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A FIBERING MAP APPROACH TO A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. We prove the existence of at least two positive solutions for the semilinear elliptic boundary-value problem

$$-\Delta u(x) = \lambda a(x)u^q + b(x)u^p \quad \text{for } x \in \Omega; \quad u(x) = 0 \quad \text{for } x \in \partial\Omega$$

on a bounded region Ω by using the Nehari manifold and the fibering maps associated with the Euler functional for the problem. We show how knowledge of the fibering maps for the problem leads to very easy existence proofs.

1. Introduction

We shall discuss the existence of positive solutions of the semilinear elliptic boundary-value problem

$$-\Delta u(x) = \lambda a(x)u^q + b(x)u^p \quad \text{for } x \in \Omega; \tag{1.1}$$

$$u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

where Ω is a bounded region with smooth boundary in $\mathbb{R}^N$, $0 < q < 1 < p < \frac{N+2}{N-2}$, $\lambda > 0$ and $a, b : \Omega \to \mathbb{R}$ are smooth functions which are somewhere positive but which may change sign on Ω. Equation (1.1), (1.2) has been recently studied in [3] by using the Mountain Pass Lemma and in [5] and [7] using the Nehari manifold.

In [4] and [2] it was shown that the Nehari manifold for an equation such as (1.1) is closely related to the fibering maps for the problem. In this paper we show how a fairly complete knowledge of all possible forms of the fibering maps provides a very simple and comparatively elementary means of establishing results similar to those proved in [5] and [7] on the existence of multiple solutions of (1.1), (1.2). In section 2 we recall the properties which we shall require of fibering maps and of the Nehari manifold. In section 3 we give a fairly complete description of the fibering maps associated with (1.1) and in section 4 we use this information to give a very simple variational proof of the existence of at least two positive solutions of (1.1), (1.2) for sufficiently small λ.

We shall throughout use the function space $W_0^{1,2}(\Omega)$ with norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 \, dx \Big)^{1/2}$$

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and the standard $L^p(\Omega)$ spaces whose norms we denote by $\|u\|_p$.

2. Fibering Maps and the Nehari manifold

The Euler functional associated with (1.1), (1.2) is

$$J_\lambda(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{\lambda}{q+1}\int_\Omega a(x)|u|^{q+1}\,dx-\frac{1}{p+1}\int_\Omega b(x)|u|^{p+1}\,dx$$

for all $u\in W_0^{1,2}(\Omega)$.

As J_λ is not bounded below on $W_0^{1,2}(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$M_\lambda(\Omega)=\{u\in W_0^{1,2}(\Omega):\langle J_\lambda'(u),u\rangle=0\}$$

where $\langle,\rangle$ denotes the usual duality. Thus $u\in M_\lambda(\Omega)$ if and only if

$$\int_\Omega|\nabla u|^2\,dx-\lambda\int_\Omega a(x)|u|^{q+1}\,dx-\int_\Omega b(x)|u|^{p+1}\,dx=0 \tag{2.1}$$

Clearly $M_\lambda(\Omega)$ is a much smaller set than $W_0^{1,2}(\Omega)$ and, as we shall show, J_λ is much better behaved on $M_\lambda(\Omega)$. In particular, on $M_\lambda(\Omega)$ we have that

$$\begin{aligned}J_\lambda(u)&=(\frac{1}{2}-\frac{1}{q+1})\int_\Omega|\nabla u|^2+(\frac{1}{q+1}-\frac{1}{p+1})\int_\Omega b(x)|u|^{p+1}\\&=(\frac{1}{2}-\frac{1}{p+1})\int_\Omega|\nabla u|^2-\lambda(\frac{1}{q+1}-\frac{1}{p+1})\int_\Omega a(x)|u|^{q+1}\end{aligned} \tag{2.2}$$

Theorem 2.1. *J_λ is coercive and bounded below on $M_\lambda(\Omega)$.*

Proof. It follows from (2.2) and the Sobolev embedding theorems that there exist positive constants c_1,c_2 and c_3 such that

$$J_\lambda(u)\ge c_1\|u\|^2-c_2\int_\Omega|u|^{q+1}\,dx\ge c_1\|u\|^2-c_3\|u\|^{q+1}$$

and so J_λ is coercive and bounded below on $M_\lambda(\Omega)$. □

The Nehari manifold is closely linked to the behaviour of the functions of the form $\phi_u:t\to J_\lambda(tu)$ $(t>0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [4] and are also discussed in Brown and Zhang [2]. If $u\in W_0^{1,2}(\Omega)$, we have

$$\phi_u(t)=\frac{1}{2}t^2\int_\Omega|\nabla u|^2-\lambda\frac{t^{q+1}}{q+1}\int_\Omega a|u|^{q+1}-\frac{t^{p+1}}{p+1}\int_\Omega b|u|^{p+1} \tag{2.3}$$

$$\phi_u'(t)=t\int_\Omega|\nabla u|^2-\lambda t^q\int_\Omega a|u|^{q+1}-t^p\int_\Omega b|u|^{p+1} \tag{2.4}$$

$$\phi_u''(t)=\int_\Omega|\nabla u|^2-\lambda qt^{q-1}\int_\Omega a|u|^{q+1}-pt^{p-1}\int_\Omega b|u|^{p+1} \tag{2.5}$$

It is easy to see that $u\in M_\lambda(\Omega)$ if and only if $\phi_u'(1)=0$ and, more generally, that $\phi_u'(t)=0$ if and only if $tu\in M_\lambda(\Omega)$, i.e., elements in $M_\lambda(\Omega)$ correspond to stationary points of fibering maps. Thus it is natural to subdivide $M_\lambda(\Omega)$ into sets corresponding to local minima, local maxima and points of inflection and so we define

$$\begin{aligned} M_\lambda^+(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) > 0\}, \\ M_\lambda^-(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) < 0\}, \\ M_\lambda^0(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) = 0\}, \end{aligned}$$

and note that if $u \in M_\lambda(\Omega)$, i.e., $\phi_u'(1) = 0$, then

$$\begin{aligned} \phi_u''(1) &= (1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int b(x)|u|^{p+1}\,dx \\ &= (1-p)\int_\Omega |\nabla u|^2\,dx - \lambda(q-p)\int a(x)|u|^{q+1}\,dx\,. \end{aligned} \tag{2.6}$$

Also, as proved in Binding, Drabek and Huang [1] or in Brown and Zhang [2], we have the following lemma.

Lemma 2.2. *Suppose that u_0 is a local maximum or minimum for J_λ on $M_\lambda(\Omega)$. Then, if $u_0 \notin M_\lambda^0(\Omega)$, u_0 is a critical point of J_λ.*

3. Analysis of the Fibering Maps

In this section we give a fairly complete description of the fibering maps associated with the problem. As we shall see the essential nature of the maps is determined by the signs of $\int a(x)|u|^{q+1}\,dx$ and $\int_\Omega b(x)|u|^{p+1}\,dx$. We will find it useful to consider the function

$$m_u(t) = t^{1-q}\int_\Omega |\nabla u|^2\,dx - t^{p-q}\int_\Omega b(x)|u|^{p+1}\,dx.$$

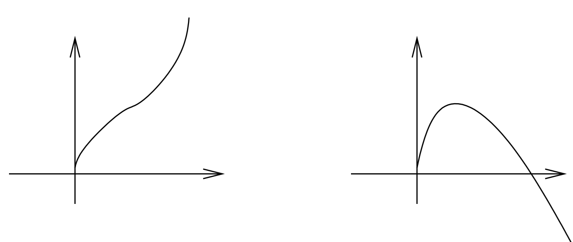
Clearly, for $t > 0$, $tu \in M_\lambda(\Omega)$ if and only if t is a solution of

$$m_u(t) = \lambda\int_\Omega a(x)|u|^{q+1}\,dx. \tag{3.1}$$

Morever,

$$m_u'(t) = (1-q)t^{-q}\int_\Omega |\nabla u|^2\,dx - (p-q)t^{p-q-1}\int_\Omega b(x)|u|^{p+1}\,dx. \tag{3.2}$$

It is easy to see that m_u is a strictly increasing function for $t \geq 0$ whenever $\int_\Omega b(x)|u|^{p+1}\,dx \leq 0$ and m_u is initially increasing and eventually decreasing with a single turning point as in Figure 1(b) when $\int_\Omega b(x)|u|^{p+1}\,dx > 0$.



(a) (b)

Figure 1. Possible forms of $m(u)$

Suppose $tu \in M_\lambda(\Omega)$. It follows from (2.6) and (3.2) that $\phi_{tu}''(1) = t^{q+2}m_u'(t)$ and so $tu \in M_\lambda^+(\Omega)(M_\lambda^-(\Omega))$ provided $m_u'(t) > 0$ (< 0).

We shall now describe the nature of the fibering maps for all possible signs of $\int_\Omega b(x)|u|^{p+1}\,dx$ and $\int_\Omega a(x)|u|^{q+1}\,dx$. If $\int_\Omega b(x)|u|^{p+1}\,dx \le 0$ and $\int_\Omega a(x)|u|^{q+1}\,dx \le 0$, clearly ϕ_u is an increasing function of t and so has graph as shown in Figure 2(a); thus in this case no multiple of u lies in $M_\lambda(\Omega)$. If $\int_\Omega b(x)|u|^{p+1}\,dx \le 0$ and $\int_\Omega a(x)|u|^{q+1}\,dx > 0$, then m_u has graph as in Figure 1(a), and it is clear that there is exactly one solution of (3.1). Thus there is a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda(\Omega)$. Clearly $m_u'(t(u)) > 0$ and so $t(u)u \in M_\lambda^+(\Omega)$. Thus the fibering map ϕ_u has a unique critical point at $t = t(u)$ which is a local minimum. Since $\lim_{t\to\infty}\phi_u(t) = \infty$, it follows that ϕ_u has graph as shown in Figure 2(c).

Suppose now $\int_\Omega b(x)|u|^{p+1}\,dx > 0$ and $\int_\Omega a(x)|u|^{q+1}\,dx \le 0$. Then m_u has graph as shown in Figure 1(b) and it is clear that there is exactly one positive solution of (3.1). Thus there is again a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda(\Omega)$ and since $m_u'(t(u)) < 0$ in this case $t(u)u \in M_\lambda^-(\Omega)$. Hence the fibering map ϕ_u has a unique critical point which is a local maximum. Since $\lim_{t\to\infty}\phi_u(t) = -\infty$, it follows that ϕ_u has graph as shown in Figure 2(b).

Finally we consider the case $\int_\Omega b(x)|u|^{p+1}\,dx > 0$ and $\int_\Omega a(x)|u|^{q+1}\,dx > 0$ where the situation is more complicated. As in the previous case m_u has a graph as shown in Figure 1(b). If $\lambda > 0$ is sufficiently large, (3.1) has no solution and so ϕ_u has no critical points - in this case ϕ_u is a decreasing function. Hence no multiple of u lies in $M_\lambda(\Omega)$. If, on the other hand, $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1(u) < t_2(u)$ of (3.1) with $m_u'(t_1(u)) > 0$ and $m_u'(t_2(u)) < 0$. Thus there are exactly two multiples of $u \in M_\lambda(\Omega)$, namely $t_1(u)u \in M_\lambda^+(\Omega)$ and $t_2(u)u \in M_\lambda^-(\Omega)$. It follows that ϕ_u has exactly two critical points - a local minimum at $t = t_1(u)$ and a local maximum at $t = t_2(u)$; moreover ϕ_u is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and decreasing in (t_2, ∞) as in Figure 2(d).

The following result ensures that when λ is sufficiently small the graph of ϕ_u must be as shown in Figure 2(d) for all non-zero u.

Lemma 3.1. *There exists $\lambda_1 > 0$ such that, when $\lambda < \lambda_1$, ϕ_u takes on positive values for all non-zero $u \in W_0^{1,2}(\Omega)$.*

Proof. If $\int_\Omega b(x)|u|^{p+1}\,dx \le 0$, then $\phi_u(t) > 0$ for t sufficiently large. Suppose $u \in W_0^{1,2}(\Omega)$ and $\int_\Omega b(x)|u|^{p+1}\,dx > 0$. Let
$$h_u(t) = \frac{t^2}{2}\int_\Omega |\nabla u|^2\,dx - \frac{t^{p+1}}{p+1}\int_\Omega b(x)|u|^{p+1}\,dx.$$
Then elementary calculus shows that h_u takes on a maximum value of
$$\frac{p-1}{2(p+1)}\Big\{\frac{(\int_\Omega |\nabla u|^2\,dx)^{p+1}}{(\int_\Omega b(x)|u|^{p+1}\,dx)^2}\Big\}^{\frac{1}{p-1}} \quad \text{when } t = t_{\max} = \Big(\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega b(x)|u|^{p+1}\,dx}\Big)^{\frac{1}{p-1}}.$$
However
$$\frac{(\int_\Omega |\nabla u|^2\,dx)^{p+1}}{(\int_\Omega |u|^{p+1}\,dx)^2} \ge \frac{1}{S_{p+1}^{2(p+1)}}$$
where S_{p+1} denotes the Sobolev constant of the embedding of $W_0^{1,2}(\Omega)$ into $L^{p+1}(\Omega)$. Hence
$$h_u(t_{\max}) \ge \frac{p-1}{2(p+1)}\Big(\frac{1}{\|b^+\|_\infty^2\, S_{p+1}^{2(p+1)}}\Big)^{\frac{1}{p-1}} = \delta$$

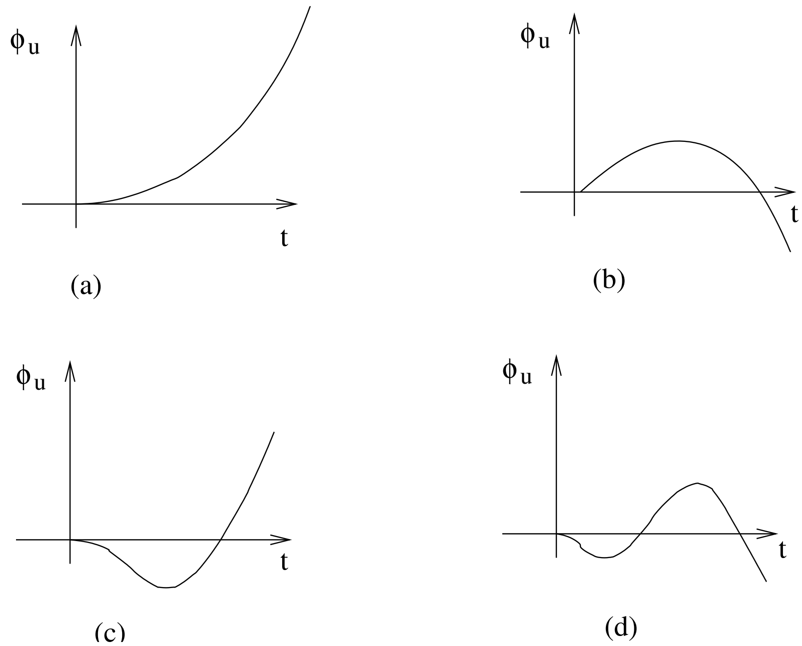


FIGURE 2. Possible forms of fibering maps

where δ is independent of u.

We shall now show that there exists $\lambda_1 > 0$ such that $\phi_u(t_{\max}) > 0$, i.e.,

$$h_u(t_{\max}) - \frac{\lambda\,(t_{\max})^{q+1}}{q+1}\int_\Omega a(x)|u|^{q+1}\,dx > 0$$

for all $u \in W_0^{1,2}(\Omega) - \{0\}$ provided $\lambda < \lambda_1$. We have

$$\begin{aligned}
&\frac{(t_{\max})^{q+1}}{q+1}\int_\Omega a(x)|u|^{q+1}\,dx \\
&\le \frac{1}{q+1}\|a\|_\infty\, S_{q+1}^{q+1}\Big(\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega b(x)|u|^{p+1}\,dx}\Big)^{\frac{q+1}{p-1}}\Big(\int_\Omega |\nabla u|^2\,dx\Big)^{\frac{q+1}{2}} \\
&= \frac{1}{q+1}\,\|a\|_\infty\, S_{q+1}^{q+1}\Big\{\frac{\big(\int_\Omega |\nabla u|^2\,dx\big)^{p+1}}{\big(\int_\Omega b(x)|u|^{p+1}\,dx\big)^2}\Big\}^{\frac{q+1}{2(p-1)}} \\
&= \frac{1}{q+1}\,\|a\|_\infty\, S_{q+1}^{q+1}\,\Big[\frac{2(p+1)}{p-1}\Big]^{\frac{q+1}{2}} h_u(t_{\max})^{\frac{q+1}{2}} = c\,h_u(t_{\max})^{\frac{q+1}{2}}
\end{aligned}$$

where c is independent of u. Hence

$$\phi_u(t_{\max}) \ge h_u(t_{\max}) - \lambda c h_u(t_{\max})^{\frac{q+1}{2}} = h_u(t_{\max})^{\frac{q+1}{2}}\big(h_u(t_{\max})^{\frac{1-q}{2}} - \lambda c\big)$$

and so, since $h_u(t_{\max}) \ge \delta$ for all $u \in W_0^{1,2}(\Omega) - \{0\}$, it follows that $\phi_u(t_{\max}) > 0$ for all non-zero u provided $\lambda < \delta^{\frac{1-q}{2}}/2c = \lambda_1$. This completes the proof. $\square$

It follows from the above lemma that when $\lambda < \lambda_1$, $\int_\Omega a(x)|u|^{q+1}\,dx > 0$ and $\int_\Omega b(x)|u|^{p+1}\,dx > 0$ then ϕ_u must have exactly two critical points as discussed in the remarks preceding the lemma.

Thus when $\lambda < \lambda_1$ we have obtained a complete knowledge of the number of critical points of ϕ_u, of the intervals on which ϕ_u is increasing and decreasing and of the multiples of u which lie in $M_\lambda(\Omega)$ for every possible choice of signs of $\int_\Omega b(x)|u|^{p+1}\,dx$ and $\int_\Omega a(x)|u|^{q+1}\,dx$. In particular we have the following result.

Corollary 3.2. *$M_\lambda^0(\Omega) = \emptyset$ when $0 < \lambda < \lambda_1$.*

Corollary 3.3. *If $\lambda < \lambda_1$, then there exists $\delta_1 > 0$ such that $J_\lambda(u) \geq \delta_1$ for all $u \in M_\lambda^-(\Omega)$.*

Proof. Consider $u \in M_\lambda^-(\Omega)$. Then ϕ_u has a positive global maximum at $t = 1$ and $\int b(x)|u|^{p+1}\,dx > 0$. Thus

$$\begin{aligned} J_\lambda(u) &= \phi_u(1) \geq \phi_u(t_{\max}) \\ &\geq h_u(t_{\max})^{\frac{q+1}{2}} \Big(h_u(t_{\max})^{\frac{1-q}{2}} - \lambda c \Big) \\ &\geq \delta^{\frac{q+1}{2}} (\delta^{\frac{1-q}{2}} - \lambda c) \end{aligned}$$

and the left hand side is uniformly bounded away from 0 provided that $\lambda < \lambda_1$. $\square$

4. Existence of Positive Solutions

In this section using the properties of fibering maps we shall give simple proofs of the existence of two positive solutions, one in $M_\lambda^+(\Omega)$ and one in $M_\lambda^-(\Omega)$.

Theorem 4.1. *If $\lambda < \lambda_1$, there exists a minimizer of J_λ on $M_\lambda^+(\Omega)$.*

Proof. Since J_λ is bounded below on $M_\lambda(\Omega)$ and so on $M_\lambda^+(\Omega)$, there exists a minimizing sequence $\{u_n\} \subseteq M_\lambda^+(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u).$$

Since J_λ is coercive, $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Thus we may assume, without loss of generality, that $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 < r < \frac{2N}{N-2}$.

If we choose $u \in W_0^{1,2}(\Omega)$ such that $\int_\Omega a(x)|u|^{q+1}\,dx > 0$, then the graph of the fibering map ϕ_u must be of one of the forms shown in Figure 2(c) or (d) and so there exists $t_1(u)$ such that $t_1(u)u \in M_\lambda^+(\Omega)$ and $J_\lambda(t_1(u)u) < 0$. Hence, $\inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u) < 0$. By (2.2),

$$J_\lambda(u_n) = (\frac{1}{2} - \frac{1}{p+1}) \int_\Omega |\nabla u_n|^2\,dx - \lambda\,(\frac{1}{q+1} - \frac{1}{p+1}) \int_\Omega a(x)|u_n|^{q+1}\,dx$$

and so

$$\lambda\,(\frac{1}{q+1} - \frac{1}{p+1}) \int_\Omega a(x)|u_n|^{q+1}\,dx = (\frac{1}{2} - \frac{1}{p+1}) \int_\Omega |\nabla u_n|^2\,dx - J_\lambda(u_n).$$

Letting $n \to \infty$, we see that $\int_\Omega a(x)|u_0|^{q+1}\,dx > 0$.

Suppose $u_n \not\to u_0$ in $W_0^{1,2}(\Omega)$. We shall obtain a contradiction by discussing the fibering map ϕ_{u_0}. Since $\int_\Omega a(x)|u_0|^{q+1}\,dx > 0$, the graph of ϕ_{u_0} must be either of the form shown in Figure 2(c) or (d). Hence there exists $t_0 > 0$ such that $t_0 u_0 \in M_\lambda^+(\Omega)$ and ϕ_{u_0} is decreasing on $(0, t_0)$ with $\phi'_{u_0}(t_0) = 0$.

Since $u_n \not\to u_0$ in $W_0^{1,2}(\Omega)$, $\int_\Omega |\nabla u_0|^2\,dx < \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx$. Thus, as

$$\phi'_{u_n}(t) = t \int_\Omega |\nabla u_n|^2\,dx - \lambda t^q \int_\Omega a(x)|u_n|^{q+1}\,dx - t^p \int_\Omega b(x)|u_n|^{p+1}\,dx$$

and

$$\phi'_{u_0}(t) = t\int_\Omega |\nabla u_0|^2\,dx - \lambda t^q \int_\Omega a(x)|u_0|^{q+1}\,dx - t^p\int_\Omega b(x)|u_0|^{p+1}\,dx,$$

it follows that $\phi'_{u_n}(t_0) > 0$ for n sufficiently large. Since $\{u_n\} \subseteq M_\lambda^+(\Omega)$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all n. Hence we must have $t_0 > 1$. But $t_0u_0 \in M_\lambda^+(\Omega)$ and so

$$J_\lambda(t_0u_0) < J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u)$$

and this is a contradiction. Hence $u_n \to u_0$ in $W_0^{1,2}(\Omega)$ and so

$$J_\lambda(u_0) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u).$$

Thus u_0 is a minimizer for J_λ on $M_\lambda^+(\Omega)$. □

Theorem 4.2. *If $\lambda < \lambda_1$, there exists a minimizer of J_λ on $M_\lambda^-(\Omega)$.*

Proof. By Corollary 3.3 we have $J_\lambda(u) \ge \delta_1 > 0$ for all $u \in M_\lambda^-(\Omega)$ and so $\inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) \ge \delta_1$. Hence there exists a minimizing sequence $\{u_n\} \subseteq M_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) > 0.$$

As in the previous proof, since J_λ is coercive, $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and we may assume, without loss of generality, that $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 < r < \frac{2N}{N-2}$. By (2.2)

$$J_\lambda(u_n) = (\frac{1}{2} - \frac{1}{q+1})\int_\Omega |\nabla u_n|^2\,dx + (\frac{1}{q+1} - \frac{1}{p+1})\int_\Omega b(x)|u_n|^{p+1}\,dx$$

and, since $\lim_{n\to\infty} J_\lambda(u_n) > 0$ and

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{p+1}\,dx = \int_\Omega b(x)|u_0(x)|^{p+1}\,dx,$$

we must have that $\int_\Omega b(x)|u_0(x)|^{p+1}\,dx > 0$. Hence the fibering map ϕ_{u_0} must have graph as shown in Figure 2(b) or (d) and so there exists $\hat{t} > 0$ such that $\hat{t}u_0 \in M_\lambda^-(\Omega)$.

Suppose $u_n \not\to u_0$ in $W_0^{1,2}(\Omega)$. Using the facts that

$$\int_\Omega |\nabla u_0|^2\,dx < \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx$$

and that, since $u_n \in M_\lambda^-(\Omega)$, $J(u_n) \geq J(su_n)$ for all $s \geq 0$, we have

$$\begin{aligned} J(\hat{t}u_0) &= \frac{1}{2}\hat{t}^2 \int_\Omega |\nabla u_0|^2\,dx - \frac{\lambda \hat{t}^{q+1}}{q+1}\int_\Omega a(x)|u_0|^{q+1}\,dx - \frac{\hat{t}^{p+1}}{p+1}\int_\Omega b(x)|u_0|^{p+1}\,dx \\ &< \lim_{n\to\infty}\Big[\frac{1}{2}\,\hat{t}^2\int_\Omega |\nabla u_n|^2\,dx - \frac{\lambda\hat{t}^{q+1}}{q+1}\int_\Omega a(x)|u_n|^{q+1}\,dx \\ &\quad - \frac{\hat{t}^{p+1}}{p+1}\int_\Omega b(x)|u_n|^{p+1}\,dx\Big] \\ &= \lim_{n\to\infty} J(\hat{t}u_n) \\ &\leq \lim_{n\to\infty} J(u_n) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) \end{aligned}$$

which is a contradiction. Hence $u_n \to u_0$ in $W_0^{1,2}(\Omega)$ and the proof can be completed as in the previous theorem. □

Corollary 4.3. *Equation* (1.1), (1.2) *has at least two positive solutions whenever* $0 < \lambda < \lambda_1$.

Proof. By Theorems 4.1 and 4.2 there exist $u^+ \in M_\lambda(\Omega)$ and $u^- \in M_\lambda^-(\Omega)$ such that $J(u^+) = \inf_{u\in M_\lambda^+(\Omega)} J(u)$ and $J(u^-) = \inf_{u\in M_\lambda^-(\Omega)} J(u)$. Moreover $J(u^\pm) = J(|u^\pm|)$ and $|u^\pm| \in M_\lambda^\pm(\Omega)$ and so we may assume $u^\pm \geq 0$. By Lemma 2.2 $u^\pm$ are critical points of J on $W_0^{1,2}(\Omega)$ and hence are weak solutions (and so by standard regularity results classical solutions) of (1.1), (1.2). Finally, by the Harnack inequality due to Trudinger [6], we obtain that $u^\pm$ are positive solutions of (1.1), (1.2). □

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