Electronic Journal of Differential Equations, Vol. 2007(2007), No. 36, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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STABILITY OF LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS WITH MULTIVALUED DELAY FEEDBACK

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Abstract. We consider a controlled linear functional differential system with linear feedback without delay and assume that the closed system is exponentially stable. Then we assume a non-ideality in the feedback loop such that it has an unknown delay, which may be distributed or not. We suppose that this delay is sufficiently small. In such a case, the disturbed system is presented by a functional differential inclusion of special type.

We prove that this inclusion remains exponentially stable. To do this, we use the exponential estimate, which is valid uniformly for all Cauchy functions of some class of linear functional differential equations that are close to given one.

1. Introduction: Notation and basic assumptions

When we consider a controlled functional differential system with feedback control

$$\dot{x}(t) = \int_0^t d_s A(t,s)\, x(s) + u(t) + f(t),$$
$$u(t) = Bx(t),$$

the closed system is described by the functional differential equation

$$\dot{x}(t) = \int_0^t d_s A(t,s)\, x(s) + Bx(t) + f(t). \tag{1.1}$$

The kernel $A : \Delta := \{(t,s) : t \in [0,\infty),\ s \in [0,t]\} \to \mathbb{R}^{n\times n}$ is supposed to satisfy the standard condition, named condition (R) in [3], which is the conjunction of the following three assumptions:

- (i) A is measurable on Δ;
- (ii) the functions $t \mapsto A(t,s)$, for any $s \geq 0$, are locally summable on $[s,\infty)$, the function $t \mapsto A(t,t)$ is locally summable on $[0,\infty)$ too;
- (iii) for almost every (a.e.) $t \geq 0$ the variation $v(t) := \operatorname{Var}_0^t A(t,\cdot)$ is finite, the function v is locally summable.

2000 *Mathematics Subject Classification.* 34K20, 34K35.
Key words and phrases. Linear functional differential equations; Cauchy function; functional differential inclusions; exponential stability.

Submitted January 18, 2007. Published February 27, 2007.
Supported by grants 03-01-00255 and 06-01-00744 from RFBR.

This condition ensures that if z is a continuous function on $[0,\infty)$, then $\mathbf{A}z$ is locally summable on $[0,\infty)$.

The problem of stabilization for the feedback control system is, for given class of matrices B of feedback coefficients, to determine the matrix B in order for the exponential stability of the closed system.

We start our study under the basic assumption that this problem is solved successfully. More precisely, we suppose that

(H1) the Cauchy matrix function $W(t,s)$ for equation (1.1) satisfies the estimate

$$|W(t,s)| \le Ke^{-\lambda(t-s)}, \quad t \ge s \ge 0, \tag{1.2}$$

with positive K and λ.

We denote norms of finite-dimensional vectors and matrices by $|\cdot|$. Furthermore, we use the following notation. Let $\gamma \ge 0$. $\mathcal{C}_\gamma$ is the space of continuous functions $x:[0,\infty)\to\mathbb{R}^n$ with finite norms $\|x\|_\gamma = \sup_{t\in[0,\infty)} e^{\gamma t}|x(t)|$. $\mathcal{M}_\gamma$ is the space of measurable functions $f:[0,\infty)\to\mathbb{R}^n$ with the norm $\|f\|_\gamma = \operatorname{ess\,sup}_{t\in[0,\infty)} e^{\gamma t}|f(t)|$. As distinct from [3], we write $\mathcal{C}_0$ and $\mathcal{M}_0$ in the case of $\gamma = 0$.

For operator $\mathbf{Q}:\mathcal{C}_\gamma\to\mathcal{M}_\gamma$, or for $\mathbf{Q}:\mathcal{M}_\gamma\to\mathcal{C}_\gamma$, etc, we denote by $\|\mathbf{Q}\|_\gamma$ the norm of operator generated by the γ-norms of functions.

We describe the property of bounded aftereffect as follows. The function A satisfies *condition* $\mathsf{D}(\delta)$, where $\delta > 0$, if $A(t,s) = 0$ for $s < t-\delta$. As distinct from [9, 3], here we ought to emphasize the value of the parameter δ.

Also, A is said to satisfy condition (V) if

$$V(A) := \operatorname*{ess\,sup}_{t\ge 0} \operatorname{Var}_0^t A(t,\cdot) < \infty.$$

Naturally we also suppose that

(H2) The kernel A satisfies both conditions $\mathsf{D}(\delta_A)$ and (V).

Let us take into consideration a non-ideality of feedback, due to which we have, instead of $Bx(t)$, something like $Bx(t-\delta)$, or $\frac{1}{2}B\big(x(t-\delta)+x(t)\big)$, or $\frac{1}{\delta}B\int_{t-\delta}^t x(s)\,ds$ (where $\delta > 0$) and we do not know exactly what we have. To describe such a situation, we introduce a multivalued delay feedback with delay not greater than δ. In this way we obtain a functional differential inclusion

$$\dot{x}(t) \in \int_0^t d_sA(t,s)\,x(s) + B\,\mathrm{co}\,x[\max(0,t-\delta),t] + f(t)$$

(see details in section 3 below).

Our goal is to prove that, under natural assumptions on the operator $\mathbf{A}$, if δ is sufficiently small, then inclusion (3.1) remains exponentially stable. It will be done in a few steps.

The first step is to show that every solution x of (3.1) satisfies the linear functional differential equation

$$\dot{x}(t) = \int_0^t d_sA(t,s)\,x(s) + B\int_{t-\delta}^t d_sr(t,s)\,x(s) + f(t),$$

which is close to equation (1.1) provided that δ is sufficiently small. This is done in section 3.

Then in section 5 we demonstrate that all these equations (3.2) have Cauchy functions that satisfy the exponential estimate

$$|C(t,s)| \le Ne^{-\beta(t-s)}, \quad t \ge s \ge 0,$$

uniformly with respect of the chosen solution x.

To show this inequality, we need a result that ensures the exponential estimate under some assumptions concerning action of operator $\mathbf{R} = \mathbf{A} + B\mathbf{r}$ and its Cauchy operator $\mathbf{C}$, such as [3, Theorem 3.3.2, p. 102]. This theorem claims the exponential estimate of the Cauchy function for all $(t,s) \in \Delta$, but has an inexplicable proof. Unfortunately this assertion has a fault — the example of O. Demyanchenko shows that the exponential estimate may lose validity for some values of s.

Even if we replace "all s" by "almost all s", [3, theorem 3.3.2] has still no proof. We, the author of the text and Demyanchenko, have obtained the exponential estimate of the Cauchy function for almost all s following the main idea of [3], but under some additional assumption. This is presented in section 4.

In section 6, we state a few properties of inclusion (3.1) that are commonly meant under the name of exponential stability.

Section 2 contains a brief survey of the theory of linear functional differential equations for reader's convenience.

2. Linear functional differential equations – survey

Linear functional differential equations of the type

$$\dot{x}(t) = \int_0^t d_s R(t,s)\, x(s) + f(t) \tag{2.1}$$

are comprehensively studied nowadays (see the survey and recent results in [1, 2, 3]). We review here the most interesting, for our special case, results on exponential stability of (2.1). We assume that R satisfies condition (R) and f is locally summable function $[0,\infty) \to \mathbb{R}^n$.

Function $x : [0,T) \to \mathbb{R}^n$ is a solution of (2.1) if it is absolutely continuous on every segment $[0,T'] \subset [0,T)$ and its derivative satisfies (2.1) for almost every (a.e.) $t \in [0,T)$.

1. The initial value problem $x(0) = x_0$ for equation (2.1) has a unique solution x, which is extendable to $[0,\infty)$. So we consider all solutions being defined on $[0,\infty)$.

2. Equation (2.1) has a Cauchy matrix function $C(t,s)$ such that for every $T \ge 0$ all solutions x of the equation

$$\dot{x}(t) = \int_T^t d_s R(t,s)\, x(s) + f(t)$$

are represented by the Cauchy formula

$$x(t) = C(t,T)\, x(T) + \int_T^t C(t,s) f(s)\, ds, \quad t \ge T. \tag{2.2}$$

The functions $C(t,\cdot)$ have bounded variation on $[0,t]$ and may be discontinuous. The *Cauchy operator* $(\mathbf{C}f)(t) := \int_0^t C(t,s) f(s)\, ds$ converts locally summable functions to continuous functions [1, 2, 3]).

3. We use here, for some psychological reason, the construction of the Cauchy function, which is independent from [3, Chapter 3]. In [7] the formula

$$C(t,s)=E+\int_s^t P(\tau,s)\,d\tau$$

was obtained, where E is the unit matrix,

$$P(t,s)=\sum_{n=1}^{\infty}K_n(t,s), \tag{2.3}$$

iterated kernels K_n are defined by equalities

$$K_1(t,s)=R(t,t)-R(t,s),$$
$$K_{n+1}(t,s)=\int_s^t K_n(t,\tau)\big(R(\tau,\tau)-R(\tau,s)\big)\,d\tau,\quad n\ge 1.$$

Neumann's series (2.3) is uniformly convergent for a.e. $t\in[0,\infty)$ and all $s\in[0,t]$ (see [7]). By an usual argument, for a.e. τ the equality

$$P(\tau,s)=R(\tau,\tau)-R(\tau,s)+\int_s^\tau P(\tau,\theta)\big(R(\theta,\theta)-R(\theta,s)\big)\,d\theta$$

holds. Integrating in τ from s to t and changing the order of integration, we get

$$C(t,s)=E+\int_s^t\Big(E+\int_\tau^t P(\theta,\tau)\,d\theta\Big)\big(R(\tau,\tau)-R(\tau,s)\big)\,d\tau.$$

Thus, the equality

$$C(t,s)=E+\int_s^t C(t,\tau)\big(R(\tau,\tau)-R(\tau,s)\big)\,d\tau \tag{2.4}$$

holds for all $(t,s)\in\Delta$ (see also [3, (3.1.5)].

The generalized semigroup equality

$$C(t,s)=C(t,\tau)C(\tau,s)+\int_\tau^t C(t,\xi)\int_s^\tau d_\eta R(\xi,\eta)C(\eta,s)\,d\xi, \tag{2.5}$$

for $0\le s\le\tau\le t$, is obtained in [9] (see also [3, p. 97]).

4. Action of operators in γ-spaces. We denote here the operator

$$(\mathbf{R}x)(t)=\int_0^t d_sR(t,s)\,x(s).$$

Lemma 2.1. *Let R satisfy both conditions $\mathsf{D}(\delta)$ and (V). Then, for any $\gamma\ge 0$, the operator $\mathbf{R}$ acts from $\mathcal{C}_\gamma$ to $\mathcal{M}_\gamma$, and $\|\mathbf{R}\|_\gamma\le V(R)\,e^{\gamma\delta}$.*

Proof. Let $x\in\mathcal{C}_\gamma$; then

$$\begin{aligned}|(\mathbf{R}x)(t)|&\le \operatorname{Var}_{t-\delta}^t R(t,\cdot)\cdot\max_{s\in[t-\delta,t]}|x(s)|\\&\le \operatorname{Var}_0^t R(t,\cdot)\cdot\max_{s\in[t-\delta,t]}e^{-\gamma s}\|x\|_\gamma\\&\le e^{-\gamma t}\cdot V(R)\,e^{\gamma\delta}\cdot\|x\|_\gamma\end{aligned}$$

for all $t\ge 0$. □

Note that we do not need the condition $\mathsf{D}(\delta)$ in the case $\gamma=0$.

Lemma 2.2. *Let the Cauchy function of the system* (2.1) *satisfy the exponential estimate*

$$|C(t,s)| \le N e^{-\beta(t-s)}, \quad t \ge s \ge 0,$$

with $\beta > 0$, *and* $\gamma \in [0,\beta)$. *Then the operator* $\mathbf{C}$ *maps* $\mathcal{M}_\gamma$ *into* $\mathcal{C}_\gamma$, *and* $\|\mathbf{C}\|_\gamma \le \frac{K}{\beta-\gamma}$.

Proof. Indeed, the exponential estimate yields that

$$e^{\gamma t}|(\mathbf{C}f)(t)| \le \int_0^t K e^{-(\beta-\gamma)(t-s)} e^{\gamma s}|f(s)|\,ds \le \frac{K}{\beta-\gamma}\|f\|_\gamma.$$

□

3. Multivalued delay feedback

Let us assume now that the measurement of the state $x(t)$ of the system is performed by the non-ideal unit; so instead of $Bx(t)$ we get some point of the set

$$\operatorname{co} Bx[t-\delta,t] := \operatorname{co}\{Bx(s) : s \in [t-\delta,t]\},$$

where $\operatorname{co} S$ denotes a convex hull of the set $S \subset \mathbb{R}^n$.

Thus we have, instead of the equation (1.1), the functional differential inclusion

$$\dot{x}(t) \in \int_0^t d_s A(t,s)\,x(s) + B \text{ co } x[\max(0,t-\delta),t] + f(t). \tag{3.1}$$

A function $x : [0,T) \to \mathbb{R}^n$ is said to be a *solution* of this inclusion if it is locally absolutely continuous and satisfy the inclusion (3.1) for a.e. $t \in [0,T)$. In a usual way, each solution is extendable to $[0,\infty)$. So we consider all of them being defined on $[0,\infty)$.

Let $\mathfrak{R}$ be the set of all functions $r : \Delta \to \mathbb{R}$ such that

- r satisfies condition (R);
- $r(t,s) = 0$ for $s \le \max(0,t-\delta)$, $r(t,s) = 1$ for $s \ge t$, and the function $r(t,\cdot)$ is nondecreasing on $[t-\delta,t]$;

and $\mathfrak{C}$ be the set of the corresponding Cauchy functions C for the operators

$$(\mathbf{R}x)(t) = \int_0^t d_s A(t,s)\,x(s) + B(\mathbf{r}x)(t),$$

where

$$(\mathbf{r}x)(t) := \int_0^t d_s r(t,s)\,x(s).$$

Theorem 3.1. *Given a solution* x *of the inclusion* (3.1), *there exists a function* $r \in \mathfrak{R}$ *such that* x *is a solution of the equation*

$$\dot{x}(t) = \int_0^t d_s A(t,s)\,x(s) + B\int_0^t d_s r(t,s)\,x(s) + f(t). \tag{3.2}$$

Proof. We refer the reader to [5, Chapter III] for comprehensive exposition of measurable multifunctions, their properties, and operations on them.

For any set $S \subset \mathbb{R}$ and multifunction X, we denote $X(S) = \bigcup_{t\in S} X(t)$. Consider the solution $x : [0,\infty) \to \mathbb{R}^n$. Let the function $F : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ be given by the correspondence

$$u = (\lambda_0,\dots,\lambda_n,\tau_0,\dots,\tau_n) \mapsto F(u) = B\sum_{i=0}^n \lambda_i x(\tau_i).$$

Denote by Λ the n-simplex,

$$\Lambda = \{\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) : \forall \lambda_i \ge 0,\ \sum_{i=0}^{n} \lambda_i = 1\},$$

and let

$$U(t) = \Lambda \times \underbrace{[\max(0, t-\delta), t] \times \ldots \times [\max(0, t-\delta), t]}_{n+1 \text{ times}}.$$

So F is continuous function and U is continuous multifunction with compact values. Take $g(t) = \dot{x}(t) - \int_0^t d_s A(t,s)\, x(s) - f(t)$, then for a.e. $t \in [0, \infty)$ we have

$$\dot{x}(t) = \int_0^t d_s A(t,s)\, x(s) + g(t) + f(t) \tag{3.3}$$

and

$$g(t) \in B \operatorname{co} x[\max(0, t-\delta), t].$$

Due to Carathéodory's fundamental theorem, $g(t) = B \sum_{i=0}^n \lambda_i x(\tau_i)$, with some $u = (\lambda_0, \dots, \lambda_n, \tau_0, \dots, \tau_n) \in U(t)$. So

$$g(t) \in F(U(t)) \quad \text{for a.e. } t \in [0, \infty).$$

According to Filippov's implicit function lemma, there exists a measurable function $u(t) = (\lambda_0(t), \dots, \lambda_n(t), \tau_0(t), \dots, \tau_n(t)) \in U(t)$ such that

$$g(t) = F(u(t)) = B \sum_{i=0}^{n} \lambda_i(t) x(\tau_i(t)) \quad \text{for a.e. } t.$$

Put

$$r(t,s) = \sum_{i=0}^{n} \lambda_i(t) h_i(t,s),$$

where

$$h_i(t,s) = \begin{cases} 0 & \text{if } 0 \le s \le \tau_i(t) < t \text{ or } 0 \le s < \tau_i(t) = t, \\ 1 & \text{otherwise,} \end{cases}$$

then the function r satisfies condition (R) and $g(t) = B \int_0^t d_s r(t,s)\, x(s)$ for a. e. $t \in [0, \infty)$. This, in conjunction with (3.3), proves the theorem. $\square$

4. Exponential estimate of Cauchy matrix function

We have to prove, for some class of equations (2.1), an inequality

$$|C(t,s)| \le N e^{-\beta(t-s)}$$

with $\beta \in (0, \lambda)$ and the parameter N expressed in terms of norms of the operators $\mathbf{R}$ and $\mathbf{C}$.

The main result of this section were obtained by the author of the text and O. Demyanchenko, following the way of V. V. Malygina [3, Theorem 3.3.2], but under the additional assumption.

To prove this, we need several lemmas.

Lemma 4.1. *If the operator* $\mathbf{C}$ *maps the space* $\mathcal{M}_\gamma$ *into* $\mathcal{C}_\gamma$, $\gamma \ge 0$, *then* $\|\mathbf{C}\|_\beta < \infty$ *for every* $\beta \in [0, \gamma]$.

Proof. We follow the ideas of [6, section 2.2] and [3, Lemma 3.3.1]. The inequality $\|\mathbf{C}\|_\gamma < \infty$ is deduced from the uniform boundedness principle for the system of functionals $\Phi_t f = e^{\gamma t}\int_0^t C(t,s)f(s)\,ds$.

For fixed i, $1 \le i \le n$, let $g_j(t) = e^{-\gamma s}\operatorname{sign} C_{ij}(t,s)$ and $g = \operatorname{col}(g_1,\dots,g_n)$, then $g \in \mathcal{M}_\gamma$ and $\|g\|_\gamma <$ const. Therefore,

$$\begin{aligned}\int_0^t e^{\gamma(t-s)}\sum_{j=1}^n |C_{ij}(t,s)|\,ds &= \Big|\int_0^t e^{\gamma t}\sum_{j=1}^n C_{ij}(t,s)g_j(s)\,ds\Big| \\ &= e^{\gamma t}|[(\mathbf{C}g)(t)]_i| \\ &\le \text{const}\cdot\|\mathbf{C}\|_\gamma.\end{aligned}$$

So

$$\int_0^t e^{\beta(t-s)}\sum_{j=1}^n |C_{ij}(t,s)|\,ds \le \int_0^t e^{\gamma(t-s)}\sum_{j=1}^n |C_{ij}(t,s)|\,ds < \text{const}\,.$$

From this estimate one can easily obtain that if $f \in \mathcal{M}_\beta$ then

$$\|\mathbf{C}f\|_\beta \le \text{const}\cdot\|f\|_\beta.$$

□

Analogously, from the uniform boundedness principle we get the following result.

Lemma 4.2. *If the operator* $\mathbf{R}$ *satisfy condition* (R) *and map the space* $\mathcal{C}_\gamma$ *into* $\mathcal{M}_\gamma$, $\gamma \ge 0$, *then* $\|\mathbf{R}\|_\gamma < \infty$.

If P is $n\times n$-matrix, we denote by P_i its i-th column. Let m and M be positives such that

$$m\sup_i |P_i| \le |P| \le M\sup_i |P_i|. \tag{4.1}$$

Lemma 4.3. *Let* R *satisfy conditions* (R) *and* (V). *If the operator* $\mathbf{C}$ *acts from* $\mathcal{M}_0$ *into* $\mathcal{C}_0$, *then*

$$|C_i(t,s)| \le K^* := 1 + \frac{1}{m}\|\mathbf{C}\|_0\,V(R) \tag{4.2}$$

for all $(t,s)\in\Delta$. *So*

$$\sup_{(t,s)\in\Delta}|C(t,s)| < \infty.$$

Proof. Consider the column $C_i(t,s)$ of the matrix $C(t,s)$ and let

$$z_s(\tau) = \begin{cases} 0, & \text{if } \tau < s,\\ \big(R(\tau,\tau) - R(\tau,s)\big)_i, & \text{if } \tau \ge s,\end{cases}$$

then $\|z_s\|_0 \le \frac{1}{m}V(R)$. According to Lemma 4.1, $\|\mathbf{C}\|_0 < \infty$. Due to (2.4),

$$C_i(t,s) = E_i + \int_0^t C(t,\tau)z_s(\tau)\,d\tau,$$

where E is the unit matrix. Therefore,

$$\begin{aligned}|C_i(t,s)| &\le 1 + \sup_{t\ge 0}\Big|\int_0^t C(t,\tau)z_s(\tau)\,d\tau\Big| \\ &\le 1 + \|\mathbf{C}\|_0\,\|z_s\|_0 \\ &\le 1 + \frac{1}{m}\|\mathbf{C}\|_0\,V(R).\end{aligned}$$

□

Lemma 4.4 (O. Demyanchenko). *Under the assumptions of Lemma 4.2, if $u \in \mathcal{C}_\gamma$, $0 < s \le \tau < \xi$, then*

$$\Big|e^{\gamma\xi}\int_s^\tau d_\eta R(\xi,\eta)\,u(\eta)\Big| \le \|\mathbf{R}\|_\gamma \cdot \|u\|_\gamma + e^{\gamma\xi}\Big[|R(\xi,s)-R(\xi,s-0)|\cdot|u(s)| \\ + |R(\xi,\tau+0)-R(\xi,\tau)|\cdot|u(\tau)|\Big].$$

For $0 = s < \tau < \xi$ we have

$$\Big|e^{\gamma\xi}\int_0^\tau d_\eta R(\xi,\eta)\,u(\eta)\Big| \le \|\mathbf{R}\|_\gamma \cdot \|u\|_\gamma + e^{\gamma\xi}|R(\xi,\tau+0)-R(\xi,\tau)|\cdot|u(\tau)|.$$

Proof. Let $\epsilon > 0$ be sufficiently small. Then the function

$$u_\epsilon(\eta) = \begin{cases} 0, & \text{if } 0 \le \eta \le s-\epsilon, \\ \epsilon^{-1}u(s)(\eta - s + \epsilon), & \text{if } s-\epsilon < \eta \le s, \\ u(\eta), & \text{if } s < \eta \le \tau, \\ -\epsilon^{-1}u(\tau)(\eta-\tau-\epsilon), & \text{if } \tau < \eta \le \tau+\epsilon, \\ 0, & \text{if } \tau+\epsilon < \eta < \infty, \end{cases}$$

belongs to $\mathcal{C}_\gamma$ and $\|u_\epsilon\|_\gamma \le \|u\|_\gamma$. Obviously,

$$\begin{aligned} &\Big|\int_s^\tau d_\eta R(\xi,\eta)\,u(\eta)\Big| \\ &= \Big|\int_s^\tau d_\eta R(\xi,\eta)\,u_\epsilon(\eta)\Big| \\ &\le \Big|\int_0^\xi d_\eta R(\xi,\eta)\,u_\epsilon(\eta)\Big| + \Big|\int_{s-\epsilon}^s d_\eta R(\xi,\eta)\,u_\epsilon(\eta)\Big| + \Big|\int_\tau^{\tau+\epsilon} d_\eta R(\xi,\eta)\,u_\epsilon(\eta)\Big|. \end{aligned}$$

We have

$$\Big|e^{\gamma\xi}\int_0^\xi d_\eta R(\xi,\eta)\,u_\epsilon(\eta)\Big| \le \|\mathbf{R}\|_\gamma \cdot \|u_\epsilon\|_\gamma \le \|\mathbf{R}\|_\gamma \cdot \|u\|_\gamma.$$

By integration by parts,

$$\begin{aligned} \int_{s-\epsilon}^s d_\eta R(\xi,\eta)\,u_\epsilon(\eta) &= R(\xi,s)\,u(s) - \epsilon^{-1}\int_{s-\epsilon}^s R(\xi,\eta)\,d\eta\,u(s) \\ &\to \big(R(\xi,s)-R(\xi,s-0)\big)\,u(s). \end{aligned}$$

Analogously,

$$\int_\tau^{\tau+\epsilon} d_\eta R(\xi,\eta)\,u_\epsilon(\eta) \to \big(R(\xi,\tau+0)-R(\xi,\tau)\big)\,u(\tau)$$

as $\epsilon \downarrow 0$. Combining these estimates, we complete the proof of the first assertion. The second one is obtained in the same way, with obvious simplification. □

Lemma 4.5. *Suppose that R satisfies condition* (R). *Almost every $s \in [0,\infty)$ has the following property: for a.e. $\xi \in [s,\infty)$ the function $R(\xi,\cdot)$ is continuous at the point s.*

Proof. By condition (R), the function $R(\xi,\eta)$ is measurable on (ξ,η) in Δ. Let $R(\xi,\eta)=R(\xi,0)$ for $\eta<0$ and $R(\xi,\eta)=R(\xi,\xi)$ for $\eta>\xi$. Then the functions $R_\delta(\xi,\eta)=R(\xi,\eta+\delta)$ are measurable on (ξ,η). The limits $R(\xi,\eta+0)=\lim_{\delta\downarrow 0}R_\delta(\xi,\eta)$ and $R(\xi,\eta-0)=\lim_{\delta\uparrow 0}R_\delta(\xi,\eta)$ exist for a.e. ξ, because the functions $R(\xi,\cdot)$ have locally bounded variation. So $R(\xi,\eta+0)$ and $R(\xi,\eta-0)$ are measurable functions. Therefore, the set

$$E=\{(\xi,s)\in\Delta: R(\xi,s-0)\neq R(\xi,s)\quad\text{or}\quad R(\xi,s)\neq R(\xi,s+0)\}$$

is measurable. For given ξ, since the function $R(\xi,\cdot)$ is a.e. continuous, $\operatorname{meas}\{\eta:(\xi,\eta)\in E\}=0$. Hence

$$\operatorname{meas}(E)=\int_0^\infty \operatorname{meas}\{\eta:(\xi,\eta)\in E\}\,d\xi=0.$$

Thus, for a.e. $s\in[0,\infty)$ the set $\{\xi:(\xi,s)\in E\}$ has zero measure. □

Theorem 4.6. *Suppose that*

(i) *the function R satisfies conditions* (R) *and* (V) *and the operator $\mathbf{R}$ maps the space $\mathcal{C}_\gamma$ into $\mathcal{M}_\gamma$;*
(ii) *the operator $\mathbf{C}$ maps $\mathcal{M}_\gamma$ into $\mathcal{C}_\gamma$.*

Then for every $\beta\in(0,\gamma)$ there exists $N>0$ such that for all $t\geq 0$ the estimate

$$|C(t,s)|\leq Ne^{-\beta(t-s)} \tag{4.3}$$

is valid for $s=0$ and a.e. $s\in(0,t]$.

Proof. We use here the notation introduced in the proof of the previous lemmas.
1. Let us multiply both sides of equality (2.5) by $e^{\beta(t-\tau)}$ and integrate in τ from s to t. For the i-th columns of the matrices we get

$$\begin{aligned}
&\frac{1}{\beta}C_i(t,s)(e^{\beta(t-s)}-1)\\
&=e^{\gamma t}\int_s^t C(t,\tau)\,e^{-\gamma\tau}\,e^{-(\gamma-\beta)(t-\tau)}C_i(\tau,s)\,d\tau+\\
&\quad+\int_s^t e^{-(\gamma-\beta)(t-\tau)}e^{\gamma t}\int_\tau^t C(t,\xi)\int_s^\tau d_\eta R(\xi,\eta)\left(e^{-\gamma\eta}\,e^{-\gamma(\tau-\eta)}C_i(\eta,s)\right)d\xi\,d\tau.
\end{aligned} \tag{4.4}$$

2. The first integral is estimated as follows. Let

$$u_s(\tau)=\begin{cases}0, & \text{if } 0\leq\tau<s,\\ e^{-\gamma\tau}\,e^{-(\gamma-\beta)(t-\tau)}C_i(\tau,s), & \text{if } \tau\geq s,\end{cases}$$

then, due to Lemma 4.3, $u_s\in\mathcal{M}_\gamma$, $\|u_s\|_\gamma\leq K^*$, and

$$\int_s^t C(t,\tau)\,e^{-\gamma\tau}\,e^{-(\gamma-\beta)(t-\tau)}C_i(\tau,s)\,d\tau=(\mathbf{C}u_s)(t).$$

Since $\mathbf{C}\mathcal{M}_\gamma\subset\mathcal{C}_\gamma$, Lemma 4.1 implies that

$$\begin{aligned}
\Big|e^{\gamma t}\int_s^t C(t,\tau)\,e^{-\gamma\tau}\,e^{-(\gamma-\beta)(t-\tau)}C_i(\tau,s)\,d\tau\Big| &\leq\|\mathbf{C}u_s\|_\gamma\\
&\leq\|\mathbf{C}\|_\gamma\cdot\|u_s\|_\gamma\\
&\leq\|\mathbf{C}\|_\gamma\cdot K^*.
\end{aligned}$$

3. Now we estimate the second integral. Note that, according to Lemma 4.1, $\|\mathbf{C}\|_0 < \infty$. Let
$$u_{s,\tau}(\eta) = e^{-\gamma\eta}\, e^{-\gamma(\tau-\eta)} C_i(\eta, s),$$
then, by Lemma 4.3, $\|u_{s,\tau}\|_\gamma \le K^*$ (we recall that $\eta \le \tau \le \xi$).

Let $\mathfrak{A}$ be a set of points $s \in [0,\infty)$ such that the functions $R(\xi,\cdot)$ are continuous at s for a.e. $\xi \in [s,\infty)$. Suppose that $s, \tau \in \mathfrak{A}$. Then, according to Lemma 4.4, the function
$$U_{s,\tau}(\xi) := \int_s^\tau d_\eta R(\xi,\eta)\, u_{s,\tau}(\eta)$$
satisfies, for $\xi > \tau$, the estimate
$$e^{\gamma\xi} |U_{s,\tau}(\xi)| \le \|\mathbf{R}\|_\gamma \cdot \|u_{s,\tau}\|_\gamma \le \|\mathbf{R}\|_\gamma\, K^*,$$
where $\|\mathbf{R}\|_\gamma < \infty$ due to Lemma 4.2. Also let
$$v_{s,\tau}(\xi) = \begin{cases} 0, & \text{if } a \le \xi \le \tau, \\ U_{s,\tau}(\xi), & \text{if } \xi > \tau, \end{cases}$$
then $v_{s,\tau} \in \mathcal{M}_\gamma$, $\|v_{s,\tau}\|_\gamma \le \|\mathbf{R}\|_\gamma\, K^*$ and
$$V_{s,\tau}(t) := \int_\tau^t C(t,\xi) U_{s,\tau}(\xi)\, d\xi = (\mathbf{C} v_{s,\tau})(t).$$
¿From Lemma 4.1 we have $\|V_{s,\tau}\|_\gamma \le \|\mathbf{C}\|_\gamma\, \|\mathbf{R}\|_\gamma\, K^*$. By Lemma 4.5, the set $[0,\infty) \setminus \mathfrak{A}$ has measure 0. Therefore, for all $s \in \mathfrak{A}$ the second integral in (4.4) is estimated with
$$\int_s^t e^{-(\gamma-\beta)(t-\tau)} \|\mathbf{C}\|_\gamma\, \|\mathbf{R}\|_\gamma\, K^*\, d\tau \le \frac{1}{\gamma-\beta} \|\mathbf{C}\|_\gamma\, \|\mathbf{R}\|_\gamma\, K^*.$$
4. Thus, for a.e. s,
$$\frac{1}{\beta} |C_i(t,s)|\, (e^{\beta(t-s)} - 1) \le \|\mathbf{C}\|_\gamma\, K^* \Big(1 + \frac{1}{\gamma-\beta}\, \|\mathbf{R}\|_\gamma\Big).$$
Hence, in accordance with (4.2), we get
$$e^{\beta(t-s)} |C_i(t,s)| \le \beta \|\mathbf{C}\|_\gamma\, K^* \Big(1 + \frac{1}{\gamma-\beta}\, \|\mathbf{R}\|_\gamma\Big) + K^*.$$
5. Eventually, combining this with (4.1) and (4.2), we get the constant N for (4.3):
$$N = M\Big[\beta \|\mathbf{C}\|_\gamma \Big(1 + \frac{1}{\gamma-\beta}\, \|\mathbf{R}\|_\gamma\Big) + 1\Big] \Big(1 + \frac{1}{m} \|\mathbf{C}\|_0\, V(R)\Big).$$
□

Remark 4.7. The estimate of N is uniform for a class of equations with uniformly bounded norms $\|\mathbf{R}\|_\gamma$, $V(R)$, $\|\mathbf{C}\|_\gamma$, and, therefore, $\|\mathbf{C}\|_0$.

Remark 4.8. Theorem 3.3.2 of the book [3] states a more strong assertion – the exponential estimate of the Cauchy function is claimed without the hypothesis (V) – but unfortunately this theorem has still no proof.

Remark 4.9. In [8] we have used [3, Theorem 3.3.2] without criticism. This result clears a gap.

5. Uniform exponential estimate for a class of Cauchy functions

We shall get (4.3) for all $C \in \mathfrak{C}$, provided that $\gamma \in (0, \lambda)$ and δ is sufficiently small. For every $r \in \mathfrak{R}$ denote

$$R(t,s) = A(t,s) + Br(t,s),$$

then R satisfies condition (R) and

$$V(R) \le V(A) + |B|.$$

By Lemma 2.1, the operator $\mathbf{R}$ acts from $\mathcal{C}_\gamma$ to $\mathcal{M}_\gamma$ and

$$\|\mathbf{R}\|_\gamma \le (V(A) + |B|)\, e^{\gamma \max(\delta, \delta_A)},$$

so the condition (i) of Theorem 4.6 is valid.

For the second condition of Theorem 4.6, we use the following lemma. In the sequel, unless otherwise mentioned, we consider $\gamma \in [0, \lambda)$, i.e., we allow γ be zero too.

Lemma 5.1. *Let $\delta > 0$ be such that*

$$q := |B| \Big(\frac{K}{\lambda - \gamma} (\|\mathbf{A}\|_\gamma + |B|) + 1 \Big)\, e^{\gamma\delta} \delta < 1, \tag{5.1}$$

where the value of K is given by hypothesis (H1). Then every operator $\mathbf{C} \in \mathfrak{C}$ maps $\mathcal{M}_\gamma$ into $\mathcal{C}_\gamma$, and

$$\|\mathbf{C}\|_\gamma \le \frac{K}{\lambda - \gamma} \, \frac{1}{1 - q}. \tag{5.2}$$

Proof. We use a general idea of W-method [1, 2, 3]. Namely, consider a model problem

$$\begin{gathered} \dot{x}(t) = \int_0^t d_s A(t,s)\, x(s) + Bx(t) + z(t), \\ x(0) = 0. \end{gathered} \tag{5.3}$$

Its solution

$$x(t) = (\mathbf{W}z)(t) = \int_0^t W(t,s) z(s)\, ds.$$

The operator $\mathbf{W}$, due to Lemma 2.2, acts from $\mathcal{M}_\gamma$ to $\mathcal{C}_\gamma$ and

$$\|\mathbf{W}\|_\gamma \le \frac{K}{\lambda - \gamma}. \tag{5.4}$$

The equality $x = \mathbf{C}f$ takes place if and only if x is a solution of the problem

$$\begin{gathered} \dot{x}(t) = \int_0^t d_s A(t,s)\, x(s) + B \int_0^t d_s r(t,s)\, x(s) + f(t), \\ x(0) = 0 \end{gathered} \tag{5.5}$$

with the corresponding kernel $r \in \mathfrak{R}$. Let $f \in \mathcal{M}_\gamma$. We shall seek the solution of the latter problem represented as $x = \mathbf{W}z$, where $z \in \mathcal{M}_\gamma$; so $x \in \mathcal{C}_\gamma$.

Substituting $\mathbf{W}z$ for x in (5.3) and (5.5), we get the equation

$$z - B(\mathbf{r} - I)\mathbf{W}z = f \tag{5.6}$$

in the space $\mathcal{M}_\gamma$. In order to solvability of (5.6), it is sufficient that

$$|\mathbf{B}| \cdot \|(\mathbf{r} - I)\mathbf{W}\|_\gamma < 1. \tag{5.7}$$

We estimate this norm as follows.

Let $x = \mathbf{W}z$. From (5.4) we have $\|x\|_\gamma \le \frac{K}{\lambda-\gamma}\|z\|_\gamma$; so

$$e^{\gamma t}\Big|\int_0^t d_sA(t,s)\,x(s)\Big| \le \|\mathbf{A}\|_\gamma \frac{K}{\lambda-\gamma}\|z\|_\gamma.$$

Therefore, due to (5.3),

$$|\dot{x}(t)| \le \Big(\frac{K}{\lambda-\gamma}(\|\mathbf{A}\|_\gamma + |B|) + 1\Big)e^{-\gamma t}\|z\|_\gamma.$$

So we have

$$\begin{aligned}
\big|\big((\mathbf{r}-I)\mathbf{W}z\big)(t)\big| &= \Big|\int_{t-\delta}^t d_s r(t,s)\,x(s) - x(t)\Big| \\
&\le \Big|\int_{t-\delta}^t r(t,s)\dot{x}(s)\,ds\Big| \\
&\le \sup_{s\in[t-\delta,t]} |\dot{x}(s)|\,\delta \\
&\le e^{-\gamma t}\Big(\frac{K}{\lambda-\gamma}(\|\mathbf{A}\|_\gamma + |B|) + 1\Big)\,e^{\gamma\delta}\,\delta\,\|z\|_\gamma.
\end{aligned}$$

Hence condition (5.1) implies (5.7). For $f \in \mathcal{M}_\gamma$ the equation (5.6) has a solution $z \in \mathcal{M}_\gamma$ and $\mathbf{C}f = x = \mathbf{W}z \in \mathcal{C}_\gamma$. Thus $\mathbf{C} : \mathcal{M}_\gamma \to \mathcal{C}_\gamma$.

To get (5.2), note that, since (5.6), $\|z\|_\gamma \le \frac{1}{1-q}\,\|f\|_\gamma$. Due to (5.4),

$$\|x\|_\gamma \le \|\mathbf{W}\| \cdot \|z\|_\gamma \le \frac{K}{\lambda-\gamma}\,\frac{1}{1-q}\|f\|_\gamma.$$

□

All these estimates hold uniformly with respect to $r \in \mathfrak{R}$. If $\beta \in (0,\lambda)$, choose some $\gamma \in (\beta,\lambda)$. Then we can apply Lemma 5.1 and Theorem 4.6. Thus, for all $r \in \mathfrak{R}$ we have inequality (4.3) with constant N independent on $r \in \mathfrak{R}$. As a result, we obtain the following theorem.

Theorem 5.2. *For every $\beta \in (0,\lambda)$, if $\delta > 0$ is sufficiently small, then there exists a constant N such that every Cauchy function $C \in \mathfrak{C}$ possesses inequality* (4.3).

6. Exponential stability of inclusion (3.1)

We recall that the hypotheses (H1) and (H2) are assumed. We also assume that δ is such that estimate (4.3) holds for every $C \in \mathfrak{C}$ (see Theorem 5.2), and $\beta \in (0,\lambda)$.

Theorem 6.1. *If $\delta > 0$ is sufficiently small, then there exists $N > 0$ such that every solution x of inclusion* (3.1) *obeys the estimate*

$$|x(t)| \le Ne^{-\beta t}|x(0)| + \int_0^t Ne^{-\beta(t-s)}|f(s)|\,ds. \tag{6.1}$$

Proof. For arbitrary solution x of inclusion (3.1), let $r \in \mathfrak{R}$ be such that x is solution of equation (3.2). If C is the Cauchy function for equation (3.2), then it satisfies the estimate (4.3) (Theorem 5.2). Then the desired inequality follows immediately from (2.2). □

This result implies a few consequences, which we usually mean talking about exponential stability (see [3]).

Let $U \subset \mathbb{R}^n$ be a bounded subset. Denote $\|U\| = \sup_{u\in U} |u|$ and let $X(U)$ be the set of all solutions x of inclusion (3.1) such that $x(0) \in U$.

1. In the proof of Lemma 2.2 we have done calculation, which leads to following result.

Corollary 6.2. *Let $\gamma \in [0, \beta)$. If $f \in \mathcal{M}_\gamma$, then solutions x of* (3.1) *belong to $\mathcal{C}_\gamma$ and*

$$\sup_{t\geq 0} e^{\gamma t}|x(t)| \leq N\|U\| + \frac{N}{\beta-\gamma}\|f\|_\gamma$$

$$\limsup_{t\to\infty} e^{\gamma t}|x(t)| \leq \frac{N}{\beta-\gamma} \lim_{T\to\infty} \operatorname*{ess\,sup}_{t\geq T} e^{\gamma t}|f(t)|.$$

2. The another way to estimate the integral term in the right of (6.1) is as follows. Let k be an integer such that $t \in (k, k+1]$. Then, summing the geometric progression, we get

$$\begin{aligned}
\Big|e^{\gamma t}\int_0^t C(t,s)f(s)\,ds\Big| &\leq e^{\gamma t}\sum_{i=1}^{k+1}\int_{i-1}^{i} Ne^{-\beta(t-s)}|f(s)|\,ds\\
&\leq Ne^{-(\beta-\gamma)t}\sum_{i=1}^{k+1} e^{(\beta-\gamma)i}\int_{i-1}^{i} e^{\gamma s}|f(s)|\,ds\\
&\leq \frac{Ne^{2(\beta-\gamma)}}{e^{\beta-\gamma}-1}\sup_{t\geq 0}\int_t^{t+1} e^{\gamma s}|f(s)|\,ds.
\end{aligned}$$

Thus we have the following result.

Corollary 6.3. *Let $\gamma \in [0, \beta)$. Then there exists a constant $M > 0$ such that all solutions $x \in X(U)$ of* (3.1) *satisfy the estimates*

$$|x(t)| \leq N\|U\|e^{-\beta t} + Me^{-\gamma t}\sup_{t\geq 0}\int_t^{t+1} e^{\gamma s}|f(s)|\,ds,$$

$$\limsup_{t\to\infty} e^{\gamma t}|x(t)| \leq M\limsup_{t\to\infty}\int_t^{t+1} e^{\gamma s}|f(s)|\,ds.$$

3. We shall say that system (2.1), or (3.1), is homogeneous since $t = T$ if $f(t) = 0$ for all $t > T$. We assume this for some $T > \delta$.

Let x be a solution of inclusion (3.1) and r be a corresponding kernel in accordance with Theorem 3.1. Denote

$$\phi(t) = \int_{\min(t-\delta,T)}^{T} d_sA(t,s)\,x(s) + B\int_{\min(t-\delta,T)}^{T} d_sr(t,s)\,x(s),$$

then

$$|\phi(t)| \leq (V(A) + |B|)\sup_{s\in[T-\delta,T]}|x(s)| \quad \text{for } t \in [T, T+\delta],$$
$$\phi(t) = 0 \quad \text{for } t > T+\delta,$$

and

$$\dot x(t) = \int_T^t d_sA(t,s)\,x(s) + B\int_T^t d_sr(t,s)\,x(s) + \phi(t)$$

for $t \geq T$.

The Cauchy formula (2.2) and the exponential estimate (4.3) imply

$$\begin{aligned}|x(t)| &\leq Ne^{-\beta(t-T)}|x(T)| + \int_T^{T+\delta} Ne^{-\beta(t-s)}|\phi(s)|\,ds \\ &\leq N\Big(1 + \big(V(A) + |B|\big)\frac{e^{\beta\delta}-1}{\beta}\Big)e^{-\beta(t-T)} \sup_{s\in[T-\delta,T]}|x(s)|.\end{aligned}$$

Corollary 6.4. *If system* (3.1) *is homogeneous since $t = T > \delta$, then there is a constant $M > 0$ such that*

$$\sup_{s\in[t-\delta,t]}|x(s)| \leq Me^{-\beta(t-T)} \sup_{s\in[T-\delta,T]}|x(s)|$$

for all $t \geq T$.

Acknowledgements. The author would like to express his gratitude to V. V. Malygina for the idea of the proof of Theorem 4.6. The helpful discussion on the seminar of Prof. Z. B. Tsalyuk, at Kuban State University, has yielded a respectable improvement of this result.

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