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EIGENCURVES OF THE p-LAPLACIAN WITH WEIGHTS AND THEIR ASYMPTOTIC BEHAVIOR

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Abstract. In this paper we study the existence of the eigencurves of the p-Laplacian with indefinite weights. We obtain also their variational formulations and asymptotic behavior.

1. Introduction and preliminaries

We consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, $m \in \mathcal{L}^\infty(\Omega)$ is a weight function which can change sign and verifies

$$\operatorname{meas}\{x \in \Omega : m(x) > 0\} \neq 0\,.$$

We denote

$$\mathcal{M}^+(\Omega) = \big\{m \in \mathcal{L}^\infty(\Omega) : \operatorname{meas}\{x \in \Omega : m(x) > 0\} \neq 0\big\}\,.$$

We say that λ is a eigenvalue of p-Laplacian with weight m, when the problem (1.1) has at least a nontrivial solution u in $\mathcal{W}_0^{1,p}(\Omega)$. The set of positive eigenvalues constitutes the spectrum $\sigma^+(-\Delta_p, m, \Omega)$ of p-Laplacian with weight m in the domain Ω. This spectrum contains an infinite sequence given by $\lambda_1 < \lambda_2 \le \dots \le \lambda_n \to +\infty$ and formulated as follows

$$\frac{1}{\lambda_n} = \frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \int_\Omega m|u|^p\,, \tag{1.2}$$

where Γ_n is defined by

$$\Gamma_n = \{K \subset S : K \text{ is symetric, compact and } \gamma(K) \ge n\},$$

$S = \{u \in \mathcal{W}_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p = 1\}$ is the sphere unity of $\mathcal{W}_0^{1,p}(\Omega)$ and γ is the genus function. We may also define the negative spectrum when $-m \in \mathcal{M}^+(\Omega)$ by

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$-\sigma^+(-\Delta_p, -m, \Omega)$ which contains an infinite sequence $\lambda_{-1} > \lambda_{-2} \geq \cdots \geq \lambda_{-n} \to -\infty$ such that

$$\lambda_{-n} = \lambda_{-n}(m) := -\lambda_n(-m) \quad . \tag{1.3}$$

The variational characterization (1.2) and the properties of λ_n depending on weight m was the subject of several works of which we cite for example [1, 2, 3, 6].

In this note, we study the following problem: Find all the real numbers α, β such that $\lambda_n(\alpha m_1 + \beta m_2) = 1$.

This last equation comes from the problem of eigencurves of Sturm-Liouville. Several applications of these problems can be found in the bifurcation domain and other, making reference [5]. In [4] we find the properties related to the first eigencurve such as concavity, differentiability and the asymptotic behavior. The authors wished to have information about the other eigencurves, especially their asymptotic behavior. This will be the object of our study. Let $m_1, m_2 \in \mathcal{M}^+(\Omega)$ so that $\operatorname{ess\,inf}_\Omega m_2 > 0$. we define the graph of the n^{th} eigencurve by

$$C_n = \{(\alpha, \beta) \in \mathbb{R}^2 : \lambda_n(\alpha m_1 + \beta m_2) = 1\}, \tag{1.4}$$

We note that this definition differs from that given in [5], which is

$$\beta_n(\alpha) = \inf_{K \in \Gamma_n} \max_{u \in K} \frac{\int_\Omega |\nabla u|^p - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p} \quad . \tag{1.5}$$

This paper is organized as follows. First, we are interested to the existence of eigencurve C_n . Then we show that $(\alpha, \beta_n(\alpha)) \in C_n$. This would allow us to affirm the coincidence of the two definitions (1.4) et (1.5) and also present the variational formulation of that eigencurve. We would end up with the study of the asymptotic behavior of the eigencurves C_n. And finally we affirm that all eigencurves have the same asymptotic behavior.

2. Existence of the eigencurve C_n

We first recall the following

Proposition 2.1. (1) *Let $m, m' \in \mathcal{M}^+(\Omega)$. If $m \leq m'$ (resp. $m < m'$), then $\lambda_n(m) \geq \lambda_n(m')$ (resp. $\lambda_n(m) > \lambda_n(m')$).*
(2) *$\lambda_n : m \mapsto \lambda_n(m)$ is continuous in $(\mathcal{M}^+(\Omega), \|.\|_\infty)$.*

For the proof, see for example [6].

Next we can establish the following

Proposition 2.2. *Let $(m_k)_k$ be a sequence in $\mathcal{M}^+(\Omega)$ such that $m_k \to m$ in $\mathcal{L}^\infty(\Omega)$. Then $\lim_k \lambda_n(m_k) = +\infty$ if and only if $m \leq 0$ almost everywhere in Ω.*

Proof. Let $(m_k)_k$ be a sequence in $\mathcal{M}^+(\Omega)$ such that m_k converges to m in $\mathcal{L}^\infty(\Omega)$. Assume first that $\lim_k \lambda_n(m_k) = +\infty$, we claim that $m \leq 0$ almost everywhere in Ω; otherwise

$$\operatorname{meas}\{x \in \Omega : m(x) > 0\} \neq 0$$

it then follows that $\lim_k \lambda_n(m_k) = \lambda_n(m)$, is a finite, a contradiction.

Inversely, if $m \leq 0$ almost everywhere in Ω, suppose that $\lim_k \lambda_n(m_k)$ is finite, then there exist $\lambda > 0$ such that

$$\lambda_n(m_k) \leq \lambda \quad \text{for all } k \in \mathbb{N}. \tag{2.1}$$

We put $r = 2\lambda/\lambda_n(2)$ and $\varepsilon = 1/r$. Then there exist $k = k(r)$, such that
$$\|m_k - m\|_\infty < \varepsilon\,.$$
We consider the following weights
$$m_{k,r}(x) = \begin{cases} m_k(x) & \text{if } x \in \Omega \setminus B_r \cap \Omega_k^- \\ \frac{1}{r} & \text{if } x \in B_r \cap \Omega_k^- \end{cases}$$
and
$$m_r(x) = \begin{cases} m(x) & \text{if } x \in \Omega \setminus B_r \cap \Omega_k^- \\ \frac{1}{r} & \text{if } x \in B_r \cap \Omega_k^- \end{cases}$$
where $B_r = B(x_k, \frac{1}{r})$ is a ball and $x_k \in \Omega_k^- = \{x \in \Omega : m_k(x) < 0\}$. It is clear that
$$\|m_{k,r} - m_r\|_\infty \le \|m_k - m\|_\infty$$
so that
$$m_{k,r} \le m_r + \varepsilon \quad \text{almost everywhere in } \Omega\,.$$
Observe that $m_k \le m_{k,r}$. Since $m \le 0$ almost everywhere in Ω, we have $m_r \le 1/r$ almost everywhere in Ω, and
$$m_k \le \frac{1}{r} + \varepsilon \quad \text{almost everywhere in } \Omega\,. \tag{2.2}$$
It follows from (2.1) and (2.2) that
$$\lambda \ge \lambda_n(m_k) \ge \lambda_n(\frac{1}{r} + \varepsilon)\,.$$
Since $\frac{1}{r} + \varepsilon = \frac{2}{r}$,
$$2\lambda = r\lambda_n(2) = \lambda_n(\frac{1}{r} + \varepsilon) \le \lambda_n(m_k) \le \lambda\,,$$
which is a contradiction. So $\lim_k \lambda_n(m_k) = +\infty$. □

Now we can state the main theorem of this section.

Theorem 2.3. *Let m_1, $m_2 \in \mathcal{M}^+(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega m_2 > 0$. So for all $\alpha \in \mathbb{R}$ there exist a unique real $t_n(\alpha)$ which satisfies $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*

Proof. Let $\alpha \in \mathbb{R}$. We consider the function $f_\alpha : t \mapsto \lambda_n(\alpha m_1 + t m_2)$. According to the proposition 2.1 we affirm that f_α is decreasing continuous. Consequently f_α is injective. To show that the equation $f_\alpha(t) = 1$ has a solution (hence only one), we distinguish three cases.
Case 1: $\lambda_{-n}(m_1) < \alpha < \lambda_n(m_1)$ It is clear that when $\alpha = 0$, the unique real t_0 that verifies $\lambda_n(\alpha m_1 + t_0 m_2) = 1$ is $t_0 = \lambda_n(m_2)$. Suppose that α is not nil, in this case we have
$$f_\alpha(0) = \frac{\lambda_n(m_1)}{\alpha} \quad \text{if } \alpha > 0 \quad \text{and} \quad f_\alpha(0) = \frac{\lambda_{-n}(m_1)}{\alpha} \quad \text{if } \alpha < 0\,;$$
So that $f_\alpha(0) > 1$. Now $\frac{\alpha}{t} m_1 + m_2 \to m_2$ in $\mathcal{L}^\infty(\Omega)$ as $t \to +\infty$;
$$\lim_{t \to +\infty} f_\alpha(t) = \lim_{t \to +\infty} \frac{1}{t} \lambda_n(\frac{\alpha}{t} m_1 + m_2) = 0\,,$$
so there exist a unique real $t_n(\alpha) \in]0, +\infty[$ which verifies $f_\alpha(t_n(\alpha)) = 1$.
Case 2: $\alpha > \lambda_n(m_1)$. In this case $\alpha > 0$ and
$$0 < f_\alpha(0) = \frac{\lambda_n(m_1)}{\alpha} < 1\,. \tag{2.3}$$

Let $\gamma_\alpha = \frac{-\alpha\|m_1\|_\infty}{\operatorname{ess\,inf}_\Omega\, m_2}$, it is easy to see that

$$\alpha m_1 + \gamma_\alpha m_2 \le 0 \quad \text{almost everywhere in } \Omega\,,$$

so

$$\gamma_\alpha \in \{t < 0 : \alpha m_1(x) + t m_2(x) \le 0 \quad \text{a.e. } x \in \Omega\} = A_\alpha$$

we put $\tau_\alpha = \sup A_\alpha$. We prove that $\tau_\alpha \in A_\alpha$.

We first show that $\tau_\alpha < 0$. Since $f_\alpha(0) > 0$, and f_α, is a continuous function then there exist $\eta < 0$ such that $f_\alpha(t) > 0$ for all $t \in [\eta, 0]$. so $\lambda_n(\alpha m_1 + t m_2) > 0$ for all $t \in [\eta, 0]$; i.e.,

$$\operatorname{meas}\{x \in \Omega : \alpha m_1(x) + t m_2(x) > 0\} \ne 0 \quad \forall t \in [\eta, 0]\,;$$

hence $\tau_\alpha \le \eta < 0$. Moreover, for all $n \in \mathbb{N}$, there exist $t_n \in A_\alpha$ such that $\tau_\alpha - \frac{1}{n} < t_n$. It follows that

$$\alpha m_1(x) + \tau_\alpha m_2(x) \le \alpha m_1(x) + t_n m_2(x) + \frac{1}{n} m_2(x) \quad \le \frac{1}{n}\|m_2\|_\infty \quad \text{a.e. } x \in \Omega, \forall n \in \mathbb{N}$$

thus we have $\alpha m_1 + \tau_\alpha m_2 \le 0$ almost everywhere in Ω. Then Proposition 2.2 implies

$$\lim_{t \to \tau_\alpha} f_\alpha(t) = +\infty\,. \tag{2.4}$$

It then follows from (2.3) and (2.4) that there exist a unique real $t_n(\alpha) \in]\tau_\alpha, 0[$ which verifies $f_\alpha(t_n(\alpha)) = 1$.

case 3: $\alpha < \lambda_{-n}(m_1)$. In this case we have $\alpha < 0$ and

$$0 < f_\alpha(0) = \frac{\lambda_{-n}(m_1)}{\alpha} < 1\,.$$

Let $\delta_\alpha = \frac{\alpha\|m_1\|_\infty}{\operatorname{ess\,inf}_\Omega m_2}$,

$$B_\alpha = \{t < 0 : \alpha m_1(x) + t m_2(x) \le 0 \text{ a.e. } x \in \Omega\} \quad \text{and} \quad \rho_\alpha = \sup B_\alpha.$$

Obviously

$$\alpha m_1 + \delta_\alpha m_2 \le 0 \quad \text{almost everywhere in } \Omega\,.$$

The rest of the proof can be carried out in a similar manner to that of the case 2. The proof is complete. ∎

3. Variational formulation of the eigencurve C_n

We consider the formula (1.5) of $\beta_n(\alpha)$. By the Sobolev inequality and hypothesis $\operatorname{ess\,inf}_\Omega m_2 > 0$, it is easy to see that $\beta_n(\alpha)$ is finite. Our objective in this section is to show that the graph of $\beta_n(\alpha)$ is exactly C_n.

Theorem 3.1. *We take again the notation of Theorem 2.3. So we have*

$$t_n(\alpha) = \beta_n(\alpha) \quad \textit{for all } \alpha \in \mathbb{R}\,.$$

Proof. We have on one hand, according to (1.5) for all $K \in \Gamma_n$, there is $u_K \in K$ such that

$$\beta_n(\alpha) \le \max_{u \in K} \frac{\int_\Omega |\nabla u|^p - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p} = \frac{\int_\Omega |\nabla u_K|^p - \alpha \int_\Omega m_1 |u_K|^p}{\int_\Omega m_2 |u_K|^p},$$

then

$$\alpha \int_\Omega m_1 |u_K|^p + \beta_n(\alpha) \int_\Omega m_2 |u_K|^p \le \int_\Omega |\nabla u_K|^p = 1\,.$$

So that

$$\min_{u\in K}\int_\Omega(\alpha m_1+\beta_n(\alpha)m_2)|u|^p \le \alpha\int_\Omega m_1|u_K|^p+\beta_n(\alpha)\int_\Omega m_2|u_K|^p\le 1,$$

for all $K\in\Gamma_n$, this implies

$$\sup_{K\in\Gamma_n}\min_{u\in K}\int_\Omega(\alpha m_1+\beta_n(\alpha)m_2)|u|^p\le 1\,.$$

Since

$$\frac{1}{\lambda_n(\alpha m_1+\beta_n(\alpha)m_2)}=\sup_{K\in\Gamma_n}\min_{u\in K}\int_\Omega(\alpha m_1+\beta_n(\alpha)m_2)|u|^p,$$

it follows that

$$\lambda_n(\alpha m_1+\beta_n(\alpha)m_2)\ge 1\,. \tag{3.1}$$

On the other hand, from Theorem 2.3, we have

$$\lambda_n(\alpha m_1+t_n(\alpha)m_2)=1\,,$$

so for all $K\in\Gamma_n$, there is $u_K\in K$ such that

$$\alpha\int_\Omega m_1|u_K|^p+t_n(\alpha)\int_\Omega m_2|u_K|^p=\min_{u\in K}\int_\Omega(\alpha m_1+t_n(\alpha)m_2)|u|^p,$$

and

$$\min_{u\in K}\int_\Omega(\alpha m_1+t_n(\alpha)m_2)|u|^p\le\lambda_n(\alpha m_1+t_n(\alpha)m_2)=1\,.$$

Since $1=\int_\Omega|\nabla u_K|^p$,

$$\alpha\int_\Omega m_1|u_K|^p+t_n(\alpha)\int_\Omega m_2|u_K|^p\le\int_\Omega|\nabla u_K|^p\,.$$

This implies

$$t_n(\alpha)\le\frac{\int_\Omega|\nabla u_K|^p-\alpha\int_\Omega m_1|u_K|^p}{\int_\Omega m_2|u_K|^p}\le\max_{u\in K}\frac{\int_\Omega|\nabla u|^p-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p},$$

for all $K\in\Gamma_n$, thus we deduce

$$t_n(\alpha)\le\inf_{K\in\Gamma_n}\max_{u\in K}\frac{\int_\Omega|\nabla u|^p-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p}=\beta_n(\alpha)\,.$$

Using the monotony of λ_n with respect to the weight, it follows that

$$\lambda_n(\alpha m_1+\beta_n(\alpha)m_2)\le\lambda_n(\alpha m_1+t_n(\alpha)m_2)=1\,. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\lambda_n(\alpha m_1+\beta_n(\alpha)m_2)=1\,.$$

Since $t_n(\alpha)$ is unique, we then conclude that $t_n(\alpha)=\beta_n(\alpha)$. $\square$

4. Asymptotic behavior of C_n

The fact of considering $\beta_n(\alpha)$ by its expression given in the variational formulation (1.5) makes regrettably the study of its asymptotic behavior difficult. So, our aim in this section is to determine the asymptotic behavior with the help of Theorem 3.1 and the definition of C_n (cf. (1.4)).

Theorem 4.1. *Let $m_1, m_2 \in \mathcal{M}^+(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega m_2 > 0$. So we have the following asymptotic behavior:*

(i) $\lim_{\alpha\to+\infty} \beta_n(\alpha)/\alpha = -\operatorname{ess\,sup}_\Omega m_1/m_2$,
(ii) $\lim_{\alpha\to-\infty} \beta_n(\alpha)/\alpha = -\operatorname{ess\,inf}_\Omega m_1/m_2$.

Proof. To prove (i), we consider $\alpha > \lambda_n(m_1)$: The formula

$$\lambda_n(\alpha m_1 + \beta_n(\alpha)m_2) = 1$$

then implies

$$\lambda_n(m_1 + \frac{\beta_n(\alpha)}{\alpha}m_2) = \alpha$$

which is a finite quantity and positive, so

$$m_1 + \frac{\beta_n(\alpha)}{\alpha}m_2 \in \mathcal{M}^+(\Omega),$$

thus there exist a subset Ω_α such that

$$\operatorname{meas}(\Omega_\alpha) \neq 0 \quad \text{and} \quad m_1(x) + \frac{\beta_n(\alpha)}{\alpha}m_2(x) > 0 \quad \text{a.e. } x \in \Omega_\alpha\,.$$

Hence

$$-\frac{\beta_n(\alpha)}{\alpha} < \frac{m_1(x)}{m_2(x)} \quad \text{a.e. } x \in \Omega_\alpha\,;$$

thus we have

$$-\frac{\beta_n(\alpha)}{\alpha} < \mathit{ess\,sup}_{\Omega} \frac{m_1}{m_2}\,.$$

So we get

$$\limsup_{\alpha\to+\infty} -\frac{\beta_n(\alpha)}{\alpha} \leq \operatorname*{ess\,sup}_{\Omega} \frac{m_1}{m_2}. \tag{4.1}$$

On the other hand, suppose that

$$l = \liminf_{\alpha\to+\infty} -\frac{\beta_n(\alpha)}{\alpha}\,.$$

We choose a sequence $\alpha_k \to +\infty$, so that

$$m_1 + \frac{\beta_n(\alpha_k)}{\alpha_k}m_2 \to m_1 - l\,m_2 \quad \text{in } \mathcal{L}^\infty(\Omega)\,.$$

Since

$$\lambda_n(m_1 + \frac{\beta_n(\alpha_k)}{\alpha_k}m_2) = \alpha_k \to +\infty,$$

according to Proposition 2.2, we obtain $m_1 - lm_2 \leq 0$ almost everywhere in Ω, i.e.,

$$\frac{m_1}{m_2} \leq l \quad \text{almost everywhere in } \Omega\,;$$

so that

$$\operatorname*{ess\,sup}_{\Omega} \frac{m_1}{m_2} \leq l = \liminf_{\alpha\to+\infty} -\frac{\beta_n(\alpha)}{\alpha}\,. \tag{4.2}$$

Then (4.1) and (4.2) yield the result (i).

The proof of (ii) can be carried out as that of (i). This concludes the proof . $\square$

Remarks.

(i) All eigencurves of the p-Laplacian have the same asymptotic behavior.

(ii) The asymptotic behavior of the first eigencurve of the p-Laplacian is already established in [4], but their method which uses the properties of the first eigenfunction is not generalised to the higher orders.

(iii) The results established in this paper can also be generalised to eigencurves of order ≥ 2 of the p-Laplacian with Neumann condition.

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