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MAXIMUM PRINCIPLE AND EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR SYSTEMS INVOLVING DEGENERATE P-LAPLACIAN OPERATORS

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Abstract. We study the maximum principle and existence of positive solutions for the nonlinear system

$$\begin{gathered} -\Delta_{p,P} u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad \text{in } \Omega, \\ -\Delta_{Q,q} v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where the degenerate p-Laplacian defined as $\Delta_{p,P} u = \operatorname{div}[P(x)|\nabla u|^{p-2}\nabla u]$. We give necessary and sufficient conditions for having the maximum principle for this system and then we prove the existence of positive solutions for the same system by using an approximation method.

1. Introduction

One of the most useful and best known tools employed in the study of partial differential equations is the maximum principle, since they are an useful tool to prove many results such as existence, multiplicity and qualitative properties for their solutions.

The maximum principle have been studied for linear elliptic systems. In particular, de Figueiredo and Mitidieri [6, 7, 8] gave a necessary and sufficient conditions for the maximum principle. In [12, 13] the authors proved sufficient and necessary conditions for having the maximum principle and the existence of positive solutions for linear systems involving Laplace operator with variable coefficients. These results have been extended in [11], to the nonlinear system

$$\begin{gathered} -\Delta_p u_i = \sum_{j=1}^{n} a_{ij}|u_j|^{p-2}u_j + f_i(x) \quad \text{in } \Omega, \\ u_i = 0, \quad i = 1, 2, \dots n \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Boushkief, Serag and de Thélin [5], proved the validity of the maximum principle and the existence of positive solutions for the following nonlinear elliptic system of

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two equations involving different operators Δ_p, Δ_q defined on bounded domain Ω of $\mathbb{R}^n$, with constant coefficients a, b, c and d

$$\begin{gathered} -\Delta_p u = a|u|^{p-2}u + b|u|^{\alpha}|v|^{\beta}v + f \quad \text{in } \Omega, \\ -\Delta_q v = c|u|^{\alpha}|v|^{\beta}u + d|v|^{q-2}v + g \quad \text{in } \Omega, \\ u = u = 0 \quad \text{on } \Omega. \end{gathered} \tag{1.2}$$

These results have been extended in [16] to the following nonlinear system defined on unbounded domain with variable coefficients

$$\begin{gathered} -\Delta_p u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad x \in \mathbb{R}^n, \\ -\Delta_q v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad x \in \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \quad u, v > 0 \quad \text{in } \mathbb{R}^n. \end{gathered} \tag{1.3}$$

Here, we consider nonlinear system involving degenerated p-Laplacian operators. We study the following nonlinear system

$$\begin{gathered} -\Delta_{p,P} u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad \text{in } \Omega, \\ -\Delta_{Q,q} v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where Ω is a bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\Delta_{p,P}$ with $p > 1$, $p \neq 2$ and $P(x)$ a weight function, denotes the degenerate p-Laplacian defined by $\Delta_{p,P} u = \operatorname{div}[P(x)|\nabla u|^{p-2}\nabla u]$, $\alpha, \beta \geq 0$, f, g are given functions and $a(x), b(x), c(x)$ and $d(x)$ are bounded variable coefficients. We consider here a generalization for the p-Laplacian to the degenerated p-Laplacian. We obtain necessary and sufficient conditions on the variable coefficients for having the maximum principle for system (1.4) and then we prove the existence of positive solutions for this system by using an approximation method.

This paper is organized as follows: In section 2, we give some assumptions on the coefficients $a(x), b(x), c(x)$ and $d(x)$, and on the functions f, g to insure the existence of solution for system (1.4) in a suitable weighted Sobolev space. We also introduce some technical results and some notations, which are established in [1, 2, 3, 10]. Section 3 is devoted to the maximum principle of system (1.4). Finally, in section 4, we prove the existence of solutions for system (1.4) using an approximation method already used in [4].

2. Technical Results

Now, we introduce some technical results [10] concerning the degenerated homogeneous eigenvalue problem

$$\begin{gathered} -\Delta_{H,p} u = \operatorname{div}[H(x)|\nabla u|^{p-2}\nabla u] = \lambda G(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $H(x)$ and $G(x)$ are measurable functions satisfying

$$\frac{\nu(x)}{c_1} < H(x) < c_1\nu(x), \tag{2.2}$$

for a.e. $x \in \Omega$ with some constant $c_1 \geq 1$, where $\nu(x)$ is a weight function, i.e., a function which is measurable and positive a.e. in Ω, satisfying the conditions

$$\nu \in L^1_{\mathrm{Loc}}(\Omega), \quad \nu^{-\frac{1}{p-1}} \in L^1_{\mathrm{Loc}}(\Omega), \quad \nu^{-s} \in L^1(\Omega), \tag{2.3}$$

with

$$s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \tag{2.4}$$

$$G(x) \in L^{\frac{k}{k-p}}(\Omega), \tag{2.5}$$

for some constant k satisfying $p < k < p_s^*$, where $p_s^* = \frac{NP_s}{N-P_s}$ with $P_s = \frac{ps}{s+1} < p < p_s^*$.

Lemma 2.1. *There exists the least (i.e. the first or principal) eigenvalue $\lambda = \lambda_G(p, \Omega) > 0$ and at least one corresponding eigenfunction $u = u_G \geq 0$ a.e. in Ω of the eigenvalue problem* (2.1).

Theorem 2.2. *Let $H(x)$ satisfy* (2.2) *and $G(x)$ satisfy* (2.5), *then* (2.1) *admits a positive principal eigenvalue $\lambda_G(p)$. Moreover, it is characterized by*

$$\lambda_G(p) \int_\Omega G(x)|u|^p \leq \int_\Omega H(x)|\nabla u|^p. \tag{2.6}$$

Now, let us introduce the weighted Sobolev space $W^{1,p}(\nu, \Omega)$ which is the set of all real valued functions u defined in Ω for which (see [3, 10])

$$\|u\|_{W^{1,p}(\nu,\Omega)} = \Big[\int_\Omega |u|^p + \int_\Omega \nu(x)|\nabla u|^p\Big]^{1/p} < \infty. \tag{2.7}$$

Since we are dealing with the Dirichlet problem, we introduce also the space $W_0^{1,p}(\nu, \Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu, \Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(\nu,\Omega)} = \Big[\int_\Omega \nu(x)|\nabla u|^p\Big]^{1/p} < \infty, \tag{2.8}$$

which is equivalent to the norm given by (2.7). Both spaces $W^{1,p}(\nu, \Omega)$ and $W_0^{1,p}(\nu, \Omega)$ are well defined reflexive Banach Spaces. The space $W_0^{1,p}(\nu, \Omega)$ is compactly imbedding into the space $L^p(\Omega)$, under the conditions given by (2.3) and (2.4), i.e.

$$W_0^{1,p}(\nu, \Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \tag{2.9}$$

which means that

$$\int_\Omega |u|^p \leq c_2 \int_\Omega \nu(x)|\nabla u|^p, \text{ i.e., } \|u\|_{L^p(\Omega)} \leq c\, \|u\|_{W_0^{1,p}(\nu,\Omega)}. \tag{2.10}$$

3. Maximum principle

In this paper, we assume that

$$\begin{gathered} \alpha, \beta \geq 0; \quad p, q > 1, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \\ f \in L^{p^*}(\Omega), \quad g \in L^{q^*}(\Omega), \quad \frac{1}{p} + \frac{1}{p^*} = 1, \quad \frac{1}{q} + \frac{1}{q^*} = 1. \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} P(x) \in L^1_{\rm Loc}(\Omega), \quad (P(x))^{-\frac{1}{p-1}} \in L^1_{\rm Loc}(\Omega), \quad (P(x))^{-s} \in L^1(\Omega) \\ \text{with } s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \\ Q(x) \in L^1_{\rm Loc}(\Omega), \quad (Q(x))^{-\frac{1}{q-1}} \in L^1_{\rm Loc}(\Omega), \quad (Q(x))^{-t} \in L^1(\Omega) \\ \text{with } t \in (\frac{N}{q}, \infty) \cap [\frac{1}{q-1}, \infty), \end{gathered} \tag{3.2}$$

We also assume that the variable coefficients $a(x), b(x), c(x)$, and $d(x)$ are bounded smooth positive functions such that

$$\begin{aligned} a(x) &\in L^{\frac{k}{k-p}}(\Omega)\cap L^p(\Omega), \quad \text{with } p<k<p_s^*,\\ d(x) &\in L^{\frac{l}{l-q}}(\Omega)\cap L^q(\Omega), \quad \text{with } q<l<q_t^* \end{aligned} \tag{3.3}$$

and

$$b(x) < (a(x))^{\frac{\alpha+1}{p}}(d(x))^{\frac{\beta+1}{q}}, \quad c(x) < (a(x))^{\frac{\alpha+1}{p}}(d(x))^{\frac{\beta+1}{q}}. \tag{3.4}$$

We say that system (1.4) satisfies the maximum principle if $f\ge 0$, $g\ge 0$ implies $u\ge 0, v\ge 0$ for any solution (u,v) for system (1.4)

Theorem 3.1. *Assume that* (3.1)–(3.4) *are satisfied. Then, the maximum principle holds for system* (1.4) *if*

$$\lambda_a(p)>1, \quad \lambda_d(q)>1, \tag{3.5}$$

$$(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}-1>0. \tag{3.6}$$

Conversely, if the maximum principle holds, then (3.5) and (3.7) are satisfied, where

$$(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}} > \Theta \inf_{x\in\Omega}(\frac{b(x)}{a(x)})^{\frac{\alpha+1}{p}} \inf_{x\in\Omega}(\frac{c(x)}{d(x)})^{\frac{\beta+1}{q}}, \tag{3.7}$$

where

$$\Theta = \frac{\inf_\Omega(\frac{\phi^p}{\psi^q})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\sup_\Omega(\frac{\phi^p}{\psi^q})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}<1,$$

and ϕ (respectively ψ) is the positive eigenfunction associated to $\lambda_a(p)$ (respectively $\lambda_d(q)$) normalized by $\|\phi\|_\infty=\|\psi\|_\infty=1$.

Proof. The condition is necessary: If $\lambda_a(p)\le 1$, then the functions $f:=a(x)(1-\lambda_a(p))\phi^{p-1}$, $g:=0$ are nonnegative, nevertheless $(-\phi,0)$ satisfies (1.4), which contradicts the maximum principle.

Similarly, if $\lambda_d(q)\le 1$, then the functions $f:=0$, $g:=d(x)(1-\lambda_d(q))\psi^{q-1}$ are nonnegative, nevertheless $(0,-\psi)$ satisfies (1.4), which means that the maximum principle does not hold.

Now suppose that $\lambda_a(p)>1$, $\lambda_d(q)>1$ and (3.7), and hence (3.6), does not hold, i.e.

$$(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}} \le \Theta \inf_{x\in\Omega}\big(\frac{b(x)}{a(x)}\big)^{\frac{\alpha+1}{p}} \inf_{x\in\Omega}\big(\frac{c(x)}{d(x)}\big)^{\frac{\beta+1}{q}}.$$

Now, we want to fined a positive real number ξ such that

$$\begin{aligned} A(\frac{\phi^p}{\psi^q})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} &\le \xi, \quad A>0,\\ B(\frac{\psi^q}{\phi^p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} &\le \frac{1}{\xi}, \quad B>0. \end{aligned} \tag{3.8}$$

Equation (3.8) is satisfied if

$$A\sup_\Omega\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \xi \le \frac{1}{B}\inf_\Omega\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{3.9}$$

Let

$$A = \big[(\lambda_a(p)-1)\sup_{x\in\Omega}(\frac{a(x)}{b(x)})\big]^{\frac{\alpha+1}{p}} \quad \text{and} \quad B = \big[(\lambda_d(q)-1)\sup_{x\in\Omega}(\frac{d(x)}{c(x)})\big]^{\frac{\beta+1}{q}},$$

then (3.9) becomes

$$\begin{aligned}&\big[(\lambda_a(p)-1)\sup_{x\in\Omega}(\frac{a(x)}{b(x)})\big]^{\frac{\alpha+1}{p}}\sup_{\Omega}\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\\ &\le \xi \le \frac{1}{\big[(\lambda_d(q)-1)\sup_{x\in\Omega}(\frac{d(x)}{c(x)})\big]^{\frac{\beta+1}{q}}}\inf_{\Omega}\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}.\end{aligned}$$

Then, ξ exists if

$$\begin{aligned}&(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}\\ &\le \frac{1}{(\sup_{x\in\Omega}(\frac{a(x)}{b(x)}))^{\frac{\alpha+1}{p}}(\sup_{x\in\Omega}(\frac{d(x)}{c(x)}))^{\frac{\beta+1}{q}}}\frac{\inf_\Omega\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\sup_\Omega\big(\frac{\phi^p}{\psi^q}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}\end{aligned}$$

Therefore,

$$(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}} \le \Theta\inf_{x\in\Omega}(\frac{b(x)}{a(x)})^{\frac{\alpha+1}{p}}\inf_{x\in\Omega}(\frac{c(x)}{d(x)})^{\frac{\beta+1}{q}}.$$

So, ζ exists if (3.7), and hence (3.6), does not hold.

If $\xi = \big(\frac{D^q}{C^p}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$ with $C, D > 0$, then (3.8) implies

$$\begin{aligned}&\big[(\lambda_a(p)-1)\sup_{x\in\Omega}(\frac{a(x)}{b(x)})\big]^{\frac{\alpha+1}{p}}(\frac{(C\phi)^p}{(D\psi)^q})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\\ &\le 1 \le \frac{1}{\big[(\lambda_d(q)-1)\sup_{x\in\Omega}(\frac{d(x)}{c(x)})\big]^{\frac{\beta+1}{q}}}(\frac{(C\phi)^p}{(D\psi)^q})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}},\end{aligned}$$

and hence, for some $x\in\Omega$, we have

$$\big[(\lambda_a(p)-1)(\frac{a(x)}{b(x)})\big]\big(\frac{(C\phi)^p}{(D\psi)^q}\big)^{\frac{\beta+1}{q}} \le \big[(\lambda_a(p)-1)\sup_{x\in\Omega}(\frac{a(x)}{b(x)})\big]\big(\frac{(C\phi)^p}{(D\psi)^q}\big)^{\frac{\beta+1}{q}} \le 1,$$

$$1 \le \frac{1}{\big[(\lambda_d(q)-1)\sup_{x\in\Omega}(\frac{d(x)}{c(x)})\big]}(\frac{(C\phi)^p}{(D\psi)^q})^{\frac{\alpha+1}{p}} \le \frac{(\frac{c(x)}{d(x)})}{(\lambda_d(q)-1)}(\frac{(C\phi)^p}{(D\psi)^q})^{\frac{\alpha+1}{p}},$$

which implies

$$\begin{aligned}a(x)(\lambda_a(p)-1)((C\phi)^p)^{\frac{\beta+1}{q}} &\le b(x)(D\psi)^{\beta+1},\\ d(x)(\lambda_d(q)-1)((D\psi)^q)^{\frac{\alpha+1}{p}} &\le c(x)(C\phi)^{\alpha+1}.\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}a(x)(\lambda_a(p)-1)(C\phi)^{p-1} &\le b(x)(D\psi)^{\beta+1}(C\phi)^{\alpha},\\ d(x)(\lambda_d(q)-1)(D\psi)^{q-1} &\le c(x)(C\phi)^{\alpha+1}(D\psi)^{\beta}.\end{aligned}$$

Then

$$\begin{aligned}f &= -a(x)(\lambda_a(p)-1)(C\phi)^{p-1} + b(x)(D\psi)^{\beta+1}(C\phi)^{\alpha} \ge 0,\\ g &= -d(x)(\lambda_d(q)-1)(D\psi)^{q-1} + c(x)(C\phi)^{\alpha+1}(D\psi)^{\beta} \ge 0,\end{aligned}$$

are nonnegative functions, nevertheless $(-C\phi, -D\psi)$ is a solution of (1.4), and the maximum principle does not hold.

The condition is sufficient: Assume that (3.5) and (3.6) hold; if (u, v) is a solution of (1.4) for $f, g \geq 0$, we obtain by multiplying the first equation of (1.4) by $u^- := \max(0, -u)$ and integrating over Ω

$$\begin{aligned}&\int_\Omega P(x)|\nabla u^-|^p\\&= \int_\Omega a(x)|u^-|^p - \int_\Omega b(x)|u^-|^{\alpha+1}|v^+|^{\beta+1} + \int_\Omega b(x)|u^-|^{\alpha+1}|v^-|^{\beta+1} - \int_\Omega fu^-,\end{aligned}$$

then

$$\int_\Omega P(x)|\nabla u^-|^p \leq \int_\Omega a(x)|u^-|^p + \int_\Omega b(x)|u^-|^{\alpha+1}|v^-|^{\beta+1}.$$

By using (2.6) and (3.4), we have

$$\begin{aligned}(\lambda_a(p) - 1)\int_\Omega a(x)|u^-|^p &\leq \int_\Omega b(x)|u^-|^{\alpha+1}|v^-|^{\beta+1}\\&\leq \int_\Omega (a(x)|u^-|^p)^{\frac{\alpha+1}{p}}(d(x)|v^-|^q)^{\frac{\beta+1}{q}}.\end{aligned}$$

Applying Hölder inequality, we get

$$(\lambda_a(p) - 1)\int_\Omega a(x)|u^-|^p \leq \Big[\int_\Omega (a(x)|u^-|^p)\Big]^{\frac{\alpha+1}{p}} \Big[\int_\Omega (d(x)|v^-|^q)^{\frac{\beta+1}{q}}\Big]^{\frac{\beta+1}{q}},$$

and hence

$$\Big[(\lambda_a(p) - 1)\Big(\int_\Omega a(x)|u^-|^p\Big)^{\frac{\beta+1}{q}} - \Big(\int_\Omega (d(x)|v^-|^q)\Big)^{\frac{\beta+1}{q}}\Big]\Big(\int_\Omega a(x)|u^-|^p\Big)^{\frac{\alpha+1}{p}} \leq 0.$$

Now, if

$$\int_\Omega a(x)|u^-|^p = 0,$$

then $u^- = 0$, (where $a(x) \neq 0$ for any x), which implies that $u \geq 0$. If not, we get

$$(\lambda_a(p) - 1)^{\frac{\alpha+1}{p}}\Big[\int_\Omega a(x)|u^-|^p\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \leq \Big[\int_\Omega d(x)|v^-|^q\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{3.10}$$

Similarly, from the second equation of (1.4), we deduce that

$$(\lambda_d(q) - 1)^{\frac{\beta+1}{q}}\Big[\int_\Omega d(x)|v^-|^q\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \leq \Big[\int_\Omega a(x)|u^-|^p\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{3.11}$$

Multiplying (3.10) by (3.11), we obtain

$$((\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}-1)\Big[\int_\Omega a(x)|u^-|^p\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\Big[\int_\Omega d(x)|v^-|^q\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \leq 0.$$

Using (3.6), we have $u^- = v^- = 0$, which implies that $u \geq 0$, $v \geq 0$, i.e. the maximum principle holds. $\square$

4. Existence of Positive Solutions

Now, we shall prove that the system (1.4) has a solution in the space $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$, by an approximation method.

Following [4], for $\epsilon \in (0,1)$, we introduce the system

$$
\begin{gathered}
-\Delta_{P,p}u_\epsilon = a(x)\frac{(|u_\epsilon|^{p-2}u_\epsilon)}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{p-1})} + b(x)\frac{|v_\epsilon|^{\beta}v_\epsilon}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta+1})}\frac{|u_\epsilon|^{\alpha}}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{\alpha})} + f,\\
-\Delta_{Q,q}v_\epsilon = c(x)\frac{|v_\epsilon|^{\beta}}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta})}\frac{|u_\epsilon|^{\alpha}u_\epsilon}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{\alpha+1})} + d(x)\frac{(|v_\epsilon|^{q-2}v_\epsilon)}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{q-1})} + g,\\
u_\epsilon = v_\epsilon = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.1}
$$

Letting $(\zeta,\eta) = (u_\epsilon, v_\epsilon)$, then the system above can be written in the form

$$
\begin{gathered}
-\Delta_{P,p}\zeta = h(\zeta,\eta) + f \quad \text{in } \Omega,\\
-\Delta_{Q,q}\eta = k(\zeta,\eta) + g \quad \text{in } \Omega,\\
\zeta = \eta = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

where

$$
\begin{gathered}
h(\zeta,\eta) = a(x)\frac{(|\zeta|^{p-2}\zeta)}{(1+|\epsilon^{1\backslash p}\zeta|^{p-1})} + b(x)\frac{|\eta|^{\beta}\eta}{(1+|\epsilon^{1\backslash q}\eta|^{\beta+1})}\frac{|\zeta|^{\alpha}}{(1+|\epsilon^{1\backslash p}\zeta|^{\alpha})},\\
k(\zeta,\eta) = c(x)\frac{|\eta|^{\beta}}{(1+|\epsilon^{1\backslash q}\eta|^{\beta})}\frac{|\zeta|^{\alpha}\zeta}{(1+|\epsilon^{1\backslash p}\zeta|^{\alpha+1})} + d(x)\frac{(|\eta|^{q-2}\eta)}{(1+|\epsilon^{1\backslash q}\eta|^{q-1})}.
\end{gathered}
$$

It is easy to prove that $h(\zeta,\eta)$ and $k(\zeta,\eta)$ are bounded, since $a(x), b(x), c(x)$ and $d(x)$ are also bounded. Then, there exists $M > 0$ such that $|h(\zeta,\eta)| \le M$ and $|k(\zeta,\eta)| \le M$ for all ζ,η.

Lemma 4.1. *System* (4.1) *has a solution* $U_\epsilon = (u_\epsilon, v_\epsilon)$ *in* $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$.

Proof. We complete the proof in the following steps
(a) Construction of sub-super solutions for system (4.1) as follows; Let

$$
\begin{aligned}
&\zeta^0 \in W_0^{1,p}(P,\Omega) \text{ be a solution of } -\Delta_{P,p}\zeta^0 = M + f,\\
&\eta^0 \in W_0^{1,q}(Q,\Omega) \text{ be a solution of } -\Delta_{Q,q}\eta^0 = M + g,\\
&\zeta_0 \in W_0^{1,p}(P,\Omega) \text{ be a solution of } -\Delta_{P,p}\zeta_0 = -M + f,\\
&\eta_0 \in W_0^{1,q}(Q,\Omega) \text{ be a solution of } -\Delta_{Q,q}\eta_0 = -M + g.
\end{aligned} \tag{4.2}
$$

Then , as in [14], we say that (ζ^0,η^0) is a super solution of (4.1) and (ζ_0,η_0) is a sub solution of the same system, since we have

$$
\begin{gathered}
-\Delta_{P,p}\zeta^0 - h(\zeta^0,\eta) - f \ge -\Delta_{P,p}\zeta^0 - (M+f) = 0 \quad \forall\eta\in[\eta_0,\eta^0] \text{ in } \Omega,\\
-\Delta_{Q,q}\eta^0 - k(\zeta,\eta^0) - g \ge -\Delta_{Q,q}\eta^0 - (M+g) = 0 \quad \forall\zeta\in[\zeta_0,\zeta^0] \text{ in } \Omega.\\
-\Delta_{P,p}\zeta_0 - h(\zeta_0,\eta) - f \le -\Delta_{P,p}\zeta_0 + M - f = 0 \quad \forall\eta\in[\eta_0,\eta^0] \text{ in } \Omega,\\
-\Delta_{Q,q}\eta_0 - k(\zeta,\eta_0) - g \le -\Delta_{Q,q}\eta_0 + M - g = 0 \quad \forall\zeta\in[\zeta_0,\zeta^0]\text{in } \Omega,
\end{gathered}
$$

Let us assume that $K = [\zeta_0,\zeta^0]\times[\eta_0,\eta^0]$.

(b) Definition of the operator T: We define the operator $T:(\zeta,\eta)\to(w,z)$ by

$$\begin{gathered}
-\Delta_{P,p}w = h(\zeta,\eta)+f \quad \text{in } \Omega,\\
-\Delta_{Q,q}z = k(\zeta,\eta)+g \quad \text{in } \Omega,\\
w=z=0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.3}$$

(c) $T(K)\subset K$: Since $(\zeta,\eta)\in[\zeta_0,\zeta^0]\times[\eta_0,\eta^0]$, then from(4.2) and (4.3), we get

$$-\Delta_{P,p}w+\Delta_{P,p}\zeta^0 \le h(\zeta,\eta)-M. \tag{4.4}$$

Multiplying this equation by $(w-\zeta^0)^+=\max(w-\zeta^0,0)$ and integrating over Ω, we obtain

$$\int_\Omega(-\Delta_{P,p}w+\Delta_{P,p}\zeta^0)(w-\zeta^0)^+ \le \int_\Omega(h(\zeta,\eta)-M)(w-\zeta^0)^+\le 0,$$

which implies

$$\int_\Omega P(x)\left[|\nabla w|^{p-2}\nabla w-|\nabla\zeta^0|^{p-2}\nabla\zeta^0\right]\nabla(w-\zeta^0)^+\le 0.$$

It is well known by [17], that the following inequality holds

$$|x-y|^p\le C\{(|x|^{p-2}x-|y|^{p-2}y)(x-y)\}^{\frac{\gamma}{2}}(|x|^p+|y|^p)^{1-\frac{\gamma}{2}}. \tag{4.5}$$

for all $x,y\in R^N$, where $\gamma=p$ if $1<p\le 2$ and $\gamma=2$ if $p>2$.

Applying (4.5), we obtain

$$\int_\Omega P(x)|\nabla(w-\zeta^0)|^p\le 0$$

which implies that, since $P(x)$ is a weight function, $(w-\zeta^0)^+=0$, and hence $w\le\zeta^0$.

Again, as above, we can deduce that $w\ge\zeta_0$. So, we have $\zeta_0\le w\le\zeta^0$.

Similarly, we can deduce that $\eta_0\le z\le\eta^0$, and hence $(w,z)\in[\zeta_0,\zeta^0]\times[\eta_0,\eta^0]$.

(d) T is completely continuous: First, we prove that T is continuous; for this, we need the following lemma. □

Lemma 4.2. *If $(\zeta_k,\eta_k)\to(\zeta,\eta)$, in $L^p(\Omega)\times L^q(\Omega)$, then, as in* [5, 16],

$$\begin{gathered}
\Big(\int_\Omega\big(a(x)[\frac{|\zeta_k|^{p-2}\zeta_k}{(1+|\epsilon^{1\backslash p}\zeta_k|^{p-1})}-\frac{|\zeta|^{p-2}\zeta}{(1+|\epsilon^{1\backslash p}\zeta|^{p-1})}]\big)^{p^*}\Big)^{1/p^*}\to 0,\\
\Big(\int_\Omega\big(b(x)[\frac{|\zeta_k|^{\alpha}}{(1+|\epsilon^{1\backslash p}\zeta_k|^{\alpha})}\frac{|\eta_k|^{\beta}\eta_k}{(1+|\epsilon^{1\backslash q}\eta_k|^{\beta+1})}\\
-\frac{|\zeta|^{\alpha}}{(1+|\epsilon^{1\backslash p}\zeta|^{\alpha})}\frac{|\eta|^{\beta}\eta}{(1+|\epsilon^{1\backslash q}\eta|^{\beta+1})}]\big)^{p^*}\Big)^{1/p^*}\to 0,\\
\Big(\int_\Omega\Big(c(x)[\frac{|\zeta_k|^{\alpha}\zeta_k}{(1+|\epsilon^{1\backslash p}\zeta_k|^{\alpha+1})}\frac{|\eta_k|^{\beta}}{(1+|\epsilon^{1\backslash q}\eta_k|^{\beta})}\\
-\frac{|\zeta|^{\alpha}\zeta}{(1+|\epsilon^{1\backslash p}\zeta|^{\alpha+1})}\frac{|\eta|^{\beta}}{(1+|\epsilon^{1\backslash q}\eta|^{\beta})}]\Big)^{q^*}\Big)^{1/q^*}\to 0,\\
\Big(\int_\Omega\big(d(x)[\frac{|\eta_k|^{q-2}\eta_k}{(1+|\epsilon^{1\backslash q}\eta_k|^{q-1})}-\frac{|\eta|^{q-2}\eta}{(1+|\epsilon^{1\backslash q}\eta|^{q-1})}]\big)^{q^*}\Big)^{1/q^*}\to 0,
\end{gathered}$$

as $k\to+\infty$.

Proof. If $(\zeta_k) \to (\zeta)$ in $L^p(\Omega)$, then there exists a subsequence still denoted by (ζ_k) itself such that $\zeta_k(x) \to \zeta(x)$ a.e. on Ω and $|\zeta_k(x)| \leq l(x)$ a.e. on Ω, for all k, with $l \in L^p(\Omega)$. Hence,

$$\Big|\frac{|\zeta_k|^{p-2}\zeta_k}{1+|\epsilon^{1\backslash p}\zeta_k|^{p-1}}\Big| \leq |\zeta_k|^{p-1} \leq l^{p-1} \in L^{p^*}(\Omega),$$

and, since $a(x) \neq 0$ is bounded, we have

$$a(x)\frac{|\zeta_k(x)|^{p-2}\zeta_k(x)}{1+|\epsilon^{1\backslash p}\zeta_k(x)|^{p-1}} \to a(x)\frac{|\zeta(x)|^{p-2}\zeta(x)}{1+|\epsilon^{1\backslash p}\zeta(x)|^{p-1}} \quad \text{a.e. on } \Omega \quad \text{as} \quad k \to +\infty.$$

Thus from the Dominated Convergence Theorem, we obtain the first statement of this theorem.

Prove of the second statement: If $(\eta_k) \to (\eta)$ in $L^q(\Omega)$, then there exists a subsequence still denoted also by (η_k), such that $\eta_k(x) \to \eta(x)$ a.e. on Ω and $|\eta_k(x)| \leq m(x)$ a.e. on Ω, for all k, with $m \in L^q(\Omega)$.

Now, from (3.1) and (3.2), we obtain

$$\frac{\alpha p^*}{p} + \frac{(\beta+1)p^*}{q} = 1,$$

and hence,

$$\int_\Omega [l^\alpha m^{\beta+1}]^{p^*} \leq \int_\Omega |l|^{\alpha p^*}|m|^{(\beta+1)p^*} \leq \Big(\int_\Omega |l|^p\Big)^{p^*\alpha/p}\Big(\int_\Omega |m|^q\Big)^{p^*(\beta+1)/q} < \infty.$$

So that

$$|(|\zeta_k|^\alpha|\eta_k|^{\beta+1})| \leq l^\alpha m^{\beta+1} \in L^{p^*}(\Omega).$$

Hence, since $b(x)$ is bounded, we have

$$b(x)\frac{|\zeta_k|^\alpha}{\big(1+|\epsilon^{1\backslash p}\zeta_k|^\alpha\big)}\frac{|\eta_k|^\beta\eta_k}{\big(1+|\epsilon^{1\backslash q}\eta_k|^{\beta+1}\big)} \to b(x)\frac{|\zeta|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta|^\alpha)}\frac{|\eta|^\beta\eta}{\big(1+|\epsilon^{1\backslash q}\eta|^{\beta+1}\big)},$$

a.e. on Ω as $k \to +\infty$. Thus from the Dominated Convergence Theorem, we obtain the second statement in this theorem. Similarly we prove the third and fourth statements.

Now, we prove the continuity of T. Assume that $(\zeta_k, \eta_k) \to (\zeta, \eta)$, in $L^p(\Omega) \times L^q(\Omega)$, then we have from the first equation of (4.3)

$$\begin{aligned}
&-\Delta_{P,p}w_k + \Delta_{P,p}w \\
&= a(x)\Big[\frac{(|\zeta_k|^{p-2}\zeta_k)}{(1+|\epsilon^{1\backslash p}\zeta_k|^{p-1})} - \frac{(|\zeta|^{p-2}\zeta)}{(1+|\epsilon^{1\backslash p}\zeta|^{p-1})}\Big] \\
&\quad + b(x)\Big[\frac{|\zeta_k|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta_k|^\alpha)}\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1\backslash q}\eta_k|^{\beta+1})} - \frac{|\zeta|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta|^\alpha)}\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1\backslash q}\eta|^{\beta+1})}\Big],
\end{aligned}$$

multiplying this equation by $(w_k - w)$ and integrating over Ω, we obtain

$$
\begin{aligned}
&\int_\Omega P(x)\left[|\nabla w_k|^{p-2}\nabla w_k - |\nabla w|^{p-2}\nabla w\right]\nabla(w_k - w)\\
&= \int_\Omega a(x)\Big[\frac{(|\zeta_k|^{p-2}\zeta_k)}{(1+|\epsilon^{1\backslash p}\zeta_k|^{p-1})} - \frac{(|\zeta|^{p-2}\zeta)}{(1+|\epsilon^{1\backslash p}\zeta|^{p-1})}\Big](w_k - w)\\
&\quad + \int_\Omega b(x)\Big[\frac{|\zeta_k|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta_k|^\alpha)}\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1\backslash q}\eta_k|^{\beta+1})}\\
&\quad - \frac{|\zeta|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta|^\alpha)}\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1\backslash q}\eta|^{\beta+1})}\Big](w_k - w).
\end{aligned}
$$

Using Hölder's inequality, we get

$$
\begin{aligned}
&\int_\Omega P(x)\left[|\nabla w_k|^{p-2}\nabla w_k - |\nabla w|^{p-2}\nabla w\right]\nabla(w_k - w)\\
&\le \Big(\int_\Omega \big((a(x)[\frac{|\zeta_k|^{p-2}\zeta_k}{(1+|\epsilon^{1\backslash p}\zeta_k|^{p-1})} - \frac{|\zeta|^{p-2}\zeta}{(1+|\epsilon^{1\backslash p}\zeta|^{p-1})}]\big)^{p^*}\Big)^{1/p^*}\Big(\int_\Omega |w_k - w|^p\Big)^{1/p}\\
&\quad + \Big(\int_\Omega \big(b(x)[\frac{|\zeta_k|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta_k|^\alpha)}\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1\backslash q}\eta_k|^{\beta+1})}\\
&\quad - \frac{|\zeta|^\alpha}{(1+|\epsilon^{1\backslash p}\zeta|^\alpha)}\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1\backslash q}\eta|^{\beta+1})}]\big)^{p^*}\Big)^{1/p^*}\Big(\int_\Omega |w_k - w|^p\Big)^{1/p}.
\end{aligned}
$$

Applying (4.5) and lemma 4.2, we obtain

$$
\int_\Omega P(x)|\nabla(w_k - w)|^p \to 0 \quad \text{as } k \to +\infty,
$$

which implies that $w_k \to w$ in $W_0^{1,p}(P,\Omega)$. Similarly, we can deduce that $z_k \to z$ in $W_0^{1,q}(Q,\Omega)$. Then, $(w_k, z_k) \to (w, z)$ in $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$.

To prove that T is compact, let (ζ_j, η_j) be a bounded sequence in K. Multiplying the first equation in (4.3) by w_j and integrating over Ω, we obtain

$$
\begin{aligned}
&\int_\Omega P(x)|\nabla w_j|^p\\
&= \int_\Omega \Big[a(x)\frac{|\zeta_j|^{p-2}\zeta_j}{1+|\epsilon^{1\backslash p}\zeta_j|^{p-1}} + b(x)\frac{|\zeta_j|^\alpha}{1+|\epsilon^{1\backslash p}\zeta_j|^\alpha}\frac{|\eta_j|^\beta\eta_j}{1+|\epsilon^{1\backslash q}\eta_j|^{\beta+1}}\Big]w_j + \int_\Omega f w_j\\
&\le \int_\Omega a(x)|\zeta_j|^{p-2}\zeta_j w_j + \int_\Omega b(x)|\zeta_j|^\alpha|\eta_j|^\beta\eta_j w_j + \int_\Omega f w_j\\
&\le \Big(\int_\Omega [a(x)|\zeta_j|^{p-1}]^{p^*}\Big)^{1/p^*}\Big(\int_\Omega (w_j)^p\Big)^{1/p}\\
&\quad + \Big(\int_\Omega [b(x)|\zeta_j|^\alpha|\eta_j|^{\beta+1}]^{p^*}\Big)^{1/p^*}\Big(\int_\Omega (w_j)^p\Big)^{1/p} + \Big(\int_\Omega |f|^{P^*}\Big)^{1/p^*}\Big(\int_\Omega (w_j)^p\Big)^{1/p}.
\end{aligned}
$$

Hence, (w_j) is bounded in $W_0^{1,p}(P,\Omega)$ and it possesses a strongly convergent subsequence in $L^p(\Omega)$. The same is true for (z_j) in $L^q(\Omega)$.

Since K is a convex, bounded, closed subset of $L^p(\Omega) \times L^q(\Omega)$, we can apply Schauder's Fixed Point Theorem to obtain the existence of a fixed point for T, which gives the existence of solution $U_\epsilon = (u_\epsilon, v_\epsilon)$ of (4.1), and this completes the proof. □

Now, we are in a position to prove the existence of a solution for system (1.4).

Theorem 4.3. *Assume that* (2.2)-(2.6) *and* (2.10) *are satisfied, then system* (1.4) *admits a solution* (u, v) *in* $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$.

Proof. This proof is done in three steps:
(a) First, we proof that $(\epsilon^{1\backslash p} u_\epsilon, \epsilon^{1\backslash q} v_\epsilon)$ is bounded in $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$. Multiplying the first equation of (4.1) by (ϵu_ϵ) and integrating over Ω , we obtain

$$\begin{aligned}
&\int_\Omega P(x)|\nabla(\epsilon^{1\backslash p} u_\epsilon)|^p \\
&\leq \int_\Omega a(x)|(\epsilon^{1\backslash p} u_\epsilon)|^p + \int_\Omega b(x)|(\epsilon^{1\backslash p} u_\epsilon)| + \epsilon^{1\backslash p^*} \int_\Omega |f| \times |(\epsilon^{1\backslash p} u_\epsilon)|.
\end{aligned} \tag{4.6}$$

From (2.6), we get

$$(\lambda_a(p) - 1) \int_\Omega a(x)|(\epsilon^{1\backslash p} u_\epsilon)|^p \leq \int_\Omega b(x)|(\epsilon^{1\backslash p} u_\epsilon)| + \epsilon^{1\backslash p^*} \int_\Omega |f| \times |(\epsilon^{1\backslash p} u_\epsilon)|.$$

Using Hölder's inequality, we have

$$(\lambda_a(p) - 1)\Big(\int_\Omega |(\epsilon^{1\backslash p} u_\epsilon)|^p\Big)^{1/p^*} \leq c \quad \text{with } c > 0,$$

which implies that $(\epsilon^{1\backslash p} u_\epsilon)$ is bounded in $L^p(\Omega)$ and from (4.6) it is bounded in $W_0^{1,p}(P,\Omega)$. similarly $(\epsilon^{1\backslash q} v_\epsilon)$ is bounded in $W_0^{1,q}(Q,\Omega)$.
(b) $(\epsilon^{1\backslash p} u_\epsilon, \epsilon^{1\backslash q} v_\epsilon)$ converges to $(0,0)$ strongly in $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$.

From (a), $(\epsilon^{1\backslash p} u_\epsilon, \epsilon^{1\backslash q} v_\epsilon)$ converges to (u_*, v_*) strongly in $L^P(\Omega) \times L^q(\Omega)$ and weakly in $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$.

Multiplying the first equation of (4.1) by $(\epsilon^{1\backslash p^*})$, we get

$$\begin{aligned}
&-\Delta_{P,p}(\epsilon^{1\backslash p} u_\epsilon) \\
&= a(x)\frac{|\epsilon^{1\backslash p} u_\epsilon|^{p-2}(\epsilon^{1\backslash p} u_\epsilon)}{(1 + |\epsilon^{1\backslash p} u_\epsilon|^{p-1})} + b(x)\frac{|\epsilon^{1\backslash p} u_\epsilon|^\alpha}{(1 + |\epsilon^{1\backslash p} u_\epsilon|^\alpha)} \frac{(\epsilon^{1\backslash q}|v_\epsilon|)^\beta(\epsilon^{1\backslash q} v_\epsilon)}{(1 + |\epsilon^{1\backslash q} v_\epsilon|^{\beta+1})} + f\epsilon^{1\backslash p^*},
\end{aligned}$$

since the sequence $(\epsilon^{1\backslash p} u_\epsilon)$ is bounded in $W_0^{1,p}(P,\Omega)$, then, we can find subsequence $(\epsilon^{1\backslash p} u_\epsilon)$ such that

$$\epsilon^{1\backslash p} u_\epsilon \to u_* \quad \text{weakly in } W_0^{1,p}(P,\Omega) \quad \text{and} \quad \epsilon^{1\backslash p} u_\epsilon \to u_* \quad \text{a.e. on } \Omega.$$

Again, using Dominated Convergence Theorem as in Lemma 4.2, we have

$$a(x)\frac{|\epsilon^{1\backslash p} u_\epsilon|^{p-2}(\epsilon^{1\backslash p} u_\epsilon)}{(1 + |\epsilon^{1\backslash p} u_\epsilon|^{p-1})} \to a(x)\frac{|u_*|^{p-2} u_*}{(1 + |u_*|^{p-1})},$$

strongly in $L^{P^*}(\Omega)$ and

$$b(x)\frac{|\epsilon^{1\backslash p} u_\epsilon|^\alpha}{(1 + |\epsilon^{1\backslash p} u_\epsilon|^\alpha)} \frac{(\epsilon^{1\backslash q}|v_\epsilon|)^\beta(\epsilon^{1\backslash q} v_\epsilon)}{(1 + |\epsilon^{1\backslash q} v_\epsilon|^{\beta+1})} \to b(x)\frac{|u_*|^\alpha}{(1 + |u_*|^\alpha)} \frac{|v_*|^\beta v_*}{(1 + |v_*|^{\beta+1})},$$

strongly in $L^{P^*}(\Omega)$. Using a classical result in [15], and passing to the limit, we obtain

$$-\Delta_{P,p}(u_*) = a(x)\frac{|u_*|^{p-2} u_*}{(1 + |u_*|^{p-1})} + b(x)\frac{|u_*|^\alpha}{(1 + |u_*|^\alpha)} \frac{|v_*|^\beta v_*}{(1 + |v_*|^{\beta+1})}. \tag{4.7}$$

Multiplying this equation by (u_*^-) and integrating over Ω, then applying (2.6), we get

$$(\lambda_a(p) - 1) \int_\Omega a(x)|u_*^-|^p \leq \int_\Omega b(x)|u_*^-|^{\alpha+1}|v_*^-|^{\beta+1}.$$

Using (3.4) and applying Hölder's inequality, as in the proof of theorem 3.1, we deduce

$$(\lambda_a(p)-1)^{\frac{\alpha+1}{p}}\Big(\int_\Omega a(x)|u_*^-|^p\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \Big(\int_\Omega d(x)|v_*^-|^q\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}, \tag{4.8}$$

similarly, from the second equation of (4.1), we have

$$(\lambda_d(q)-1)^{\frac{\beta+1}{q}}\Big(\int_\Omega d(x)|v_*^-|^q\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \Big(\int_\Omega a(x)|u_*^-|^p\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{4.9}$$

Multiplying (4.8) by (4.9), we obtain

$$((\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}-1)\Big(\int_\Omega a(x)|u^-|^p\int_\Omega d(x)|v^-|^q\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le 0.$$

From (3.6), we have $u_*^- = v_*^- = 0$, which implies that $u^*, v^* \ge 0$.

Now, we show that $u_* = v_* = 0$. Multiplying equation (4.7) by (u_*) and integrating over Ω, we get, as above

$$((\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}-1)\Big(\int_\Omega a(x)|u^*|^p\int_\Omega d(x)|v^*|^q\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le 0,$$

which implies that $u^* = v^* = 0$.

(c) (u_ϵ, v_ϵ) is bounded in $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$. Assume that

$$\|u_\epsilon\|_{W_0^{1,p}(P,\Omega)} \to \infty \quad \text{or} \quad \|v_\epsilon\|_{W_0^{1q}(Q,\Omega)} \to \infty.$$

Set

$$t_\epsilon = \max(\|u_\epsilon\|^p_{W_0^{1,p}(P,\Omega)}, \|v_\epsilon\|^q_{W_0^{1q}(Q,\Omega)}), \quad z_\epsilon = u_\epsilon t_\epsilon^{-1\backslash p}, \quad w_\epsilon = v_\epsilon t_\epsilon^{-1\backslash q}.$$

Dividing the first equation of (4.1) by ($t_\epsilon^{1\backslash p^*}$) and the second by ($t_\epsilon^{1\backslash q^*}$), we obtain

$$\begin{aligned}
-\Delta_{P,p} z_\epsilon &= a(x)\frac{(|z_\epsilon|^{p-2}z_\epsilon)}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{p-1})} + b(x)\frac{|z_\epsilon|^\alpha}{(1+|\epsilon^{1\backslash p}u_\epsilon|^\alpha)}\frac{|w_\epsilon|^\beta w_\epsilon}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta+1})}\\
&\quad + f t_\epsilon^{-1\backslash p^*},\\
-\Delta_{Q,q} w_\epsilon &= d(x)\frac{(|w_\epsilon|^{q-2}w_\epsilon)}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{q-1})} + c(x)\frac{|z_\epsilon|^\alpha z_\epsilon}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{\alpha+1})}\frac{|v_\epsilon|^\beta}{(1+|\epsilon^{1\backslash q}v_\epsilon|^\beta)}\\
&\quad + g t_\epsilon^{-1\backslash q^*}.
\end{aligned}$$

As in (b) above, we can prove that $(z_\epsilon, w_\epsilon) \to (z, w)$ strongly in $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$, and taking the limit, as $\epsilon \to 0$, we obtain

$$\begin{gathered}
-\Delta_{P,p} z = a(x)|z|^{p-2}z + b(x)|w|^\beta|z|^\alpha w,\\
-\Delta_{Q,q} w = d(x)|w|^{q-2}w + c(x)|w|^\beta|z|^\alpha z,
\end{gathered}$$

and hence, we deduce that $w = z = 0$.

Since there exists a sequence $(\epsilon_n)_{n\in N}$ such that either $\|z_{\epsilon_n}\| = 1$ or $\|w_{\epsilon_n}\| = 1$, we obtain a contradiction.

Hence, (u_ϵ, v_ϵ) is bounded in $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$, we can extract a subsequence denoted by (u_ϵ, v_ϵ) which converges to (u_0, v_0) strongly in $L^P(\Omega)\times L^q(\Omega)$ and weakly in $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$ as $\epsilon \to 0$. By using a similar procedure as above, we can prove that (u_ϵ, v_ϵ) converges strongly to (u_0, v_0) in $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$.

Indeed, since $\epsilon^{1\backslash p}u_\epsilon \to 0$ a.e. on Ω, we have

$$\Big|\frac{|u_\epsilon|^{p-2}u_\epsilon}{1+|\epsilon^{1\backslash p}u_\epsilon|^{p-1}}\Big| \le |u_\epsilon|^{p-1} \le l^{p-1} \in L^{P^*}(\Omega),$$

with $l \in L^P(\Omega)$ and

$$a(x)\frac{|u_\epsilon|^{p-2}u_\epsilon}{1+|\epsilon^{1\backslash p}u_\epsilon|^{p-1}} \to a(x)|u_0|^{p-2}u_0 \text{ a.e. on } \Omega.$$

Hence, from the Dominated Convergence Theorem, we obtain

$$\int_\Omega \big(a(x)[(|u_\epsilon|^{p-2}u_\epsilon)\big(1+|\epsilon^{1\backslash p}u_\epsilon|^{p-1}\big)^{-1} - |u_0|^{p-2}u_0]\big)^{p^*} \to 0, \quad \text{as } \epsilon \to 0.$$

Also, since $\epsilon^{1\backslash p}u_\epsilon \to 0$ and $\epsilon^{1\backslash q}v_\epsilon \to 0$ a.e. on Ω, then

$$|u_\epsilon|^\alpha\big(1+|\epsilon^{1\backslash p}u_\epsilon|^\alpha\big)^{-1}|v_\epsilon|^\beta v_\epsilon\big(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta+1}\big)^{-1} \to |u_0|^\alpha|v_0|^\beta v_0, \quad \text{a.e. on } \Omega,$$

and, as in the proof of Lemma 4.2, we have

$$\Big|\frac{|u_\epsilon|^\alpha}{\big(1+|\epsilon^{1\backslash p}u_\epsilon|^\alpha\big)}\frac{|v_\epsilon|^\beta v_\epsilon}{\big(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta+1}\big)}\Big| \le |u_\epsilon|^\alpha|v_\epsilon|^{\beta+1} \le l^\alpha m^{\beta+1} \in L^{P^*}(\Omega)$$

with $m \in L^q(\Omega)$. From the Dominated Convergence Theorem, we have

$$\int_\Omega \big(b(x)[\frac{|u_\epsilon|^\alpha}{(1+|\epsilon^{1\backslash p}u_\epsilon|^\alpha)}\frac{|v_\epsilon|^\beta v_\epsilon}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{\beta+1})} - |u_0|^\alpha|v_0|^\beta v_0]\big)^{p^*} \to 0,$$

as$\epsilon \to 0$. Similarly, we have

$$\int_\Omega \big(c(x)[\frac{|u_\epsilon|^\alpha u_\epsilon}{(1+|\epsilon^{1\backslash p}u_\epsilon|^{\alpha+1})}\frac{|v_\epsilon|\beta}{(1+|\epsilon^{1\backslash q}v_\epsilon|^\beta)} - |u_0|^\alpha u_0|v_0|^\beta]\big)^{q^*} \to 0 \quad \text{as } \epsilon \to 0,$$

$$\int_\Omega d(x)[\frac{|v_\epsilon|^{q-2}v_\epsilon}{(1+|\epsilon^{1\backslash q}v_\epsilon|^{q-1})} - |v_0|^{q-2}v_0)]^{q^*} \to 0 \quad \text{as } \epsilon \to 0.$$

Therefore, passing to the limit, $(u_\epsilon, v_\epsilon) \to (u_0, v_0)$, and hence we obtain from (4.1)

$$\begin{aligned} -\Delta_{P,p}u_0 &= a(x)|u_0|^{p-2}u_0 + b(x)|u_0|^\alpha|v_0|^\beta v_0 + f \quad \text{in } \Omega, \\ -\Delta_{Q,q}v_0 &= d(x)|v_0|^{q-2}v_0 + c(x)|u_0|^\alpha u_0|v_0|^\beta + g \quad \text{in } \Omega. \end{aligned}$$

Hence, (u_0, v_0) satisfies the system (1.4). $\square$

Remark 4.4. (i) When $P(x) = Q(x) = 1$, $p = q = 2$ and $\alpha = \beta = 0$, we obtain some results presented in [4]. (ii) When $P(x) = Q(x) = 1$ and the coefficients $a(x), b(x), c(x)$ and $d(x)$ are constants, we have some results presented in [5].

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