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DEGENERACY IN THE BLASIUS PROBLEM

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Abstract. The Navier-Stokes equations for the boundary layer are transformed, by a similarity transformation, into the ordinary Blasius differential equation which, together with appropriate boundary conditions constitutes the Blasius problem,

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, \quad f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1.$$

The well-posedness of the Navier-Stokes equations is an open problem. We solve this problem, in the case of constant flow in a boundary layer, by showing that the Blasius problem is ill-posed. If the second condition is replaced by $f'(0) = -\lambda$, then degeneracy occurs for $0 < \lambda < \lambda_c \simeq 0.354$. We investigate the problem analytically to explain this phenomenon. We derive a simple equation $g(\alpha, \lambda) = 0$, whose roots, for a fixed λ, determine the solutions of the problem. It is found that the equation has exactly two roots for $0 < \lambda < \lambda_c$ and no root beyond this point. Since an arbitrarily small perturbation of the boundary condition gives rise to an additional solution, which can be markedly different from the unperturbed solution, the Blasius problem is ill-posed.

1. Introduction

A boundary value problem is said to be *well-posed* if a solution to the problem exists, this solution is unique and depends continuously on the boundary conditions. Otherwise the problem is called *ill-posed*. A boundary value problem, describing a physical situation, must be well-posed because the conditions at a boundary are only satisfied approximately also various approximations are involved in the derivation of the equations governing a problem. Well-posedness of a problem governed by the Poisson equation with Dirichlet boundary conditions in a bounded domain follows easily by an application of the maximum principle for harmonic functions. However the above problem may be ill-posed on an unbounded domain. A famous open problem of applied mathematics is the well-posedness or otherwise of the Navier-Stokes equations, the fundamental equations of fluid mechanics . In this paper we shall investigate the well-posedness of the Blasius problem which deals with the Navier-Stokes equations specialized to the fluid flow in a boundary layer [6, 9].

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The two dimensional constant viscous flow over a semi-infinite flat plate is modeled by the Blasius problem

$$f'''(\eta)+\beta_0 f(\eta)f''(\eta) = 0, f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad \beta_0 > 0, \quad \eta \in [0,\infty) \tag{1.1}$$

Here η represents a *similarity variable* introduced by Blasius [6] to transform a pair of partial differential Navier-Stokes equations into a single ordinary differential equation contained in (1.1). For a thorough discussion of this problem see Schlichting and Gersten [9]. If the third condition in (1.1) is replaced by $f''(0) = \alpha$, where $\alpha > 0$, then the initial-value problem

$$f'''(\eta)+\beta_0 f(\eta)f''(\eta) = 0, f(0) = 0, \quad f'(0) = 0, \quad f^{''}(0) = \alpha, \quad \beta_0 > 0, \quad \eta \in [0,\infty) \tag{1.2}$$

will be called the Blasius initial value problem. It is well-known that any solution of (1.2) corresponding to a fixed α has the property that as $\eta \to \infty$, the first derivative of $f(\eta)$ approaches a constant limit. There is a unique value of α, say α_0, for which the solution of the initial-value problem (1.2) is also the solution of the Blasius problem (1.1). Several authors have devised numerical algorithms to find a good approximate value of this number, see Asaithambi [4] and references therein. For $\beta_0 = \frac{1}{2}$ it is found that

$$\alpha_0 = 0.33206 \tag{1.3}$$

Recently Wang [10] developed an analytical method for finding α_0. Fang et al. [7] showed that for arbitrary β_0,

$$\alpha_0 = 0.469600\sqrt{\beta_0}. \tag{1.4}$$

In view of the above result, it is sufficient to consider the Blasius problem for a fixed β_0, say $\beta_0 = \frac{1}{2}$. We shall do this in this paper.

Let $\lambda \geq 0$, then the boundary-value problem

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, f(0) = 0, \quad f'(0) = -\lambda, \quad f'(\infty) = 1, \quad \eta \in [0,\infty) \tag{1.5}$$

will be called the modified Blasius problem and the related problem

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, f(0) = 0, \quad f'(0) = -\lambda, \quad f^{''}(0) = \alpha, \quad \eta \in [0,\infty) \tag{1.6}$$

will be named the modified Blasius initial value problem. Physically problem (1.5) now models a boundary layer flow over a moving plate with constant velocity λ.

Every solution of the problem (1.2) increases from 0 to ∞ remaining convex throughout until $f'(\eta)$ approaches a constant limit. On the other hand a solution of the problem (1.6) first decreases, attains a minimum and then increases to infinity. A striking difference between the two problems is that whereas the Blasius problem (1.1) possesses a unique solution, the modified Blasius problem (1.5) possesses two solutions when

$$0 < \lambda < \lambda_c \simeq 0.354, \tag{1.7}$$

these solutions coalesce into a unique solution for $\lambda = \lambda_c$ and there is no solution beyond this critical value [1, 2]. Thus the modified Blasius problem becomes degenerate as soon as $\lambda > 0$. The evidence in favor of this remarkable phenomenon is based on numerical results and it is not clear why it should appear at $\lambda = 0$ and then disappear at $\lambda = \lambda_c$. In this paper we shall provide a theory for this.

In Section 2 we shall present a qualitative theory of problem (1.6) which will be used in Section 4 to derive a simple equation of the form

$$g(\lambda, \alpha) = 0. \tag{1.8}$$

For any fixed $\lambda \in (0, \lambda_c)$, this equation has exactly two solutions α_1 and α_2 both of them lying in a finite interval $(0, \alpha_0)$. When $\lambda > \lambda_c$, Equation (1.8) fails to have any real solution.

The occurrence of a second solution as soon as $\lambda > 0$, makes the Blasius problem ill-posed.

2. Qualitative theory of the modified Blasius problem

We consider the initial value problem (1.6) where $\alpha > 0$ and $\lambda > 0$. Anticipating the result $f''(\eta) > 0$ for $0 \le \eta < \infty$, we write the differential equation in the form

$$\frac{f'''(\eta)}{f''(\eta)} = -\frac{1}{2} f(\eta). \tag{2.1}$$

An integration from 0 to η gives

$$f''(\eta) = \alpha \exp\big(-\frac{1}{2}\int_0^\eta f(u)du\big). \tag{2.2}$$

It is clear that

$$f''(\eta) > 0, \quad 0 \le \eta < \infty, \tag{2.3}$$

and we have justified the step leading to (2.1). Since $f(0) = 0$ and $f'(0) < 0$, a solution of (2.1) is negative on some interval. Suppose $f(\eta) \le 0$ on $(0, \infty)$. Then (2.2) gives

$$f''(\eta) > \alpha, \quad 0 \le \eta < \infty.$$

Integration of the above result twice on $[0, \eta]$ gives

$$f(\eta) > \alpha\frac{\eta^2}{2} - \lambda\eta.$$

Since the right hand side of the above inequality is positive for sufficiently large η we have a contradiction of the assumption that $f(\eta) \le 0$ on $[0, \infty)$. Hence there is a point η_1 such that f changes sign at this point. Thus

$$f(\eta) < 0,\ 0 < \eta < \eta_1, \tag{2.4}$$

$$f(\eta_1) = 0,\ f'(\eta_1) > 0. \tag{2.5}$$

Due to (2.3), $f'(\eta)$ increases on (η_1, ∞) and remains positive, therefore the function $f(\eta)$ also remains positive on this interval. On $[0, \eta_1]$, the continuous function $f'(\eta)$ has changed sign hence, due to the well-known intermediate value theorem [5], there is a point η_0, $0 < \eta_0 < \eta_1$ where

$$f'(\eta_0) = 0 \tag{2.6}$$

Using (2.3), (2.4) and (2.5) in the differential equation (2.1), we obtain

$$f'''(\eta) > 0, \quad 0 < \eta < \eta_1, \tag{2.7}$$

$$f'''(\eta_1) = 0. \tag{2.8}$$

Thus we have proved the following result:

Every solution of problem (1.6) has a unique minimum at a point η_0 and a unique zero η_1. On $[0, \eta_1]$ the second derivative of the solution increases and has a maximum at η_1.

For a typical solution, the graphs of $f(\eta)$ and $f''(\eta)$ are shown in Fig.1. This Figure depicts a solution of the problem (1.6) for $\lambda = 0.2$ and $\alpha = 0.016$.

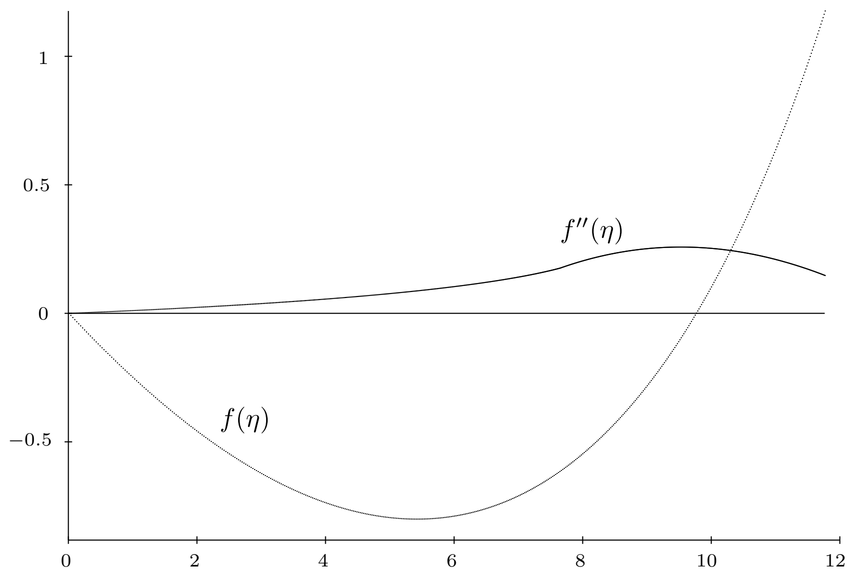

FIGURE 1. Graphs of $f(\eta)$ and $f''(\eta)$, the solution of (1.6) with $\lambda = 0.2$

Asymptotic behavior. There exist positive constants K and c such that

$$f(\eta) > c > 0, \quad \eta \geq K \tag{2.9}$$

Using this in (2.2), we find

$$f''(\eta) < \alpha \exp\Big(-\frac{1}{2}\Big[\int_0^K f(\eta)d\eta + \int_K^\eta c\, d\eta\Big]\Big), \quad \eta \geq K$$

or

$$f''(\eta) < \alpha F \exp\big(-\frac{1}{2}c(\eta - K)\big), \quad \eta \geq K, \tag{2.10}$$

where we have set $F = \exp[-\frac{1}{2}\int_0^K f(\eta)d\eta]$. From (2.3) and (2.10), we get

$$\lim_{\eta\to\infty} f''(\eta) = 0 \tag{2.11}$$

Integrating (2.10) on $[K, \eta)$. We get

$$f'(\eta) < f'(K) + \frac{2\alpha F}{c}\big[1 - e^{-\frac{1}{2}c(\eta-K)}\big], \quad \eta \geq K. \tag{2.12}$$

Hence $f'(\eta)$ is bounded above by $f'(K) + \frac{2\alpha F}{c}$ on $[K, \infty)$. Let $K_1 = \max(K, \eta_1)$. From (2.3), (2.5) and (2.12) the function $f'(\eta)$ is positive, increasing and bounded on $[K_1, \infty)$, hence

$$\lim_{\eta\to\infty} f'(\eta) = l > 0. \tag{2.13}$$

There exists a constant L such that when $\eta \geq L$, $f'(\eta) > \frac{l}{2}$. An integration from L to η gives

$$f(\eta) > f(L) + \frac{l}{2}(\eta - L), \quad \eta \geq L.$$

It follows that

$$\lim_{\eta \to \infty} f(\eta) = \infty \tag{2.14}$$

Thus solutions of (1.2) and (1.6) asymptotically behave in a similar manner.

3. Wang's transformation

Wang [10] used the change of variables

$$x = f'(\eta), \quad y = f''(\eta) \tag{3.1}$$

to transform the Blasius equation $f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0$ into the second order Wang's equation

$$yy'' + \frac{1}{2}x = 0. \tag{3.2}$$

If we make another change of variable $z = x + \lambda$ then (3.2) will become

$$yy'' + \frac{1}{2}(z - \lambda) = 0, \tag{3.3}$$

where differentiation now is with respect to z. The initial conditions are

$$y(0) = \alpha, \quad y'(0) = 0. \tag{3.4}$$

The above problem can be solved by the Adomian decomposition method [10] or by a technique used by Ahmad [3]. The solution up to the term z^8 is found to be

$$\begin{aligned} y(z) = \alpha + \frac{\lambda}{4\alpha}z^2 - \frac{1}{12\alpha}z^3 - \frac{\lambda^2}{96\alpha^3}z^4 + \frac{\lambda}{120\alpha^3}z^5 + \frac{7\lambda^3 - 8\alpha^2}{5760\alpha^5}z^6 \\ - \frac{59\lambda^2}{40320\alpha^5}z^7 - \frac{127\lambda^4 - 336\lambda\alpha^2}{645120\alpha^7}z^8 + \dots \end{aligned} \tag{3.5}$$

Substitute $z = 2\lambda$ in (3.5). This corresponds to $x = \lambda$. Keeping terms only up to λ^6 we find

$$y(2\lambda) = \alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3} \tag{3.6}$$

Concerning a solution of (1.6), Equation (3.6) indicates that there is some point $\eta = \eta_2$ such that

$$f'(\eta_2) = \lambda, \quad f''(\eta_2) = \alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3} \tag{3.7}$$

We shall utilize (3.7) in the next section to prove the existence of degeneracy in the Blasius problem.

4. Degeneracy

Consider a solution of the problem (1.6) which also happens to be a solution of the modified Blasius problem (1.5). Also let $\eta \geq \eta_2$. Keeping in view (2.13), we approximate the function $f(\eta)$ between the points η_2 and η by the equation of the line segment joining them and we take the slope of this segment as the average of the slope at the point η_2, which is λ, and the slope at η which we take as unity. The larger η_2 is, the better this approximation becomes. The area under the curve representing $f(\eta)$ from the point η_2 to η is approximately $\frac{1}{4}(1+\lambda)(\eta-\eta_2)^2$. Using this and (3.7) in (2.2), we find

$$f''(\eta) = (\alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3}) \exp\big(-\frac{1}{8}(1+\lambda)(\eta-\eta_2)^2\big), \quad \eta \geq \eta_2 \tag{4.1}$$

Integrating (4.1) from η_2 to ∞, we get

$$f'(\infty) - f'(\eta_2) = \sqrt{\frac{2\pi}{1+\lambda}}\Big(\alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3}\Big)$$

Putting values of $f'(\infty)$ and $f'(\eta_2)$ and dividing with the term under the square-root sign, we obtain

$$\frac{(1-\lambda)\sqrt{1+\lambda}}{\sqrt{2\pi}} = \alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3} \tag{4.2}$$

For a fixed λ, the number of real positive roots determines the number of solutions possessed by the modified Blasius problem under consideration. The graph of the function

$$g(\alpha) = \alpha + \frac{\lambda^3}{3\alpha} + \frac{\lambda^6}{90\alpha^3} \tag{4.3}$$

for a fixed λ is shown in Figure 2. The minimum value of $g(\alpha)$ is $1.2032\lambda^{3/2}$ and this occurs when $\alpha = 0.6433\lambda^{3/2}$. It is clear that exactly two roots of (4.2) will exist as long as

$$\frac{(1-\lambda)\sqrt{1+\lambda}}{\sqrt{2\pi}} > 0.6433\,\lambda^{3/2} \tag{4.4}$$

This happens for

$$0 < \lambda < \lambda_c = 0.386 \tag{4.5}$$

For $\lambda = \lambda_c$ the two solutions overlap and if $\lambda > \lambda_c$, Equation (4.2) has no real root implying that no solution of the modified Blasius problem exists. Thus the problem becomes degenerate between $\lambda = 0$ and $\lambda = \lambda_c$.

5. Numerical results

In Table 1 we present two pairs of values, α_1, α_2 and α_1^e, α_2^e of the parameter α each of one produces a solution of the problem (1.6) which is also a solution of the modified Blasius problem (1.5). The numbers α_1 and α_2 have been found by solving (4.2) while α_1^e and α_2^e refer to their exact values found by a numerical solution of (1.6) for various α and searching for those values for which $f'(R) \simeq 1$ for sufficiently large R.

The approximate critical value λ_c was found above as 0.386. Its exact value is 0.3546. Considering the nature of approximations involved in obtaining (4.2), agreement between approximate and exact results is remarkable.

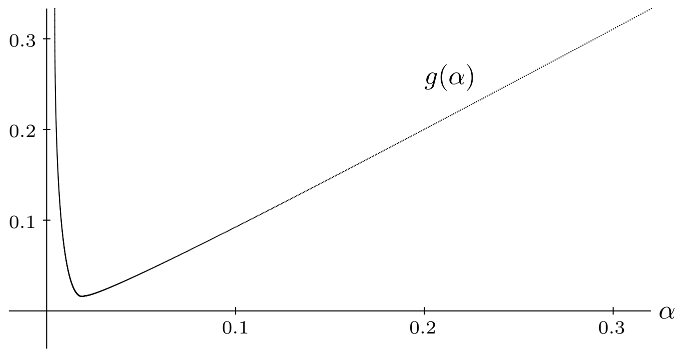

FIGURE 2. Graph of $g(\alpha)$ defined by (4.3) when $\lambda = 0.1$

λ	α_1	α_2	α_1^e	α_2^e
0.1	0.0034	0.376	0.00137	0.326
0.15	0.0083	0.361	0.0061	0.317
0.20	0.016	0.342	0.0157	0.304
0.25	0.028	0.318	0.0321	0.284
0.30	0.0468	0.287	0.0600	0.252
0.34	0.0706	0.252	0.1034	0.206
0.354	0.083	0.235	0.149	0.150

TABLE 1. Approximate and exact values of α that produce solutions of the modified Blasius problem

Discussion. A study of the qualitative behavior of the solutions of (1.6) paved the way for a proper understanding of degeneracy in the Blasius problem. All pairs of values of α_1^e and α_2^e lie in the interval $(0, 0.33206)$. In [1] an analytical solution of the problem was obtained by employing the *homotopy analysis method* [8] and it was erroneously claimed that the above interval is $(0.33206, 0.52)$.

Existence of multiple solutions of (1.5) for $\lambda \in (0, \lambda_c)$ demonstrates that the Blasius problem (1.1) is ill-posed. A solution of a well-posed problem must depend *continuously* on the initial or boundary conditions. However as soon as the condition $f'(0) = 0$ of the Blasius problem is replaced by $f'(0) = -\lambda$, λ being arbitrarily small, a solution appears which is markedly different from the unique solution of the Blasius problem. This indicates a need to re-interpret experimental results of any investigation based on the Blasius problem.

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