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# FRICTIONLESS CONTACT PROBLEM WITH ADHESION FOR NONLINEAR ELASTIC MATERIALS

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Abstract. We consider a quasistatic frictionless contact problem for a nonlinear elastic body. The contact is modelled with Signorini's conditions. In this problem we take into account of the adhesion which is modelled with a surface variable, the bonding field, whose evolution is described by a first order differential equation. We derive a variational formulation of the mechanical problem and we establish an existence and uniqueness result by using arguments of time-dependent variational inequalities, differential equations and Banach fixed point. Moreover, we prove that the solution of the Signorini contact problem can be obtained as the limit of the solution of a penalized problem as the penalization parameter converges to 0.

## 1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve a complicated surface phenomena, and are modelled with highly nonlinear initial boundary value problems. Taking into account various frictional contact conditions associated with behavior laws becoming more and more complex leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. A first tentative to study frictional contact problems within the framework of variational inequalities was made in [6]. The mathematical, mechanical and numerical state of the art can be found in [13]. In this paper, we study a mathematical model which describes a quasistatic frictionless adhesive contact problem between a deformable body and a rigid foundation. Models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 7, 17]. As in [8, 9], we use the bonding field as an additional state variable $\beta$, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [10, 12, 14, 15, 16]. In this work, we

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extend the result established in [17] for linear elastic bodies to the nonlinear elastic bodies where the adhesion is taken into account and the contact is modelled with Signorini's conditions. We derive a variational formulation of the mechanical problem which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solutions. Moreover, we study the behavior of the solution of a penalized problem as the penalization parameter converges to 0. We also wish to make it clear that from [11] it follows that the model does not allow for complete debonding field in finite time.

This paper is structured as follows. In section 2 we present some notations and give the variational formulation. In section 3 we state and prove our main existence and uniqueness result, Theorem 2.2. Finally, in section 4, we prove a convergence result of a penalized problem, Theorem 4.2.

## 2. Variational Formulation

Let $\Omega \subset \mathbb{R}^d$; $(d = 2, 3)$, be the domain initially occupied by an elastic body. $\Omega$ is supposed to be open, bounded, with a sufficiently regular boundary $\Gamma$. $\Gamma$ is partitioned into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and *meas* $\Gamma_1 > 0$. The body is acted upon by a volume force of density $\varphi_1$ on $\Omega$ and a surface traction of density $\varphi_2$ on $\Gamma_2$. On $\Gamma_3$ the body is in adhesive frictionless contact with a rigid foundation. We use a nonlinear elastic constitutive law to the material behavior and an ordinary differential equation to describe the evolution of the bonding field.

Thus, the classical formulation of the mechanical problem is written as follows.

**Problem $P_1$.** Find $u : \Omega \times [0, T] \to \mathbb{R}^d$, $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{aligned} u_\nu \le 0, \sigma_\nu + c_\nu R(u_\nu)\beta^2 \le 0 \\ (\sigma_\nu + c_\nu R(u_\nu)\beta^2)u_\nu = 0 \end{aligned}\right\} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$-\sigma_\tau = p_\tau(\beta)R^*(u_\tau) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\dot{\beta} = -c_\nu \beta_+ (R(u_\nu))^2 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.8}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which $F$ is a given function and $\varepsilon(u)$ denotes the small strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which $\nu$ denotes the unit outward normal vector on $\Gamma$ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (2.5) represent the Signorini conditions with adhesion in which $c_\nu$ is a given adhesion coefficient which may dependent on $x \in \Gamma_3$ and $R : \mathbb{R} \to \mathbb{R}$ is a truncation operator defined as

$$R(s) = \begin{cases} -L & \text{if } s \le -L, \\ s & \text{if } |s| < L, \\ L & \text{if } s \ge L. \end{cases}$$

Here $L > 0$ is the characteristic lengh of the bond, beyond which it does not offer any additional traction (see [14]). By choosing $L$ very large, we can assume that $R(u_\nu) = u_\nu$ and therefore we recover the contact condition

$$\begin{gathered} u_\nu \le 0, \sigma_\nu + c_\nu R(u_\nu)\beta^2, \\ (\sigma_\nu + c_\nu R(u_\nu)\beta^2)u_\nu = 0 \quad \text{on } \Gamma_3 \times (0,T). \end{gathered}$$

These conditions were already used in [2, 5, 12]. Next, equation (2.6) represents the adhesive contact. We assume that the resistance to tangential motion is generated by the glue, in comparison to which the frictional traction can be neglected. Therefore, the tangential contact traction depends only on the bonding field and the tangential displacement, thus,

$$-\sigma_\tau = p_\tau(\beta)R^*(u_\tau).$$

Where the truncation operator $R^*$ is defined by

$$R^*(v) = \begin{cases} v & \text{if } \|v\| \le L, \\ Lv/\|v\| & \text{if } \|v\| > L, \end{cases}$$

and $p_\tau$ is a prescribed, nonnegative tangential stiffness function. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field, in which $r_+ = \max\{r, 0\}$, and it was already used in [4]. Since $\dot\beta \le 0$ on $\Gamma_3 \times (0,T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Finally, (2.8) is the initial condition, in which $\beta_0$ denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$); and $\|\cdot\|$ represents the Euclidian norm on $\mathbb{R}^d$ and $S_d$. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $\|v\| = (v.v)^{1/2}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau.\tau)^{1/2}$. Here and below, the indices $i$ and $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$\begin{gathered} H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \\ Q = \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \\ H(\operatorname{div};\Omega) = \{\sigma \in Q; \operatorname{div}\sigma \in H\}. \end{gathered}$$

Note that $H$ and $Q$ are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad (\sigma,\tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The small strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = \{1,\dots,d\};$$

where $\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of $\sigma$. For every $v \in H_1$ we denote by $v_\nu$ and $v_\tau$ the normal and tangential components of $v$ on the boundary $\Gamma$ given by $v_\nu = v.\nu$, $v_\tau = v - v_\nu\nu$. Similary, for a regular tensor field $\sigma : \Omega \to S_d$, we define the normal and tangential components of $\sigma$ by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu$$

and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where $da$ is the surface measure element. Let $V$ be the closed subspace of $H_1$ defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq 0 \text{ on } \Gamma_3\}.$$

Since meas$\Gamma_1 > 0$, the following Korn's inequality holds [6],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where the constant $c_\Omega$ depends only on $\Omega$ and $\Gamma_1$. We equip $V$ with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

For $p \in [1, \infty]$ , we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to $X$; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \quad \varphi_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d) \tag{2.11}$$

and we denote by $f(t)$ the element of $V$ defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).vdx + \int_{\Gamma_2} \varphi_2(t).vda \quad \forall v \in V, \ t \in [0, T]. \tag{2.12}$$

Using (2.11) and (2.12) yield $f \in W^{1,\infty}(0, T; V)$.

In the study of the mechanical problem $P_1$ we assume that $F : \Omega \times S_d \to S_d$, satisfies

$$\begin{aligned}
&\text{(a) there exists } M > 0 \text{ such that } \|F(x,\varepsilon_1) - F(x,\varepsilon_2)\| \leq \\
&\quad M\|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d \text{ a.e. x in } \Omega \\
&\text{(b) there exists } m > 0 \text{ such that } (F(x,\varepsilon_1) - F(x,\varepsilon_2)).(\varepsilon_1 - \\
&\quad \varepsilon_2) \geq m\|\varepsilon_1 - \varepsilon_2\|^2, \text{ for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \text{ a.e. } x \text{ in } \Omega; \\
&\text{(c) the mapping } x \to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\
&\quad \text{for any } \varepsilon \text{ in } S_d \\
&\text{(d) } F(x,0) = 0 \text{ for all } x \text{ in } \Omega.
\end{aligned} \tag{2.13}$$

**Remark 2.1.** $F(x,\tau(x)) \in Q$, for all $\tau \in Q$ and thus it is possible to consider $F$ as an operator defined from $Q$ into $Q$.

The adhesion coefficient satisfies

$$c_\nu \in L^\infty(\Gamma_3) \quad \text{and} \quad c_\nu \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{2.14}$$

Also we assume that the initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3); \quad 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3. \tag{2.15}$$

As in [4] we assume that the tangential contact function satisfies

$$\begin{aligned}
&\text{(a) } p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+;\\
&\text{(b) there exists } L_\tau > 0 \text{ such that } |p_\tau(x,\beta_1) - p_\tau(x,\beta_2)| \leq\\
&\qquad L_\tau|\beta_1 - \beta_2| \text{ for all } \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3;\\
&\text{(c) there exists } M_r > 0 \text{ such that } |p_\tau(x,\beta)| \leq M_\tau \text{ for all}\\
&\qquad \beta \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3;\\
&\text{(d) for any } \beta \in \mathbb{R},\ x \to p_\tau(x,\beta) \text{ is measurable on } \Gamma_3;\\
&\text{(e) the mapping } x \to p_\tau(x,0) \text{ belongs to } L^2(\Gamma_3).
\end{aligned} \tag{2.16}$$

Next, we define the adhesion functional $j_T : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_T(\beta,u,v) = \int_{\Gamma_3} p_\tau(\beta) R^*(u_\tau).v_\tau da \quad \forall \beta \in L^\infty(\Gamma_3), \forall u,v \in V.$$

We note that $j_T$ satisfies the property.

$$j_T(\beta,u,-v) = -j_T(\beta,u,v).$$

On the other hand we have

$$\begin{aligned}
&j_T(\beta_1,u_1,u_2-u_1) + j_T(\beta_2,u_2,u_1-u_2)\\
&= \int_{\Gamma_3} p_\tau(\beta_1)(R^*(u_{1\tau}) - R^*(u_{2\tau})).(u_{2\tau} - u_{1\tau})da\\
&\quad + \int_{\Gamma_3} (p_\tau(\beta_1) - p_\tau(\beta_2))R^*(u_{2\tau}).(u_{2\tau} - u_{1\tau})da,
\end{aligned}$$

and since $(R^*(u_{1\tau}) - R^*(u_{2\tau})).(u_{2\tau} - u_{1\tau}) \leq 0$ a.e. on $\Gamma_3$, we obtain

$$\begin{aligned}
&j_T(\beta_1,u_1,u_2-u_1) + j_T(\beta_2,u_2,u_1-u_2)\\
&\leq \int_{\Gamma_3} (p_\tau(\beta_1) - p_\tau(\beta_2))R^*(u_{2\tau}).(u_{2\tau} - u_{1\tau})da.
\end{aligned}$$

Using now the inequality $|R^*(u_{2\tau})| \leq L$ valid a.e. on $\Gamma_3$ and the property (2.16) (b) of the function $p_\tau$ we deduce that

$$j_T(\beta_1,u_1,u_2-u_1) + j_T(\beta_2,u_2,u_1-u_2) \leq C\int_{\Gamma_3} |\beta_1 - \beta_2||u_1 - u_2|da,$$

where $C > 0$. Next, we combine the previous inequality with (2.10) to obtain

$$j_T(\beta_1,u_1,u_2-u_1) + j_T(\beta_2,u_2,u_1-u_2) \leq C\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|u_1 - u_2\|_V.$$

By choosing $\beta_1 = \beta_2 = \beta$ in the previous inequality we find

$$j_T(\beta,u_1,u_2-u_1) + j_T(\beta,u_2,u_1-u_2) \leq 0.$$

Using the Lipschitz continuity of the truncation operators $R$ and $R^*$ and the boundness of the function $p_\tau$, we find

$$j_T(\beta, u_1, v) - j_T(\beta, u_2, v) \le C\|u_1 - u_2\|_V \|v\|_V,$$

and also we have $j_T(\beta, v, v) \ge 0$.

As in [17] we define the adhesion functional $j_N : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_N(\beta, u, v) = -\int_{\Gamma_3} c_\nu \beta^2 (-R(u_\nu))_+ v_\nu da \quad \forall \beta \in L^\infty(\Gamma_3)\ \forall u, v \in V,$$

and we recall that the functional $j_N$ satisfies the same properties satisfied by the functional $j_T$. Next, we define the adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j = j_N + j_T,$$

where

$$\begin{aligned} j(\beta, u, v) &= j_N(\beta, u, v) + j_T(\beta, u, v) \\ &= -\int_{\Gamma_3} c_\nu \beta^2 (-R(u_\nu))_+ v_\nu da + \int_{\Gamma_3} p_\tau(\beta) R^*(u_\tau).v_\tau da \end{aligned}$$

for all $\beta \in L^\infty(\Gamma_3)$, for all $u, v \in V$. Then from the properties satisfied by the functionals $j_N$ and $j_T$ we deduce that the adhesion functional $j$ satisfies the following properties

$$j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2) \le C\|\beta_1 - \beta_2\|_{L^2(\Gamma_2)} \|u_1 - u_2\|_V, \tag{2.17}$$

$$j(\beta, u_1, u_2 - u_1) + j(\beta, u_2, u_1 - u_2) \le 0, \tag{2.18}$$

$$j(\beta, u_1, v) - j(\beta, u_2, v) \le C\|u_1 - u_2\|_V \|v\|_V, \tag{2.19}$$

$$j(\beta, v, v) \ge 0. \tag{2.20}$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula and techniques similar to those exposed in [6] that the problem $P_1$ has the following variational formulation.

**Problem $P_2$.** Find $(u, \beta) \in W^{1,\infty}(0, T; V) \times W^{1,\infty}(0, T; L^\infty(\Gamma_3))$ such that $u(t) \in K$ for all $t \in [0, T]$, and

$$\begin{aligned} &(F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_Q + j(\beta(t), u(t), v - u(t)) \\ &\ge (f(t), v - u(t))_V \quad \forall v \in K, t \in [0, T], \end{aligned} \tag{2.21}$$

$$\dot\beta(t) = -c_\nu(\beta(t))_+ (R(u_\nu(t)))^2 \quad \text{a.e. } t \in (0, T), \tag{2.22}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.23}$$

Our main result of this section is stated as theorem and will be established in the next section.

**Theorem 2.2.** *Let* (2.11), (2.13), (2.14), (2.15), (2.16) *hold. Then problem $P_2$ has a unique solution.*

## 3. Existence and Uniqueness Result

The proof of Theorem 2.2 is carried out in several steps. In the first step, for a given $\beta \in L^\infty(\Gamma_3)$ such that $0 \le \beta \le 1$ a.e. on $\Gamma_3$, we consider the following variational problem.

**Problem $P_{1\beta}$.** Find $u_\beta \in C([0,T];V)$ such that for all $t \in [0,T]$, $u_\beta(t) \in K$ and

$$\begin{aligned}&(F\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + j(\beta(t), u_\beta(t), v - u_\beta(t))\\ &\geq (f(t), v - u_\beta(t))_V, \quad \forall v \in K.\end{aligned} \tag{3.1}$$

We obtain the following result.

**Lemma 3.1.** *Problem $P_{1\beta}$ has a unique solution.*

*Proof.* For all $t \in [0,T]$, we consider the operator $T_t : V \to V$ defined by

$$(T_t u, v)_V = (F\varepsilon(u(t)), \varepsilon(v))_Q + j(\beta(t), u(t), v).$$

It suffices to see from the assumptions (2.13)(a)-(b) are satisfied by $F$ and the properties satisfied by $j$, that $T_t$ is strongly monotone and Lipschitz continuous; since $K$ is a closed convex subset of $V$, it follows from the theory of elliptic variational inequalities [1] that there exists a unique element $u_\beta(t)$ which solves (3.1). Thus, $T_t$ is invertible and its inverse $T_t^{-1} : V \to V$ has the same properties as $T_t$. Therefore, using the regularity of $f$, $u_\beta = T_t^{-1} f$ satisfies $u_\beta \in C([0,T];V)$. □

In the second step we consider the following problem.

**Problem $P_{2\beta}$.** Find $\beta_a \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ such that

$$\dot{\beta}_a(t) = -c_\nu(\beta_a(t))_+ (R(u_{\beta_a\nu}(t)))^2 \ a.e.\ t \in (0,T), \tag{3.2}$$

$$\beta_a(0) = \beta_0. \tag{3.3}$$

We can prove the following lemma.

**Lemma 3.2.** *Problem $P_{2\beta}$ has a unique solution.*

*Proof.* For $k > 0$ we introduce the space

$$X = \big\{\beta \in C([0,T];L^2(\Gamma_3));\ \sup_{t\in[0,T]} [\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}] < +\infty\big\}.$$

which is a Banach space for the norm

$$\|\beta\|_X = \sup_{t\in[0,T]} [\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}].$$

Consider the mapping $\Phi : X \to X$ given by

$$\Phi\beta(t) = \beta_0 - \int_0^t c_\nu(\beta(s))_+ (R(u_{\beta\nu}(s)))^2 ds.$$

Then we get

$$\begin{aligned}&\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)}\\ &\leq C\int_0^t \|(\beta_1(s))_+(R(u_{\beta_1\nu}(s)))^2 - (\beta_2(s))_+(R(u_{\beta_2\nu}(s)))^2\|_{L^2(\Gamma_3)} ds.\end{aligned}$$

Using the definition of $R$, and writing

$$(\beta_1(s))_+ = (\beta_1(s))_+ - (\beta_2(s))_+ + (\beta_2(s))_+,$$

since

$$|(\beta_1(s))_+ - (\beta_2(s))_+| \leq |\beta_1(s) - \beta_2(s)|,$$

we obtain

$$\begin{aligned}
&\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)}\\
&\le C\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds + C\int_0^t \|u_{\beta_1\nu}(s)-u_{\beta_2\nu}(s)\|_{L^2(\Gamma_3)}ds.
\end{aligned}$$

Now using (2.10), we have

$$\begin{aligned}
&\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)}\\
&\le C\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds + C\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V ds.
\end{aligned}$$

On the other hand using the inequality (3.1), the assumption (2.13)(b) on $F$ and the property (2.19) of $j$, we get

$$\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V \le C\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}. \tag{3.4}$$

Whence, we obtain

$$\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds,$$

and

$$\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)} = \int_0^t \exp(ks)[\exp(-ks)\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}]ds.$$

Since

$$\int_0^t \exp(ks)[\exp(-ks)\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}]ds \le \|\beta_1-\beta_2\|_X \int_0^t \exp(ks)ds,$$

and

$$\|\beta_1-\beta_2\|_X \int_0^t \exp(ks)ds = \|\beta_1-\beta_2\|_X \frac{\exp(kt)-1}{k} \le \|\beta_1-\beta_2\|_X \frac{\exp(kt)}{k},$$

we deduce

$$\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)} \le C\|\beta_1-\beta_2\|_X \frac{\exp(kt)}{k},$$

which implies

$$\exp(-kt)\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)} \le \frac{C}{k}\|\beta_1-\beta_2\|_X\,.$$

So we obtain

$$\|\Phi\beta_1-\Phi\beta_2\|_X \le \frac{C}{k}\|\beta_1-\beta_2\|_X. \tag{3.5}$$

This inequality shows that for $k$ sufficiently large $\Phi$ is a contraction. Then we deduce, by using the fixed point theorem that $\Phi$ has a unique fixed point $\beta_a$ which satisfies (3.2) and (3.3). Moreover from (2.15) and (3.2), see [17] for details, we deduce that

$$0 \le \beta(t) \le 1 \ \forall t \in [0,T], \quad \text{a.e. on } \Gamma_3.$$

□

Now, to prove the existence and uniqueness of the solution for Theorem 2.2, let $\beta_a$ be the fixed point of $T$ and let $u_a$ be the solution of problem $P_{1\beta}$ for $\beta = \beta_a$, i.e., $u_a = u_{\beta_a}$. Using the same arguments used in the proof of (3.4), we get

$$\|u_a(t_1) - u_a(t_2)\|_V \le c\|\beta_a(t_1) - \beta_a(t_2)\|_{L^2(\Gamma_3)} \quad \forall t_1, t_2 \in [0, T]. \tag{3.6}$$

Since $T\beta_a = \beta_a$ we deduce from lemma 3.2 that $\beta_a \in W^{1,\infty}(0, T; L^2(\Gamma_3))$ and then (3.6) implies that $u_a \in W^{1,\infty}(0, T; V)$. Moreover, we conclude by (3.1), (3.2) and (3.3) that $(u_a, \beta_a)$ is a solution to problem $P_2$. To prove the uniqueness of the solution, suppose that $(u, \beta)$ is a solution of problem $P_2$ which satisfies (2.21) and (2.22). It follows from (2.21) that $u$ is a solution to problem $P_{1\beta}$, and from lemma 3.1 that $u = u_\beta$. Take $u = u_\beta$ in (2.21) and use the initial condition (2.23), we deduce that $\beta$ is a solution to problem $P_{2\beta}$. Therefore, we get from lemma 3.2, $\beta = \beta_a$ and we conclude that $(u_a, \beta_a)$ is a unique solution to problem $P_2$.

## 4. The Penalized Problem

Let us consider the following strong formulation of the penalized problem with frictionless contact and adhesion, for $\delta > 0$, which can be seen as a frictionless contact and adhesion with a normal compliance.

**Problem $P_{1\delta}$.** Find $u_\delta : \Omega \times [0, T] \to \mathbb{R}^d$, $\beta_\delta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{4.1}$$

$$\sigma = F\varepsilon(u_\delta) \quad \text{in } \Omega \times (0, T), \tag{4.2}$$

$$u_\delta = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{4.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{4.4}$$

$$-\sigma_\nu = \frac{(u_{\delta\nu})_+}{\delta} - c_\nu \beta^2(-R(u_{\delta\nu}))_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{4.5}$$

$$-\sigma_\tau = p_\tau(\beta_\delta) R^*(u_{\delta\tau}) \text{ on } \Gamma_3 \times (0, T), \tag{4.6}$$

$$\dot\beta_\delta = -c_\nu(\beta_\delta)_+(R(u_{\delta\nu}))^2 \text{ on } \Gamma_3 \times (0, T), \tag{4.7}$$

$$\beta_\delta(0) = \beta_0 \text{ on } \Gamma_3. \tag{4.8}$$

The problem $P_{1\delta}$ has the following variational formulation.

**Problem $P_{2\delta}$.** Find $(u_\delta, \beta_\delta) \in W^{1,\infty}(0, T; V) \times W^{1,\infty}(0, T; L^\infty(\Gamma_3)))$ such that

$$\begin{aligned}&(F\varepsilon(u_\delta(t)), \varepsilon(v))_Q + \frac{1}{\delta}((u_{\delta\nu}(t))_+, v_\nu)_{L^2(\Gamma_3)} + j(\beta_\delta(t), u_\delta(t), v)\\ &= (f(t), v)_V \quad \forall v \in V, t \in [0, T],\end{aligned} \tag{4.9}$$

$$\dot\beta_\delta(t) = -c_\nu(\beta_\delta(t))_+(R(u_{\delta\nu}(t)))^2 \text{ on } \Gamma_3 \times (0, T), \tag{4.10}$$

$$\beta_\delta(0) = \beta_0 \text{ on } \Gamma_3. \tag{4.11}$$

We can prove the following result.

**Theorem 4.1.** *Problem $P_{2\delta}$ has a unique solution.*

The proof of the above theorem is similar to that of Theorem 2.2; however we omit some of the details. Here are the main steps of the proof.

(i) For any $\beta \in L^2(\Gamma_3)$ such that $0 \le \beta(t) \le 1$ for all $t \in [0,T]$, a.e. on $\Gamma_3$, we prove that there exists a unique $u_\delta \in C([0,T];V)$ such that

$$\begin{aligned}&(F\varepsilon(u_\delta(t)),\varepsilon(v))_Q + \frac{1}{\delta}((u_{\delta\nu}(t))_+, v_\nu)_{L^2(\Gamma_3)} + j(\beta(t), u_\delta(t), v)\\ &= (f(t), v)_V \quad \forall v \in V, t \in [0,T].\end{aligned} \tag{4.12}$$

To prove this step, for $t \in [0,T]$ and $u, v \in V$, we define the operator $T_t : V \to V$ by

$$(T_t u, v)_V = (F\varepsilon(u(t)),\varepsilon(v))_Q + \frac{1}{\delta}((u_\nu(t))_+, v_\nu)_{L^2(\Gamma_3)} + j(\beta(t), u(t), v)$$

In the study of the operator $T_t$ we need to recall that for $a, b \in \mathbb{R}$, we have

$$\begin{gathered}(a_+ - b_+)(a-b) \ge (a_+ - b_+)^2,\\ |a_+ - b_+| \le |a-b|.\end{gathered} \tag{4.13}$$

Using (2.13)(a), (2.13)(b), the properties (2.17)–(2.20) are satisfied by the functional $j$ and the property (4.13) to see that the operator $T_t$ is strongly monotone and Lipschitz continuous, and therefore invertible.
(ii) There exists a unique $\beta_\delta$ such that

$$\beta_\delta \in W^{1,\infty}(0,T;L^2(\Gamma_3)), \tag{4.14}$$

$$\dot\beta_\delta(t) = -c_\nu(\beta_\delta(t))_+(R(u_{\delta\nu}(t)))^2 \quad \text{a.e. } t \in (0,T), \tag{4.15}$$

$$\beta_\delta(0) = \beta_0. \tag{4.16}$$

(iii) Let $\beta_\delta$ defined in ii) and denote again by $u_\delta$ the function obtained in step i) for $\beta = \beta_\delta$. Then, by using (4.13)–(4.16) it is easy to see that $(u_\delta, \beta_\delta)$ is the unique solution to problem $P_{2\delta}$ and it satisfies $(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}(0,T;L^2(\Gamma_3))$, such that

$$0 \le \beta_\delta(t) \le 1 \ \forall t \in [0,T], \quad \text{a.e. on } \Gamma_3.$$

Now, in the theorem below we prove the convergence of the solution $(u_\delta, \beta_\delta)$ as $\delta \to 0$ to the solution $(u, \beta)$ of Problem $P_2$ as follows.

**Theorem 4.2.** *Assume that* (2.13), (2.14), (2.15) *hold. Then we have the following convergence:*

$$\lim_{\delta\to 0} \|u_\delta - u\|_{C([0,T];V} = 0, \tag{4.17}$$

$$\lim_{\delta\to 0} \|\beta_\delta - \beta\|_{C([0,T];L^2(\Gamma_3))} = 0. \tag{4.18}$$

The proof is carried out in several steps. In the first step, we show the following lemma.

**Lemma 4.3.** *There exists a function $\bar u(t) \in V$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$u_\delta(t) \to \bar u(t) \quad \text{weakly in } V \text{ for all } t \in [0,T]. \tag{4.19}$$

*Proof.* Take in (4.9) $v = u_\delta(t)$, we get

$$\begin{aligned}&(F\varepsilon(u_\delta(t)),\varepsilon(u_\delta(t)))_Q + \frac{1}{\delta}((u_{\delta\nu}(t))_+, (u_\delta(t)))_{L^2(\Gamma_3)} + j(\beta_\delta(t), u_\delta(t), u_\delta(t))\\ &= (f(t), u_\delta(t))_V\,.\end{aligned} \tag{4.20}$$

Using (4.12), we have

$$((u_{\delta\nu}(t))_+, (u_{\delta\nu}(t)))_{L^2(\Gamma_3)} \geq ((u_{\delta\nu}(t))_+, (u_{\delta\nu}(t))_+)_{L^2(\Gamma_3)} \geq 0$$

and using (2.20), we have $j(\beta_\delta(t), u_\delta(t), u_\delta(t)) \geq 0$, then from (4.20) we have

$$(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t)))_Q \leq (f(t), u_\delta(t))_V$$

and keeping, in mind (2.13)(b), we deduce that there exists a constant $C > 0$ such that

$$\|u_\delta(t)\|_V \leq C\|f(t)\|_V.$$

The sequence $(u_\delta(t))$ is bounded in $V$, then there exists a function $\bar{u}(t) \in V$ and a subsequence again denoted $(u_\delta(t))$ such that (4.19) holds. □

Now, let us consider the following auxiliary problem.

**Problem $P_a$.** Find $\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3))$, such that

$$\dot{\beta}(t) = -c_\nu(\beta(t))_+(R(\bar{u}_\nu(t)))^2 \quad \text{a.e. } t \in (0, T),$$
$$\beta(0) = \beta_0.$$

Using the same proof as in the lemma 3.2, we have the following result.

**Lemma 4.4.** *Problem $P_a$ has a unique solution $\beta$. Moreover*

$$0 \leq \beta(t) \leq 1 \quad \forall t \in [0, T], \ \text{a.e. on } \Gamma_3.$$

*Now, we show the following convergence result.*

**Lemma 4.5.** *Let $\beta$ be the solution to problem $P_a$, then we have*

$$\lim_{\delta\to 0} \|\beta_\delta - \beta\|_{C([0,T];L^2(\Gamma_3))} = 0. \tag{4.21}$$

*Proof.* As in the proof of lemma 3.2, we have

$$\|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} \leq C \int_0^t \|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} ds. \tag{4.22}$$

From (4.19) we deduce that $u_{\delta\nu}(t) \to \bar{u}_\nu(t)$ strongly in $L^2(\Gamma_3)$, as $\delta \to 0$. On the other hand we have

$$\|u_{\delta\nu}(t) - \bar{u}_\nu(t)\|_{L^2(\Gamma_3)} \leq C\|u_\delta(t) - \bar{u}(t)\|_V \leq C(\|f(t)\|_V + \|\bar{u}(t)\|_V),$$

which implies that there exists a constant $C > 0$ such that

$$\|u_{\delta\nu}(t) - \bar{u}_\nu(t)\|_{L^2(\Gamma_3)} \leq C.$$

Then it follows from Lebesgue convergence theorem that

$$\lim_{\delta\to 0} \int_0^t \|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} ds = 0. \tag{4.23}$$

So we deduce that

$$\|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} \to 0 \quad \text{for all } t \in [0, T],$$

and as

$$W^{1,\infty}(0, T; L^2(\Gamma_3)) \hookrightarrow C([0, T]; L^2(\Gamma_3)),$$

it results that (4.21) is a consequence of (4.22) and (4.23). □

**Lemma 4.6.** *We have $\bar{u}(t) = u(t)$ for all $t \in [0, T]$.*

*Proof.* From (4.19), we deduce that

$$((u_{\delta\nu}(t))_+, (u_{\delta\nu}(t)))_{L^2(\Gamma_3)} \leq \delta C,$$

and then

$$((u_{\delta\nu}(t))_+, (u_{\delta\nu}(t))_+)_{L^2(\Gamma_3)} \leq \delta C. \tag{4.24}$$

From (4.12) and (4.19), we deduce that

$$(u_{\delta\nu}(t))_+ \to (\bar{u}_\nu(t))_+ \quad \text{strongly in } L^2(\Gamma_3) \text{ as } \delta \to 0. \tag{4.25}$$

Then we deduce from (4.24) and (4.25) that

$$\|(\bar{u}_\nu(t))_+\|_{L^2(\Gamma_3)} \leq \liminf_{\delta\to 0} \|(u_{\delta\nu}(t))_+\|_{L^2(\Gamma_3)} = 0. \tag{4.26}$$

It follows from (4.26) that $(\bar{u}_\nu(t))_+ = 0$; i.e., $\bar{u}_\nu(t) \leq 0$ a.e. on $\Gamma_3$ which shows that $\bar{u}(t) \in K$. Testing with $v - u_\delta(t)$ in (4.9) and keeping in mind that for all $v \in K$,

$$((u_{\delta\nu}(t))_+, v_\nu - u_{\delta\nu}(t))_{L^2(\Gamma_3)} = ((u_{\delta\nu}(t))_+ - v_{\nu+}, v_\nu - u_{\delta\nu}(t))_{L^2(\Gamma_3)},$$

we obtain

$$\begin{aligned} &(F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t)))_Q + j(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \\ &\geq (f(t), v - u_\delta(t))_V \quad \forall v \in K. \end{aligned} \tag{4.27}$$

Next, using (2.17) and (4.21), we get

$$\begin{aligned} &|j(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) - j(\beta(t), u_\delta(t), v - u_\delta(t))| \\ &\leq C\|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)}\|v - u_\delta(t)\|_V. \end{aligned}$$

On the other hand, using the properties of $R$, we get

$$j(\beta(t), u_\delta(t), v - u_\delta(t)) \to j(\beta(t), \tilde{u}(t), v - \tilde{u}(t)) \quad \text{as } \delta \to 0,$$

for all $v \in V$. Therefore, passing to the limit in (4.27) as $\delta \to 0$, we obtain that $\bar{u}(t) \in K$ and

$$(F\varepsilon(\bar{u}(t)), \varepsilon(v - \bar{u}(t)))_Q + j(\beta(t), \bar{u}(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \quad \forall v \in K. \tag{4.28}$$

Now, setting $v = u(t)$ in (4.28) and $v = \bar{u}(t)$ in (2.21) and add them up, we get by using the assumption (2.13)(b) on $F$ that

$$m\|\bar{u}(t) - u(t)\|_V^2 \leq j(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + j(\beta(t), u(t), \bar{u}(t) - u(t)).$$

Using (2.18), we have

$$j(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + j(\beta(t), u(t), \bar{u}(t) - u(t)) \leq 0,$$

and thus

$$\bar{u}(t) = u(t). \tag{4.29}$$

Now, we have all the ingredients to prove Theorem 4.2. Indeed, from (4.29), we deduce immediately (4.18). To prove (4.17), take $v = u(t)$ in (4.27), we get by using the assumption (2.13)(b) on $F$ that

$$\begin{aligned} m\|u_\delta(t) - u(t)\|_V^2 \leq\ & j(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) - j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \\ &+ j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \\ &+ (F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t)))_Q + (f(t), u_\delta(t) - u(t))_V. \end{aligned}$$

Passing to the limit as $\delta \to 0$ in the previous inequality and using the convergence

$$j(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) - j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0,$$
$$j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0,$$
$$(F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t)))_Q + (f(t), u_\delta(t) - u(t))_V \to 0,$$

we obtain that $\|u_\delta(t) - u(t)\|_V \to 0$ for all $t \in [0, T]$, and so as

$$W^{1,\infty}(0, T; V) \hookrightarrow C([0, T]; V),$$

we deduce (4.17). $\square$

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## Addendum posted on December 22, 2011.

The author wants to make the following corrections:

(1) On page 6, line -2: Replace $\beta \in L^\infty(\Gamma_3)$, $0 \le \beta \le 1$ a.e. $\Gamma_3$ by $\beta \in X_1$ where $X_1$ is the nonempty closed subset of the space $C([0, T]; L^2(\Gamma_3))$ defined as

$$X_1 = \{\theta \in C([0, T]; L^2(\Gamma_3)) : \theta(0) = \beta_0,\ 0 \le \theta(t) \le 1\, \forall t \in [0, T],\ \text{a.e. on } \Gamma_3\},$$

and the Banach space $C([0,T];L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_k = \sup_{t\in[0,T]}[\exp(-kt)\|\theta(t)\|_{L^2(\Gamma_3)}] \quad \forall\theta\in C([0,T];L^2(\Gamma_3)),\ k>0\,.$$

(2) In Lemma 3.1: Replace

$$(T_t u, v)_V = (F\varepsilon(u(t)),\varepsilon(v))_Q + j(\beta(t),u(t),v)$$

by

$$(T_t u, v)_V = (F\varepsilon(u),\varepsilon(v))_Q + j(\beta(t),u,v) \quad \forall u,v\in V$$

To prove that $u_\beta\in C([0,T];V)$, let $t_1$, $t_2\in[0,T]$. In inequality (3.1), take $t=t_1$ and $v=u_\beta(t_2)$; then $t=t_2$ and $v=u_\beta(t_1)$, by adding the resulting inequalities we obtain

$$\|u_\beta(t_1)-u_\beta(t_2)\|_V \le c(\|f(t_1)-f(t_2)\|_V + \|\beta(t_1)-\beta(t_2)\|_{L^2(\Gamma_3)}) \quad \forall t_1,t_2\in[0,T],$$

and conclude by using $f\in C([0,T];V)$ and $\beta\in C([0,T];L^2(\Gamma_3))$.

(3) In Lemma 3.2: Replace the space $X$ by $X_1$.

(4) Page 7, line -3: Add
We have $(\beta_1(s))_+ \le (\beta_1(s)-\beta_2(s))_+ + (\beta_2(s))_+$.

(5) On page 9: Replace the inequality (3.6) by

$$\|u_a(t_1)-u_a(t_2)\|_V \le c(\|f(t_1)-f(t_2)\|_V + \|\beta_a(t_1)-\beta_a(t_2)\|_{L^2(\Gamma_3)}) \quad \forall t_1,t_2\in[0,T],$$

and conclude by using $f\in W^{1,\infty}(0,T;V)$ and $\beta_a\in W^{1,\infty}(0,T;L^2(\Gamma_3))$.

(6) On page 10, line 1, in (i): Replace
$\beta\in L^2(\Gamma_3)$, $0\le\beta(t)\le 1$ for all $t\in[0,T]$, a.e. $\Gamma_3$
by $\beta\in X_1$.

End of addendum.

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