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ITERATED ORDER OF SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

SAADA HAMOUDA

Abstract. In this paper, we investigate the iterated order of solutions of homogeneous and nonhomogeneous linear differential equations where the coefficients are entire functions and satisfy certain growth conditions.

1. Introduction and statement of results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory [5].

For $n \geq 2$, we consider the linear differential equation

$$w^{(n)} + A_{n-1}(z)w^{(n-1)} + \cdots + A_0(z)w = B(z), \tag{1.1}$$

where $A_0(z), \dots, A_{n-1}, B(z)$ are entire functions with $A_0(z) \not\equiv 0$ or $B(z) \not\equiv 0$. It is well known that all solutions of (1.1) are entire functions, and if some of the coefficients are transcendental, then (1.1) has at least one solution of infinite order. Thus, the question which arises is: What conditions on $A_0(z), \dots, A_{n-1}, B(z)$ will guarantee that every solution $w \not\equiv 0$ has infinite order? For the above question, there are many results, (see for example [2, 4, 1]). Gundersen and Steinbart [4] proved the following results.

Theorem 1.1 ([4]). *Let μ, θ_1, θ_2 be real constants satisfying $\mu > 0$ and $\theta_1 < \theta_2$. Suppose that, in the differential equation*

$$w^{(n)} + A_{n-1}(z)w^{(n-1)} + \cdots + A_0(z)w = B(z), \tag{1.2}$$

there exists a unique coefficient $A_q(z)$ such that for any $\theta \in (\theta_1, \theta_2)$ there exist real constants $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ satisfying $0 \leq \beta < \alpha$, so that the following conditions hold as $z \to \infty$ along $\arg z = \theta$:

$$\begin{gathered} |A_q(z)| \geq \exp\{(\alpha + o(1))|z|^\mu\}, \\ |A_k(z)| \leq \exp\{(\beta + o(1))|z|^\mu\} \quad \text{for all } k \neq q, \\ |B(z)| \leq \exp\{(\beta + o(1))|z|^\mu\}. \end{gathered}$$

Assume that $\alpha(\theta)$ and $\beta(\theta)$ are continuous functions on $\theta_1 < \theta < \theta_2$.

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Suppose that w is transcendental solution of (1.2) with $\sigma(w) < \infty$. If $l \geq q$ is an integer, then for any $\theta \in (\theta_1, \theta_2)$ we have

$$|w^{(l)}(z)| \leq \exp\{-(\alpha - \beta + o(1))|z|^{\mu}\}$$

as $z \to \infty$ along $\arg z = \theta$.

Corollary 1.2 ([4]). *Let $\theta_1, \theta_2, \dots, \theta_m$ be a finite set of real numbers that satisfy $\theta_1 < \theta_2 < \dots < \theta_m + 2\pi$. Suppose that for each $i = 1, 2, \dots, m-1$, there exists in (1.2) one particular coefficient $A_{q_i}(z)$ and a corresponding constant $\mu_i > 0$, such that for any $\theta \in (\theta_i, \theta_{i+1})$ there exist constants $\alpha_i = \alpha_i(\theta)$ and $\beta_i = \beta_i(\theta)$ satisfying $0 \leq \beta_i < \alpha_i$, so that the following conditions hold as $z \to \infty$ along $\arg z = \theta$:*

$$\begin{gathered}
|A_{q_i}(z)| \geq \exp\{(\alpha_i + o(1))|z|^{\mu_i}\}, \\
|A_k(z)| \leq \exp\{(\beta_i + o(1))|z|^{\mu_i}\} \quad \text{for all } k \neq q_i, \\
|B(z)| \leq \exp\{(\beta_i + o(1))|z|^{\mu_i}\}.
\end{gathered}$$

For each $i = 1, 2, \dots, m-1$, assume that $\alpha_i(\theta)$ and $\beta_i(\theta)$ are continuous functions on $\theta_i < \theta < \theta_{i+1}$.

Then every transcendental solution w of (1.2) satisfies $\sigma(w) = \infty$.

In 2001, Belaïdi and Hamouda proved the following result.

Theorem 1.3 ([1]). *Let $A_0(z), \dots, A_{n-1}$ be entire functions with $A_0(z) \not\equiv 0$, such that for some real constants $\alpha, \beta, \mu, \theta_1, \theta_2$, with $0 \leq \beta < \alpha$, $\mu > 0$, $\theta_1 < \theta_2$, we have*

$$\begin{gathered}
|A_0(z)| \geq e^{\alpha|z|^{\mu}}, \\
|A_k(z)| \leq e^{\beta|z|^{\mu}} \quad (k = 1, \dots, n-1),
\end{gathered}$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $w \not\equiv 0$ of the differential equation

$$w^{(n)} + A_{n-1}(z)w^{(n-1)} + \dots + A_0(z)w = 0 \tag{1.3}$$

has infinite order.

Another question is: For solutions of infinite order, how can we describe precisely their growth?

For $r \in [0, \infty)$, we define $\exp_0 r = r$, $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$ ($i \in \mathbb{N}$). For r sufficiently large, we define $\log_0 r = r$, $\log_1 r = \log r$, $\log_{i+1} r = \log(\log_i r)$ ($i \in \mathbb{N}$). Also, we can define $\exp_{-i} r = \log_i r$ and $\log_{-i} r = \exp_i r$. To express the rate of growth of entire function of infinite order, we introduce the notion of iterated order [9, 7, 6].

The iterated i-order of entire function w is defined by

$$\sigma_i(w) = \limsup_{r \to +\infty} \frac{\log_i T(r, w)}{\log r} = \limsup_{r \to +\infty} \frac{\log_{i+1} M(r, w)}{\log r} \quad (i \in \mathbb{N}). \tag{1.4}$$

Recently, Tu, Chen and Zheng proved the following result.

Theorem 1.4 ([10]). *Let $A_0(z), \dots, A_{n-1}$ be entire functions with $A_0(z) \not\equiv 0$, such that for real constants α, β, μ, θ_1, θ_2, and positive integer p with $0 \leq \beta < \alpha$, $\mu > 0$, $\theta_1 < \theta_2$, we have*

$$\begin{gathered}
|A_0(z)| \geq \exp_p\{\alpha|z|^{\mu}\}, \\
|A_k(z)| \leq \exp_p\{\beta|z|^{\mu}\}, \quad (k = 1, \dots, n-1),
\end{gathered}$$

as $z \to \infty$ *with* $\theta_1 \le \arg z \le \theta_2$. *Then* $\sigma_{p+1}(w) \ge \mu$ *holds for all non-trivial solutions of* (1.3).

In this paper, we will prove the following results.

Theorem 1.5. *Let* $A_0(z), \dots, A_{n-1}$ *be entire functions with* $A_0(z) \not\equiv 0$, *and let* $\theta_1, \theta_2, \dots, \theta_m$ *be a finite set of real numbers that satisfy* $0 = \theta_1 < \theta_2 < \dots < \theta_m = 2\pi$ *such that for each* $i \in \{1, 2, \dots, m-1\}$. *Then there exists in* (1.3) *one particular coefficient* $A_{s_i}(z)$ *and a corresponding constant* $\mu_i > 0$, *such that for any* $\theta \in (\theta_i, \theta_{i+1})$ *there exist constants* α_i *and* β_i *satisfying* $0 \le \beta_i < \alpha_i$, *we have*

$$|A_{s_i}(z)| \ge \exp_p\{\alpha_i |z|^{\mu_i}\}, \tag{1.5}$$

$$|A_k(z)| \le \exp_p\{\beta_i |z|^{\mu_i}\} \quad \text{for all } k \ne s_i), \tag{1.6}$$

as $z \to \infty$ *along* $\arg z = \theta$, *where* $p \ge 1$ *is an integer. Then, every transcendental solution* w *of* (1.3) *satisfies*

$$\sigma_{p+1}(w) \ge \min_{1 \le i \le m-1} \{\mu_i\}.$$

Theorem 1.6. *Suppose that* $A_0(z), \dots, A_{n-1}, B(z)$ *are entire functions such that for real constants* α, β, μ, θ_1, θ_2, *with* $0 \le \beta < \alpha$, $\mu > 0$, $0 \le \theta_1 < \theta_2 \le 2\pi$, *we have, for some integer* s $(1 \le s \le n-1)$,

$$|A_s(z)| \ge \exp_p\{\alpha |z|^{\mu}\}, \tag{1.7}$$

$$|A_k(z)| \le \exp_p\{\beta |z|^{\mu}\} \quad \text{for all } k \ne s, \tag{1.8}$$

$$|B(z)| \le \exp_p\{\beta |z|^{\mu}\} \tag{1.9}$$

as $z \to \infty$ *with* $\theta_1 \le \arg z \le \theta_2$, *where* $p \ge 1$ *is an integer. Given* $\varepsilon > 0$ *small enough. If* w *is a transcendental solution with* $\sigma_p(w) < \infty$ *of the differential equation*

$$w^{(n)} + A_{n-1}(z) w^{(n-1)} + \dots + A_0(z) w = B(z), \tag{1.10}$$

then there exists a constant $M > 0$ *such that, for each integer* $l \ge s$, *we have*

$$|w^{(l)}(z)| \le M \tag{1.11}$$

along any ray $\arg z = \theta \in S(\varepsilon) = \{\theta : \theta_1 + \varepsilon \le \arg z \le \theta_2 - \varepsilon\}$.

Corollary 1.7. *Let* $A_0(z), \dots, A_{n-1}$, $B(z)$ *be entire functions with* $B(z) \not\equiv 0$, *let* $\theta_1, \theta_2, \dots, \theta_m$ *be a finite set of real numbers that satisfy* $0 = \theta_1 < \theta_2 < \dots < \theta_m = 2\pi$ *such that for each* $i \in \{1, 2, \dots, m-1\}$ *there exists in* (1.10) *one particular coefficient* $A_{s_i}(z)$ *and a corresponding constants* μ_i, α_i, β_i *satisfying* $0 \le \beta_i < \alpha_i$, $\mu_i > 0$, *we have*

$$|A_{s_i}(z)| \ge \exp_p\{\alpha_i |z|^{\mu_i}\}, \tag{1.12}$$

$$|A_k(z)| \le \exp_p\{\beta_i |z|^{\mu_i}\} \quad \text{for all } k \ne s_i, \tag{1.13}$$

$$|B(z)| \le \exp_p\{\beta_i |z|^{\mu_i}\} \tag{1.14}$$

as $z \to \infty$ *with* $\theta_i < \arg z < \theta_{i+1}$, *where* $p \ge 1$ *is an integer. Then, every transcendental solution* w *of* (1.10) *satisfies* $\sigma_p(w) = \infty$.

In the above corollary, the differential equation (1.10) may have a polynomial solution. For example, $w(z) = z$ is a solution of

$$w^{(4)} + e^z w^{(3)} + e^{-z} w'' + w' + w = z + 1.$$

In general, $w(z) = z$ is a solution of the differential equation

$$w^{(n)} + A_{n-1}(z)w^{(n-1)} + \cdots + A_2(z)w'' + w' + w = z + 1,$$

where the coefficients $A_2(z), \dots, A_{n-1}(z)$ satisfy the conditions (1.12) and (1.13).

Our proofs depend mainly on the following Lemmas.

2. Auxiliary Lemmas

Lemma 2.1 ([3]). *Let $w(z)$ be a transcendental entire function. Let*

$$\Gamma = \{(k_1, q_1), (k_2, q_2), \dots, (k_m, q_m)\}$$

denote a finite set of distinct pairs of integers that satisfy $k_i > q_i \geq 0$ $(i = 1, \dots, m)$ and let $\alpha > 0$ be a given real constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero and a constant $c > 0$ that depend only on α, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq R_0$, and for all $(k, q) \in \Gamma$, we have

$$\Big|\frac{w^{(k)}(z)}{w^{(q)}(z)}\Big| \leq c[T(\alpha r, w)\frac{1}{r}\log^{\alpha}(r)\log T(\alpha r, w)]^{k-q}.$$

From the above Lemma, which is also [3, Theorem 2], we obtain the following result.

Lemma 2.2. *Let w be a transcendental entire function with $\sigma_p(w) = \sigma < \infty$ $(p \geq 1)$. Let $\Gamma = \{(k_1, q_1), (k_2, q_2), \dots, (k_m, q_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > q_i \geq 0$ $(i = 1, \dots, m)$. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq R_0$, and for all $(k, q) \in \Gamma$, we have*

$$\Big|\frac{w^{(k)}(z)}{w^{(q)}(z)}\Big| \leq [\exp_{p-1}\{r^{\sigma+\varepsilon}\}]^{k-q},$$

where $\varepsilon > 0$.

Proof. The definition

$$\sigma_p(w) = \limsup_{r \to +\infty} \frac{\log_p T(r, w)}{\log r} = \sigma$$

implies that for any given $\varepsilon' > 0$ there exists $r_0 > 0$ such that for all $r \geq r_0$ we have

$$\frac{\log_p T(r, w)}{\log r} < \sigma + \varepsilon';$$

which implies

$$T(r, w) < \exp_{p-1}\{r^{\sigma+\varepsilon'}\}. \tag{2.1}$$

Combining (2.1) with Lemma 2.1, for $\alpha > 0$, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero and a constant $c > 0$, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $r'_0 = r'_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq r'_0$,

$$\Big|\frac{w^{(k)}(z)}{w^{(q)}(z)}\Big| \leq c\big[\exp_{p-1}\{(\alpha r)^{\sigma+\varepsilon'}\}\exp_{p-2}\{(\alpha r)^{\sigma+\varepsilon'}\}\big]^{k-q}.$$

Then, there exists a constant $\varepsilon > \varepsilon' > 0$ and R_0 large enough, such that for $|z| = r \geq R_0$

$$|\frac{w^{(k)}(z)}{w^{(q)}(z)}| \leq [\exp_{p-1}\{r^{\sigma+\varepsilon}\}]^{k-q}.$$

□

Lemma 2.3 ([2, 4, 8]). *Let $w(z)$ be an entire function and suppose that $|w^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_j = r_j e^{i\theta}$ $(j = 1, 2, \dots)$, where $r_j \to +\infty$, such that $w^{(k)}(z_j) \to \infty$ and*

$$|\frac{w^{(q)}(z_j)}{w^{(k)}(z_j)}| \leq \frac{1}{(k-q)!}(1 + o(1))|z_j|^{k-q} \quad q = 0, \dots, k-1.$$

3. Proof of Theorems

Proof of Theorem 1.5. From (1.3) we can write

$$\begin{aligned} |A_{s_i}| \leq |\frac{w^{(n)}}{w^{(s_i)}}| + |A_{n-1}||\frac{w^{(n-1)}}{w^{(s_i)}}| + \dots + |A_{s_i+1}||\frac{w^{(s_i+1)}}{w^{(s_i)}}| \\ + |A_{s_i-1}||\frac{w^{(s_i-1)}}{w^{(s_i)}}| + \dots + |A_1||\frac{w'}{w^{(s_i)}}| + |A_0||\frac{w}{w^{(s_i)}}|. \end{aligned} \tag{3.1}$$

By Lemma 2.1 and taking into account $\frac{1}{r}\log^2(r) < 1$ and

$$\log T(2r, w^{(s_i)}) < T(2r, w^{(s_i)}),$$

there exists a constant $c > 0$ and a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) - E$, then there is a constant $r_1 = r_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq r_1$, we have

$$|\frac{w^{(k)}(z)}{w^{(s_i)}(z)}| \leq c[T(2r, w^{(s_i)})]^{2n}, \quad k = s_i + 1, \dots, n. \tag{3.2}$$

Since w is transcendental, then there exist a real constants φ_1, φ_2, with $0 \leq \varphi_1 < \varphi_2 \leq 2\pi$, such that for all $\theta \in (\varphi_1, \varphi_2)$, $|w^{(s_i)}(z)|$ is unbounded as $z \to \infty$ along $\arg z = \theta$. The sector (φ_1, φ_2) will certainly be intersected with, at least, one of the sectors (θ_i, θ_{i+1}) $(i = 1, \dots, m-1)$ in a sector (φ_1', φ_2'), $(\varphi_1 \leq \varphi_1' < \varphi_2' \leq \varphi_2)$. By Lemma 2.3, if $\theta \in (\varphi_1', \varphi_2')$, there exists an infinite sequence of points $z_j = r_j e^{i\theta}$ where $r_j \to \infty$, such that $w^{(s_i)}(z_j) \to \infty$ and

$$|\frac{w^{(k)}(z_j)}{w^{(s_i)}(z_j)}| \leq (1 + o(1))|z_j|^{s_i - k} \tag{3.3}$$

as $z_j \to \infty$, for all k satisfying $0 \leq k \leq s_i - 1$.

Combining (1.5), (1.6), (3.2) and (3.3) with (3.1), we obtain, for r_j large enough,

$$\exp_p\{\alpha_i r_j^{\mu_i}\} \leq A[T(2r_j, w^{(s_i)})]^{2n} \exp_p\{\beta_i r_j^{\mu_i}\},$$

where $A > 0$; so that

$$\exp_p\{\alpha_i r_j^{\mu_i}\} \exp\{-\exp_{p-1}\{\beta_i r_j^{\mu_i}\}\} \leq A[T(2r_j, w^{(s_i)})]^{2n},$$

which implies

$$\limsup_{r\to+\infty} \frac{\log_{p+1} T(r, w^{(s_i)})}{\log r} \geq \mu_i.$$

Since

$$T(r, w^{(s_i)}) \leq (s_i + 1)T(r, w) + S(r, w),$$

where $S(r,w)=o\{T(r,w)\}$ as $r\to\infty$, it follows that $\sigma_{p+1}(w)\geq\mu_i$ for, at least, one $i\in\{1,\dots,m-1\}$. Thus, in general, we have

$$\sigma_{p+1}(w)\geq\min_{1\leq i\leq m-1}\{\mu_i\}.$$

□

Proof of Theorem 1.6. Suppose that w is a transcendental solution with $\sigma_p(w)=\sigma<\infty$ of (1.10). From (1.10), we can write

$$\begin{aligned} w^{(s)}\Big[&\frac{w^{(n)}}{w^{(s)}}\frac{1}{A_s}+\frac{w^{(n-1)}}{w^{(s)}}\frac{A_{n-1}}{A_s}+\dots+\frac{w^{(s+1)}}{w^{(s)}}\frac{A_{s+1}}{A_s}\\ &+1\frac{w^{(s-1)}}{w^{(s)}}\frac{A_{s-1}}{A_s}+\dots+\frac{w}{w^{(s)}}\frac{A_0}{A_s}\Big]=\frac{B}{A_s}\end{aligned}\tag{3.4}$$

From Lemma 2.2, there exists a set $E\subset[0,2\pi)$ of linear measure zero such that for all $k=s+1,\dots,n$ and all $r\geq r_2$ large enough, we have

$$\Big|\frac{w^{(k)}(z)}{w^{(s)}(z)}\Big|\leq[\exp_{p-1}\{r^{\sigma+\varepsilon}\}]^{k-s},\tag{3.5}$$

along any ray $\arg z=\psi\in[0,2\pi)-E$, provided $\varepsilon>0$.

If $|w^{(s)}(z)|$ is unbounded on some ray $\arg z=\phi\in S(0)-E$, then by Lemma 2.3, there exists a sequence of points $z_j=r_je^{i\phi}$, $r_j\to\infty$ such that $w^{(s)}(z_j)\to\infty$ and

$$\Big|\frac{w^{(q)}(z_j)}{w^{(s)}(z_j)}\Big|\leq\frac{1}{(s-q)!}(1+o(1))|z_j|^{s-q}\leq 2|z_j|^n\tag{3.6}$$

for all $q=0,\dots,s-1$ and all j large enough.

Combining now (1.7), (1.8), (1.9), (3.5) and (3.6) with (3.4) yelds $w^{(s)}(z_j)\to 0$ as $z_j\to\infty$, a contradiction. Hence $|w^{(s)}(z_j)|$ must be bounded on any $\arg z=\phi\in S(0)-E$. By Phragmén-Lindelöf theorem, we conclude that $|w^{(s)}(z)|$ is bounded, say $|w^{(s)}(z)|<M$, in the whole sector $S(\frac{\varepsilon}{2})$. Using the Cauchy integral formula to deduce that for all $l>s$,

$$|w^{(l)}(z)|<M$$

in $S(\varepsilon)$. □

Proof of Corollary 1.7. Suppose that $w(z)$ is a transcendental solution of (1.10) with $\sigma_p(w)<\infty$. By Theorem 1.6, in all the sectors $(\theta_i+\varepsilon,\theta_{i+1}-\varepsilon)$ $(i=1,2,\dots,m-1)$, ($\varepsilon>0$ small enough), we obtain

$$|w^{(l)}(z)|<M\tag{3.7}$$

for all $l\geq n-1$. By the Phragmén-Lindelöf theorem, (3.7) remains valid in the whole plane; and by the Liouville theorem, $w(z)$ has to be a polynomial, a contradiction. Then $\sigma_p(w)=\infty$. □

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References

[1] B. Belaïdi, S. Hamouda; *Orders of solutions of an n-th order linear differential equations with entire coefficients*, Electron. J. Diff. Eqns, 2001 (2001), no. 61, 1–5.

[2] G. G. Gundersen; *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), 415-429.

[3] G. G. Gundersen; *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., (2) 37 (1988), 88-104.

[4] G. G. Gundersen, E. Steinbart; *Finite order solutions of nonhomogeneous linear differential equations*, Ann. Acad. Sci. Fenn. A I Math. 17, (1992), 327-341.

[5] W. K. Hayman; *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[6] G. Jank, L. Volkman; *Einführung in die Theory der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichengen*, Birkhäuser Verlag, Basel-Boston, 1985.

[7] I. Laine; *Nevanlinna theory and complex differential equations*, W. de Gruyter, Berlin, 1993.

[8] I. Laine, and R. Yang; *Finite order solutions of complexe linear differential equations*, Electron. J. Diff. Eqns, 2004 (2004), no. 65, 1-8.

[9] D. Sato; *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. 69(1963), 411-414.

[10] J. Tu, Z. X. Chen, X. Zheng; *Growth of solutions of complex differential equations with coefficients of finite iterated order*, Electron. J. Diff. Eqns, 2006 (2006), no. 54, 1-8.

Saada Hamouda

Department of Mathematics, University of Mostaganem, B. P. 227 Mostaganem, Algeria

E-mail address: `hamouda_saada@yahoo.fr`