operator F is increasing, (4.2), (4.15) and (4.16) imply

$$u_0(t) \le u_1(t) \le \dots \le u_n(t) \dots \le v_n(t) \le \dots \le v_1(t) \le v_0(t), \quad t \in I. \tag{4.17}$$

For $t_0 = m/M$, from (4.8), (4.2) and (4.16), it can obtained by induction that

$$u_n(t) \ge (t_0)^{q^n} v_n(t), \quad t \in I,\ n = 0, 1, 2, \dots. \tag{4.18}$$

From (4.17) and (4.18) we know that

$$0 \le u_{n+p}(t) - u_n(t) \le v_n(t) - u_n(t) \le \big(1 - (t_0)^{q^n}\big) Mh_0(t), \forall n, p, \tag{4.19}$$

so that there exists a function $w(t) \in D$ such that

$$u_n(t) \to w(t), \quad v_n(t) \to w(t), \quad \text{(uniformly on } I), \tag{4.20}$$

$$u_n(t) \le w(t) \le v_n(t), \quad t \in I,\ n = 0, 1, 2, \dots. \tag{4.21}$$

From the operator F being increasing and (4.16) we have

$$u_{n+1}(t) = Fu_n(t) \le Fw(t) \le Fv_n(t) = v_{n+1}(t), \quad n = 0, 1, 2, \dots.$$

This together with (4.20) and uniqueness of the limit imply that $w(t)$ satisfy (4.9), hence $w(t) \in C^1(I) \bigcap C^2(J)$ is a solution of (4.1).

Form (4.5) (4.16) and the operator F being increasing, we obtain

$$u_n(t) \le h_n(t) \le v_n(t), \quad t \in I,\ n = 0, 1, 2, \dots, \tag{4.22}$$

thus, it follows from (4.19), (4.21) and (4.22) that

$$\begin{aligned} |h_n(t) - w(t)| &\le |h_n(t) - u_n(t)| + |u_n(t) - w(t)| \\ &\le 2|v_n(t) - u_n(t)| \\ &\le \big(1 - (t_0)^{q^n}\big) M|h_0(t)|. \end{aligned}$$

Therefore,

$$\max_{t\in I} |h_n(t) - w(t)| \le \big(1 - (t_0)^{q^n}\big) M \max_{t\in I} |h_0(t)|.$$

So that (4.6) holds. From $h_0(t)$ is arbitrary in D we know that $w(t)$ is the unique solution of the equation (4.9) in D. $\square$

Remark 4.2. If $f(t, u)$ is continuous on $I \times R^+$, then it is quite evident that the condition (4.4) holds. Hence the unique solution $w(t)$ is in $C^2(I)$.

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