Electronic Journal of Differential Equations, Vol. 2007(2007), No. 20, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

POSITIVE SOLUTIONS FOR A CLASS OF NONRESONANT BOUNDARY-VALUE PROBLEMS

XUEMEI ZHANG

ABSTRACT. This paper concerns the existence and multiplicity of positive solutions to the nonresonant second-order boundary-value problem

$$Lx = \lambda w(t) f(t, x).$$

We are interested in the operator $Lx := -x'' + \rho q x$ when w is in L^p for $1 \le p \le +\infty$. Our arguments are based on fixed point theorems in a cone and Hölder's inequality. The nonexistence of positive solutions is also studied.

1. Introduction

Consider the second-order boundary-value problem (BVP)

$$\begin{gathered} Lx = \lambda w(t) f(t, x), \quad 0 < t < 1, \\ x(0) = x(1) = 0, \end{gathered} \tag{1.1}$$

where λ is a positive parameter and L denotes the linear operator

$$Lx := -x'' + \rho q x,$$

where $q \in C([0, 1], [0, \infty))$ and $\rho > 0$ such that

$$\begin{gathered} Lx = 0, \quad 0 < t < 1, \\ x(0) = x(1) = 0, \end{gathered}$$

has only the trivial solution. For the classical case $Lx = -x''$ and $f(t, x) = f(x)$, several results are available in the literature. Bandle [1] and Lin [12] established the existence of positive solutions under the assumption that f is superlinear, i.e., $f_0 = \lim_{x\to 0} \frac{f(x)}{x} = 0$, $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x} = \infty$. Wang [14] established the existence of positive solutions under the assumption that f is sublinear, i.e., $f_0 = \infty$ and $f_\infty = 0$. Eloe and Henderson [3] and Henderson and Wang [7] obtained the existence of positive solutions under the assumption that f_0 and f_∞ exist.

In the case $Lx = |x'|^{p-2}x'$, $p > 1$, i.e., the one-dimensional p-Laplacian, Jiang [9] obtained existence and multiplicity results under the assumption that f may be semilinear or superlinear at $x = \infty$ and change sign. Wang and Gao [16] established

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Positive solution; fixed point theorem; existence; complete continuity.

Submitted November 28, 2006. Published January 27, 2007.
Supported by grant 10626018 from the National Nature Science Foundation of China and the Science Foundation for Young Teachers of North China Electric Power University.

the existence of positive solutions under the assumption that $f(t,x) = f(x)$ is positive, right continuous, nonincreasing in $(0,+\infty)$ and $f_0 = \infty$.

In recent papers, when $Lx = |x'|^{p-2}x'$, $p > 1$ and $Lx = -x''$, $f(t,x) = f(x)$, where $x = (x_1, \dots, x_n)$. If $0 < f_0 < \infty$ and $0 < f_\infty < \infty$, Henderson and Wang [8] established the existence of positive solutions. Wang [15] showed that appropriate combinations of superlinearity and sublinearity of $f(x)$ guarantee the existence, multiplicity, and nonexistence of positive solutions.

However, the existence, multiplicity, and nonexistence of positive solutions of (1.1) is not available for the case when $Lx := -x'' + \rho q x$ and w is L^p-integrable for some $1 \leq p \leq +\infty$. This paper fills this gap in the literature. The purpose of this paper is to improve and generalize the results in the above mentioned references. we will show that the number of positive solutions of BVP (1.1) is determined by the parameter λ. The arguments are based upon fixed point theorems in a cone and Hölder's inequality.

The following lemmas are crucial to prove our main results. This is a fixed point theorem of cone expansion and compression of norm type [2,4,5,6].

Lemma 1.1. *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E, such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous, where θ denotes the zero element of E and P is a cone in E. Suppose that one of the following two conditions is satisfied:*

(i) *$\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_2$;*
(ii) *$\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_2$.*

Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Lemma 1.2 ([5])**.** *Let Ω_1, Ω_2 and Ω_3 be three bounded open sets in Banach space E, such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, $\bar{\Omega}_2 \subset \Omega_3$. Let operator $A : P \cap (\bar{\Omega}_3 \backslash \Omega_1) \to P$ be completely continuous, where θ denotes the zero element of E, and P is a cone in E. Suppose the following conditions are satisfied:*

(i) *$\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_1$;*
(ii) *$\|Ax\| \leq \|x\|$, $Ax \neq x$, for all $x \in P \cap \partial\Omega_2$;*
(iii) *$\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_3$.*

Then A has at least two fixed points x^*, x^{**} in $P \cap (\bar{\Omega}_3 \backslash \Omega_1)$, and $x^* \in P \cap (\Omega_2 \backslash \Omega_1)$, $x^{**} \in P \cap (\bar{\Omega}_3 \backslash \bar{\Omega}_2)$.

To obtain some of the norm inequalities in Theorems 3.1,3.2 and 3.5, we employ Hölder's inequality:

Lemma 1.3. *Let $f \in L^p[a,b]$ with $p > 1$, $g \in L^q[a,b]$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a,b]$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Let $f \in L^1[a,b]$, $g \in L^\infty[a,b]$. Then $fg \in L^1[a,b]$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

This paper is organized as follows: In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with BVP (1.1). In Section 3, the main result will be stated and proved. Finally some examples illustrate our main results.

2. Preliminaries

Let $J = [0,1]$. The basic space used in this paper is $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{t \in J} |x(t)|$.

Let K be a cone of E, $K_r = \{x \in K : \|x\| \le r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, $\bar{K}_{r,R} = \{x \in K : r \le \|x\| \le R\}$, where $0 < r < R$.

The following assumptions will stand throughout this paper:

(H1) $w \in L^p[0,1]$ for some $1 \le p \le +\infty$ and there exists $m > 0$ such that $w(t) \ge m$ a.e. on $[0,1]$;

(H2) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

In this paper, the Green's function of the corresponding homogeneous BVP is

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \phi(s)\psi(t) & \text{if } 0 \le s \le t \le 1, \\ \phi(t)\psi(s) & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{2.1}$$

Where ϕ and ψ satisfy

$$L\phi = 0, \quad \phi(0) = 0, \quad \phi'(0) = 1, \tag{2.2}$$

$$L\psi = 0, \quad \psi(1) = 0, \quad \psi'(1) = -1. \tag{2.3}$$

From [17,18], it is not difficult to show that $\Delta = -(\phi(t)\psi'(t) - \phi'(t)\psi(t)) > 0$ and $\phi'(t) > 0$ on $(0,1]$ and $\psi'(t) < 0$ on $[0,1)$. It is easy to prove that $G(t,s)$ has the following properties;

- For $t, s \in (0,1)$, we have
$$G(t,s) > 0. \tag{2.4}$$
- For $t, s \in J$, we have
$$0 \le G(t,s) \le G(s,s). \tag{2.5}$$
- Let $\theta \in (0, \frac{1}{2})$ and define $J_\theta = [\theta, 1-\theta]$. Then for all $t \in J_\theta, s \in J$ we have
$$G(t,s) \ge \sigma G(s,s), \tag{2.6}$$
where
$$\sigma(= \sigma(\theta)) = \min\big\{\frac{\psi(1-\theta)}{\psi(0)}, \frac{\phi(\theta)}{\phi(1)}\big\}. \tag{2.7}$$
In fact, for $t \in [\theta, 1-\theta]$, we have
$$\frac{G(t,s)}{G(s,s)} \ge \min\big\{\frac{\psi(1-\theta)}{\psi(s)}, \frac{\phi(\theta)}{\phi(s)}\big\} \ge \min\big\{\frac{\psi(1-\theta)}{\psi(0)}, \frac{\phi(\theta)}{\phi(1)}\big\} =: \sigma.$$
It is easy to see that $0 < \sigma < 1$.

For the sake of applying Lemma 1.1 and Lemma 1.2, we construct a cone in $E = C[0,1]$ by

$$K = \{x \in C[0,1] : x \ge 0, \min_{t \in J_\theta} x(t) \ge \sigma\|x\|\}. \tag{2.8}$$

It is easy to see K is a closed convex cone of E and $\bar{K}_{r,R} \subset K$.

Define an operator $T_\lambda : \bar{K}_{r,R} \to K$ by

$$T_\lambda x(t) = \lambda \int_0^1 G(t,s)w(s)f(s,x(s))ds, \tag{2.9}$$

From the above equality, it is well known that (1.1) has a positive solution x if and only if $x \in \bar{K}_{r,R}$ is a fixed point of T_λ.

Lemma 2.1. *Let (H1) and (H2) hold. Then $T_\lambda \bar{K}_{r,R} \subset K$ and $T_\lambda : \bar{K}_{r,R} \to K$ is completely continuous.*

Proof. For $x \in K$, by (2.9), we have $T_\lambda x(t) \ge 0$ and

$$\|T_\lambda x\| \le \lambda \int_0^1 G(s,s)w(s)f(s,x(s))ds, \tag{2.10}$$

On the other hand, by (2.9), (2.10) and (2.6), we obtain

$$\begin{aligned}
\min_{t\in J_\theta} T_\lambda x(t) &= \min_{t\in J_\theta} \lambda \int_0^1 G(t,s)w(s)f(s,x(s))ds \\
&\ge \lambda\sigma \int_0^1 G(s,s)w(s)f(s,x(s))ds \\
&\ge \sigma\|T_\lambda x\|.
\end{aligned}$$

Therefore $T_\lambda x \in K$, i.e., $T_\lambda K \subset K$. Also we have $T_\lambda \bar{K}_{r,R} \subset K$ by $\bar{K}_{r,R} \subset K$. Hence we have $T_\lambda : \bar{K}_{r,R} \to K$.

Next by standard methods and Ascoli-Arzela theorem one can prove $T_\lambda : \bar{K}_{r,R} \to K$ is completely continuous. So it is omitted. $\square$

3. Main results

Write

$$f^\beta = \limsup_{x\to\beta} \max_{t\in J} \frac{f(t,x)}{x}, \quad f_\beta = \liminf_{x\to\beta} \min_{t\in J} \frac{f(t,x)}{x},$$

where β denotes 0 or ∞. In this section, we apply the Lemmas 1.1-1.3 to establish the existence of positive solutions for BVP (1.1). We consider the following three cases for $w \in L^p[0,1]$; $p > 1$, $p = 1$ and $p = \infty$. Case $p > 1$ is treated in the following theorem.

Theorem 3.1. *Assume that (H1) and (H2) hold. In addition, letting $f_0 = \infty$ and $f^\infty = 0$ be satisfied, then, for all $\lambda > 0$, BVP (1.1) has at least one positive solution $x^*(t)$.*

Proof. Let T_λ be cone preserving, completely continuous operator that was defined by (2.9). Considering $f_0 = \infty$, there exists $r_1 > 0$ such that $f(t,x) \ge \varepsilon_1 x$, for $0 < x \le r_1$, $t \in J$, where $\varepsilon_1 > 0$ satisfies $\lambda\sigma^2 \frac{1}{\Delta} m\phi(\theta)\psi(1-\theta)\varepsilon_1 \ge 1$. So, for $x \in \partial K_{r_1}$, $t \in J$, from (2.6), we have

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda \int_0^1 G(t,s)w(s)f(s,x(s))ds \\
&\ge \lambda\varepsilon_1 \int_0^1 G(t,s)w(s)x(s)ds \\
&\ge \lambda m\varepsilon_1 \min_{t\in J_\theta} \int_0^1 G(t,s)x(s)ds \\
&\ge \lambda m\sigma\varepsilon_1 \int_0^1 G(s,s)x(s)ds \\
&\ge \lambda m\sigma^2\varepsilon_1 \|x\| \int_\theta^{1-\theta} G(s,s)ds \\
&\ge \lambda\sigma^2 \frac{1}{\Delta} m\phi(\theta)\psi(1-\theta)\varepsilon_1\|x\| \\
&\ge \|x\|.
\end{aligned} \tag{3.1}$$

Consequently, for $x \in \partial K_{r_1}$, we have

$$\|T_\lambda x\| \geq \|x\|. \tag{3.2}$$

Next, turning to $f^\infty = 0$, there exists $\bar{r}_2 > 0$ such that $f(t,x) \leq \varepsilon_2 x$, for $x \geq \bar{r}_2$, $t \in J$, where $\varepsilon_2 > 0$ satisfies $\varepsilon_2 \lambda \|G\|_q \|w\|_p \leq \frac{1}{2}$. Let

$$M = \lambda \sup_{x \in \partial K_{\bar{r}_2, t \in J}} f(t,x) \int_0^1 G(t,t)w(t)dt.$$

It is not difficult to see that $M < +\infty$.

Choosing $r_2 > max\{r_1, \bar{r}_2, 2M\}$, we get $M < \frac{1}{2}r_2$. Now, we choose $x \in \partial K_{r_2}$ arbitrary. Letting $\bar{x}(t) = \min\{x(t), \bar{r}_2\}$, we have $\bar{x} \in \partial K_{\bar{r}_2}$. In addition, writing $e(x) = \{t \in J : x(t) > \bar{r}_2\}$, for $t \in e(x)$, we get $\bar{r}_2 < x(t) \leq \|x\| = r_2$. By the choosing $\bar{r}_2$, for $t \in e(x)$, we have $f(t, x(t)) \leq \varepsilon_2 r_2$. Thus for $x \in \partial K_{r_2}$, from (2.5), we have

$$\begin{aligned}
(T_\lambda x)(t) &\leq \lambda \int_0^1 G(s,s)w(s)f(s,x(s))ds \\
&= \lambda \int_{e(x)} G(s,s)w(s)f(s,x(s))ds + \lambda \int_{[0,1]\setminus e(x)} G(s,s)w(s)f(s,x(s))ds \\
&\leq \lambda \varepsilon_2 r_2 \int_0^1 G(s,s)w(s)ds + \lambda \int_0^1 G(s,s)w(s)f(s,\bar{x}(s))ds \\
&\leq \lambda \varepsilon_2 r_2 \|G\|_q \|w\|_p + M \\
&< \frac{1}{2}r_2 + \frac{1}{2}r_2 \\
&= r_2 = \|x\|.
\end{aligned} \tag{3.3}$$

Consequently, from (3.3), for $x \in \partial K_{r_2}$, we have

$$\|T_\lambda x\| < \|x\|. \tag{3.4}$$

Applying (ii) of Lemma 1.1 to (3.2) and (3.4) yields that T_λ has a fixed point $x^* \in \bar{K}_{r_1,r_2}$, $r_1 \leq \|x^*\| \leq r_2$ and $x^*(t) \geq \sigma\|x^*\| > 0$, $t \in J_\theta$. Thus it follows that (1.1) has a positive solution x^* for all $\lambda > 0$. The proof is complete. $\square$

The following theorem studies the case $p = \infty$.

Theorem 3.2. *Suppose the conditions of Theorem 3.1 hold. Then, for all $\lambda > 0$, BVP (1.1) has at least one positive solution $x^*(t)$.*

To prove the above theorem, let $\|G\|_1\|w\|_\infty$ replace $\|G\|_p\|w\|_q$ and repeat the argument above. Finally we consider the case of $p = 1$.

Theorem 3.3. *Suppose the conditions of Theorem 3.1 hold. Then, for all $\lambda > 0$, (1.1) has at least one positive solution $x^*(t)$.*

Proof. As in the proof of Theorem 3.1, choose $r_2 = max\{r_1, \bar{r}_2, 2M\}$. For $x \in \partial K_{r_2}$, from (2.5) we have

$$
\begin{aligned}
(T_\lambda x)(t) &\le \lambda \int_0^1 G(s,s)w(s)f(s,x(s))ds \\
&= \lambda \int_{e(x)} G(s,s)w(s)f(s,x(s))ds + \lambda \int_{[0,1]\setminus e(x)} G(s,s)w(s)f(s,x(s))ds \\
&\le \lambda \varepsilon_2 r_2 \int_0^1 G(s,s)w(s)ds + \lambda \int_0^1 G(s,s)w(s)f(s,\bar{x}(s))ds \\
&\le \lambda \varepsilon_2 r_2 \frac{1}{\Delta}\phi(1)\psi(0)\|w\|_1 + M \\
&< \frac{1}{2}r_2 + \frac{1}{2}r_2 \\
&= r_2 = \|x\|,
\end{aligned}
\tag{3.5}
$$

where $\varepsilon_2 > 0$ satisfies $\varepsilon_2 \lambda \frac{1}{\Delta}\phi(1)\psi(0)\|w\|_1 \le 1$. Consequently, from (3.5), for $x \in \partial K_{r_2}$, we have $\|T_\lambda x\| < \|x\|$. This and (3.2) complete the proof. □

Corollary 3.4. *Let $f^0 = 0$ replace $f^\infty = 0$ and $f_\infty = \infty$ replace $f_0 = \infty$ in Theorems 3.1-3.3. Then the results still hold.*

In the following theorems we only consider the case of $p > 1$. The existence theorems corresponding to the cases of $p = 1$ and $p = \infty$ are similar and are omitted.

Theorem 3.5. *Assume (H1), (H2) and the following two conditions:*

(i) *$f_0 = \infty$ or $f_\infty = \infty$;*

(ii) *There exist $\rho > 0$ and $\delta > 0$, for $0 < x \le \rho$ and $t \in J$, such that $f(t,x) \le \delta$.*

Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, BVP (1.1) has at least one positive solution $x^*(t)$.

Proof. Considering $f_0 = \infty$, there exists $0 < r_3 < \rho$ such that $f(t,x) \ge \varepsilon_3 x$, for $0 < x \le r_3$, $t \in J$, where $\varepsilon_3 > 0$ satisfies $\varepsilon_3 \lambda \sigma^2 \frac{1}{\Delta} m\phi(\theta)\psi(1-\theta) \ge 1$. So, for $x \in \partial K_{r_3}$, from (2.6), we have

$$
\begin{aligned}
(T_\lambda x)(t) &= \lambda \int_0^1 G(t,s)w(s)f(s,x(s))ds \\
&\ge \lambda \varepsilon_3 \int_0^1 G(t,s)w(s)x(s)ds \\
&\ge \lambda m \varepsilon_3 \min_{t\in J_\theta} \int_0^1 G(t,s)x(s)ds \\
&\ge \lambda m \sigma \varepsilon_3 \int_0^1 G(s,s)x(s)ds \\
&\ge \lambda m \sigma \varepsilon_3 \int_\theta^{1-\theta} G(s,s)x(s)ds \\
&\ge \lambda \sigma^2 \frac{1}{\Delta} m\phi(\theta)\psi(1-\theta)\varepsilon_1 \|x\| \\
&\ge \|x\|.
\end{aligned}
\tag{3.6}
$$

Consequently, for $x \in \partial K_{r_3}$, we have

$$\|T_\lambda x\| \geq \|x\|. \tag{3.7}$$

If $f_\infty = \infty$, similar to the proof of (3.7), there exists $r_4 > \rho$ such that $f(t,x) \geq \varepsilon_4 x$, for $x \geq r_4$, $t \in J$, where $\varepsilon_4 > 0$ satisfies $\varepsilon_4 \lambda \sigma^2 \frac{1}{\Delta} m\phi(\theta)\psi(1-\theta) \geq 1$, and, for $x \in \partial K_{r_4}$, we have

$$\|T_\lambda x\| \geq \|x\|. \tag{3.8}$$

On the other hand, from (ii), when a $\rho > 0$ is fixed, then there exists a $\lambda_0 > 0$ such that $f(t,x) \leq \delta < \frac{1}{\lambda}[\|G\|_q \|w\|_p]^{-1}\rho$ for $0 < \lambda < \lambda_0$, $x \in \partial K_\rho$. Therefore for $x \in \partial K_\rho$ and $t \in J$ we have

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda \int_0^1 G(t,s)w(s)f(s,x(s))ds \\
&\leq \delta\lambda \int_0^1 G(t,s)w(s)ds \\
&\leq \delta\lambda \|G\|_q \|w\|_p \\
&< \rho = \|x\|.
\end{aligned}$$

Consequently, for $x \in \partial K_\rho$, we have

$$\|T_\lambda x\| < \|x\|. \tag{3.9}$$

By Lemma 1.1, for all $0 < \lambda < \lambda_0$, (3.7) and (3.9), (3.8) and (3.9), respectively, yield that T_λ has a fixed point $x^* \in \bar{K}_{r_3,\rho}$, $r_3 \leq \|x^*\| < \rho$ and $x^*(t) \geq \sigma\|x^*\| > 0$, $t \in J_\theta$ or $x^* \in \bar{K}_{\rho,r_4}$, $\rho_1 < \|x^*\| \leq r_4$ and $x^*(t) \geq \sigma\|x^*\| > 0$, $t \in J_\theta$. Thus it follows that BVP (1.1) has at least one positive solution x^* for all $0 < \lambda < \lambda_0$. □

Theorem 3.6. *Assume (H1), (H2) and the following two conditions:*

(i) $f_0 = \infty$ *and* $f_\infty = \infty$;
(ii) *There exist* $\rho > 0$, $\delta > 0$, *for* $0 < x \leq \rho$ *and* $t \in J$ *such that* $f(t,x) \leq \delta$.

Then there exists $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$, *BVP (1.1) has at least two positive solutions* $x^*(t), x^{**}(t)$.

Proof. The proof is similar to that of Theorem 3.5. Lemma 1.2, (3.7)-(3.9) yield that T_λ has at least two fixed points x^*, x^{**}, where $x^* \in \bar{K}_{r_3,\rho}$, $r_3 \leq \|x^*\| < \rho$ and $x^*(t) \geq \sigma\|x^*\| > 0$, $t \in J_\theta$, $x^{**} \in \bar{K}_{\rho,r_4}$, $\rho < \|x^*\| \leq r_4$ and $x^{**}(t) \geq \sigma\|x^{**}\| > 0$, $t \in J_\theta$. Thus it follows that BVP 1.1 has at least two positive solutions x^*, x^{**} for all $0 < \lambda < \lambda_0$. □

Corollary 3.7. *Assume (H1), (H2) and the following two conditions:*

(i) $f^0 = 0$ *or* $f^\infty = 0$;
(ii) *There exist* $\rho > 0$, $\delta > 0$, *such that* $f(t,x) \geq \delta$ *for* $x \geq \rho$ *and* $t \in J$.

Then there exists $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$, *BVP (1.1) has at least one positive solution* $x^*(t)$.

Corollary 3.8. *Assume (H1), (H2) and the following two conditions:*

(i) $f^0 = 0$ *and* $f^\infty = 0$;
(ii) *There exist* $\rho > 0$ *and* $\delta > 0$, *such that* $f(t,x) \geq \delta$ *for* $x \geq \rho$ *and* $t \in J$.

Then there exists $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$, *BVP (1.1) has at least two positive solutions* $x^*(t), x^{**}(t)$.

Our last result corresponds to the case when (1.1) has no positive solution.

Theorem 3.9. *Assume (H1), (H2), $f_0 > 0$ and $f_\infty > 0$. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, BVP (1.1) has no positive solution.*

Proof. Since $f_0 > 0$ and $f_\infty > 0$, then there exist $\eta_1 > 0$, $\eta_2 > 0$, $h_1 > 0$ and $h_2 > 0$ such that $h_1 < h_2$ and for $t \in J$, $0 < x \le h_1$, we have

$$f(t,x) \ge \eta_1 x, \tag{3.10}$$

and for $t \in J$, $x \ge h_2$, we have

$$f(t,x) \ge \eta_2 x. \tag{3.11}$$

Let

$$\eta = min\{\eta_1, \eta_2, \quad min\{\frac{f(t,x)}{x} : t \in J, \sigma h_1 \le x \le h_2\}\} > 0.$$

Thus, for $t \in J$, $x \ge \sigma h_1$, we have

$$f(t,x) \ge \eta x, \tag{3.12}$$

and for $t \in J$, $x \le h_1$, we have

$$f(t,x) \ge \eta x. \tag{3.13}$$

Assume y is a positive solution of (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0 = [\eta\sigma^2 \int_\theta^{1-\theta} G(s,s)w(s)ds]^{-1}$. In fact, if $\|y\| \le h_1$, (3.13) implies that

$$f(t,y) \ge \eta y, \ \ for\ t \in J.$$

On the other hand, if $\|y\| > h_1$, then $\min_{t\in J_\theta} y(t) \ge \sigma\|y\| > \sigma h_1$, which, together with (3.12), implies that, for $t \in J_\theta$, we get $f(t,y) \ge \eta y$. Since $(Ty)(t) = y(t)$, it follows that, for $\lambda > \lambda_0$, $t \in J$,

$$\begin{aligned}
\|y\| &= \|(T_\lambda y)\| \\
&= \max_{t\in J} \lambda \int_0^1 G(t,s)w(s)f(s,y(s))ds \\
&\ge \min_{t\in J_\theta} \lambda \int_\theta^{1-\theta} G(t,s)w(s)\eta y(s)ds \\
&\ge \lambda\eta\sigma\|y\|\sigma \int_\theta^{1-\theta} G(s,s)w(s)ds \\
&\ge \lambda\eta\|y\|\sigma^2 \int_\theta^{1-\theta} G(s,s)w(s)ds \\
&> \|y\|,
\end{aligned}$$

which is a contradiction. The proof is complete. □

Corollary 3.10. *Let $f^0 < \infty$ and $f^\infty < \infty$ replace $f_0 > 0$ and $f_\infty > 0$ in Theorem 3.9. Then the results are still valid.*

Remark 3.11. We did not use Hölder's inequality in the proof of Theorem 3.3 and Theorem 3.9.

It is clear that the results obtained here improve the results of [2,3,11,12,13,14,15]. To illustrate how our main results can be used in practice we present two examples.

Example 3.12. We set $\omega(t) = |2t - \frac{1}{8}|^{-1/15}$. Then $w \in L^p$ for $1 < p < 15$. For this function, $m = (\frac{8}{15})^{1/15}$. In addition, we define $f(t, x(t)) = \sqrt[3]{t^2+1}x^{1/n}$, $n > 1$. It is not difficult to see that

$$f_0 = \liminf_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty, \quad f^\infty = \limsup_{x\to\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0.$$

Hence the conditions of the Theorem 3.1 are satisfied.

Example 3.13. We set $\omega(t) = |t - \frac{1}{8}|^{-1/8}$. Then $w \in L^p$ for $1 < p < 8$. For this function, $m = (\frac{8}{7})^{1/8}$. In addition, we define $f(t, x(t)) = (1 + t^2)x^n + \frac{3+t}{16}x^{1/n}$, $n > 1$. It is not difficult to see that

$$f_0 = \liminf_{x\to 0} \min_{t\in J} \frac{f(t,x)}{x} = \infty, \quad f_\infty = \liminf_{x\to\infty} \min_{t\in J} \frac{f(t,x)}{x} = \infty.$$

Therefore, conditions (H1), (H2) and (i) of the Theorem 3.6 are satisfied. Finally we verify (ii) of the Theorem 3.6. Choosing $q(t) = 0$, then we get $\phi(t) = t, \psi(t) = 1-t$, $\phi(0) = 0$, $\psi(1) = 0$, $\phi'(0) = 1 = a$, $\psi'(1) = -1 = -c$, $\Delta = -(-t - (1-t)) = 1 > 0$ and

$$G(t,s) = \begin{cases} s(1-t) & \text{if } 0 \le s \le t \le 1, \\ t(1-s) & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Clearly, for $\rho = 1$, $\delta = 9/4$, we obtain

$$f(t,x) \le 2x^2 + \frac{1}{4}x^{1/2} \le 2 + \frac{1}{4} = \frac{9}{4} = \delta$$

for $0 < x \le \rho$, which implies the condition (ii) of the Theorem 3.6 also holds.

Acknowledgement. The authors thank the anonymous referee for his/her valuable comments and suggestions.

References

[1] C. Bandle, C. V. Coffman, M. Marcus; *Nonlinear elliptic problems in annular domains*, J.Differential Equations 69(1987)322-345.

[2] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[3] Paul W. Eloe and Johnny Henderson; *Positive solutions and nonlinear multipoint conjugate eigenvalue problems*, Electronic Journal of Differential Equations, Vol. 1997(1997), No. 03, pp. 1-11.

[4] D. J. Guo, V. Lakshmikantham, X. Z. Liu; *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers,Dordrecht, 1996.

[5] D. J. Guo, *Nonlinear functional analysis*, Jinan Shandong Science and Technology Press, 1985.

[6] D. J. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., New York, 1988.

[7] J. Henderson and H. Y. Wang; *Positive solutions for nonlinear eigenvalue problems*, J. Math. Anal. Appl. 208(1997), 252-259.

[8] J. Henderson and H. Y. Wang; *Nonlinear eigenvalue problems for quasilinear systems*, Computers & Mathematics with Applications, 2004,1-17.

[9] D. Q. Jiang, *Positive solutions to singular semilinear and superlinear boundary value problems for the one-dimensional p-Laplacian*, Acta Mathematica Sinica, 45A 4(2002), 731-736 (in Chinese).

[10] E. R. Kaufmann, N. Kosmatov; *A multiplicity result for a boundary value problem with infinitely many singularities*, J. Math. Anal. Appl. 269(2002), 444-453.

[11] Y. S. Liu, B. Q. Yan; *Multiple positive solutions for a class of nonresonant singular boundary-value problems*, Electronic Journal of Differential Equations, Vol. 2006 (2006), No. 42, pp. 1-11.

[12] S. S. Lin, *On the existence of positive radial solutions for semilinear elliptic equations in annular domains*, J. Differential Equations 81(1989), 221-233.
[13] D. O'Regan, *Singular Dirichlet boundary-value problems-I, Superlinear and nonresonance case*, Nonlinear Anal., 29(1997), 221-249.
[14] H. Y. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations 109(1994), 1-7.
[15] H. Y. Wang, *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl. 281(2003), 287-306.
[16] J. Y. Wang and W. J. Gao; *A singular boundary value problem for the one-dimensional p-laplacian*, J. Math. Anal. Appl. 201(1996), 851-866.
[17] Z. L. Wei, C. C. Pang; *Positive solutions of nonresonant singular boundary-value problem of second order differential equations*, Nagoya Math. J., 162(2001), 127-148.
[18] Z. L. Wei, *Positive solutions of singular Dirichlet boundary-value problems at nonresonance*, Chinese Annals of Mathematics, 20(1999), 543-552 (in Chinese).
[19] Q. L. Yao, *Positive solutions for generalized Gelfend models*, Appl. Math. J. Chinese Univ. Ser. 16A, 4(2001), 407-413.

Xuemei Zhang
Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
E-mail address: zxm74@sina.com