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A NOTE ON EXTREMAL FUNCTIONS FOR SHARP SOBOLEV INEQUALITIES

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Abstract. In this note we prove that any compact Riemannian manifold of dimension $n \geq 4$ which is non-conformal to the standard n-sphere and has positive Yamabe invariant admits infinitely many conformal metrics with nonconstant positive scalar curvature on which the classical sharp Sobolev inequalities admit extremal functions. In particular we show the existence of compact Riemannian manifolds with nonconstant positive scalar curvature for which extremal functions exist. Our proof is simple and bases on results of the best constants theory and Yamabe problem.

1. Introduction and statement of main results

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. We denote by $H_1^2(M)$ the standard first-order Sobolev space defined as the completion of $C^\infty(M)$ under the norm

$$\|u\|_{H_1^2(M)} = \left(\int_M |\nabla_g u|^2 \, dv_g + \int_M |u|^2 \, dv_g \right)^{1/2} .$$

The Sobolev embedding theorem ensures that the inclusion $H_1^2(M) \subset L^{2^*}(M)$ is continuous for $2^* = \frac{2n}{n-2}$. So, there exist constants $A, B \in \mathbb{R}$ such that, for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^*} \, dv_g \right)^{2/2^*} \leq A \int_M |\nabla_g u|^2 \, dv_g + B \int_M |u|^2 \, dv_g \, . \tag{I_g^2}$$

In this case, we say simply that (I_g^2) is valid. The first Sobolev best constant associated to (I_g^2) is

$$A_0(2, g) = \inf\{ A \in \mathbb{R} : \text{ there exists } B \in \mathbb{R} \text{ such that } (I_g^2) \text{ is valid } \}$$

and, by Aubin [1], its value is given by $K(n, 2)^2$, where

$$K(n,2) = \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{1/2^*}}{\left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{1/2}},$$

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and $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\Big)^{1/2}.$$

Moreover, Aubin [1] and Talenti [9] found the explicit value of $K(n,2)$. The optimal L^2-Riemannain Sobolev inequality on $H_1^2(M)$ states that

$$\Big(\int_M |u|^{2^*} \, dv_g\Big)^{2/2^*} \le K(n,2)^2 \int_M |\nabla_g u|^2 \, dv_g + B \int_M |u|^2 \, dv_g . \qquad (I^2_{g,opt})$$

We say that $(I^2_{g,opt})$ is valid if there exists a constant $B \in \mathbb{R}$ such that $(I^2_{g,opt})$ holds for all $u \in H_1^2(M)$. A first question is the validity or not of $(I^2_{g,opt})$. The optimal inequality was proved to be valid by Hebey and Vaugon [6]. Thus, consider the second Sobolev best constant

$$B_0(2,g) = \inf\{B \in \mathbb{R} : \ (I^2_{g,opt}) \text{ is valid}\}.$$

Clearly, for any $u \in H_1^2(M)$, one has

$$\Big(\int_M |u|^{2^*} \, dv_g\Big)^{2/2^*} \le K(n,2)^2 \int_M |\nabla_g u|^2 \, dv_g + B_0(2,g) \int_M |u|^2 \, dv_g . \qquad (J^2_{g,opt})$$

A function $u_0 \in H_1^2(M)$, $u_0 \neq 0$, is said to be extremal for $(J^2_{g,opt})$ if

$$\Big(\int_M |u_0|^{2^*} \, dv_g\Big)^{2/2^*} = K(n,2)^2 \int_M |\nabla_g u_0|^2 \, dv_g + B_0(2,g) \int_M |u_0|^2 \, dv_g .$$

Two important questions are the existence of extremal functions for $(J^2_{g,opt})$ and the explicit value of $B_0(2,g)$. These questions were completely solved on compact manifolds (M,g) conformal to the canonical n-sphere $(\mathbb{S}^n, g_0)$. Indeed, on such manifolds, Hebey proved in [5] that there exists an extremal function for $(J^2_{h,opt})$ if, and only if, h is isometric to g and, in this case, the scalar curvature of the metric h is constant. Djadli and Druet [3] showed that on compact Riemannian manifolds of dimension $n \ge 4$, at least, one of the following assertions holds:

(a) an extremal function for $(J^2_{g,opt})$ exists, or
(b) $B_0(2,g) = \frac{n-2}{4(n-1)} K(n,2)^2 \max_M \mathrm{Scal}_g$,

where Scal_g stands for the scalar curvature of g. Hebey and Vaugon [7], introduced the notion of critical function in order to study the dichotomy between (a) and (b). In particular, they showed that it is not exclusive, see [4] for an overview of this subject. Combining the assertion (b) with the solution of Yamabe problem given by Aubin [2] and Schoen [8], one easily concludes that there exist extremal functions for $(J^2_{g,opt})$ when either $\mathrm{Scal}_g \le 0$ or Scal_g is constant. However, the existence of extremal functions is an open question in the nonconstant positive scalar curvature case. In this note we are interested in discussing this case. This discussion is motivated by the fact of existing examples of compact Riemannian manifolds (M,g) for which $(J^2_{g,opt})$ possesses no extremal function.

To state our main result, we recall the definition of the Yamabe invariant. Consider the functional $I_g(u)$ on $H_1^2(M) \setminus \{0\}$ given by

$$I_g(u) = \frac{\int_M |\nabla_g u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_g \ u^2 \, dv_g}{(\int_M |u|^{2^*} \, dv_g)^{2/2^*}} .$$

The Yamabe invariant on (M, g) is defined by

$$\mu_g = \inf_{H_1^2(M)\setminus\{0\}} I_g(u)\,.$$

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ non-conformal to $(\mathbb{S}^n, g_0)$ such that $\mu_g > 0$. Then exist an infinitely many metrics h conformal to g with nonconstant positive scalar curvature such that $(J^2_{h,opt})$ admits an extremal function.*

The following results is a direct consequence of Theorem 1.1.

Corollary 1.2. *There exist compact Riemannian manifolds (M, g) of dimension $n \geq 4$ with nonconstant positive scalar curvature such that $(J^2_{g,opt})$ possesses an extremal function.*

One easily constructs concrete examples of such manifolds. For instance, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and the projective space $\mathbb{P}^n$, with their usual metrics, are non-conformal to $(\mathbb{S}^n, g_0)$ and possesses positive Yamabe invariant.

The proof of Theorem 1.1 is simple and short. The ideas are based on standard minimization techniques and use a well-known existence result due to Aubin [2]. Results of the works [2] and [8] about the solution of the Yamabe problem and of the work [3] about the existence of extremal functions play an essential role in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Construction of the metric h. Let $w_0 \in C^\infty(M)$ be a positive solution of the Yamabe problem on (M, g). The scalar curvature of the metric $h_0 = w_0^{2^*-2} g$ is a positive constant R, since $\mu_g > 0$. Moreover, one has $\mu_g = \frac{n-2}{4(n-1)} R v_{h_0}^{\frac{2}{n}}$, so that

$$\Big(\frac{1}{K(n,2)^2}\Big)^{2^*/n} \mu_g^{-\frac{2^*}{2}} = \Big(\frac{4(n-1)}{n-2}\frac{1}{K(n,2)^2 R}\Big)^{n/(n-2)} K(n,2)^2 v_{h_0}^{-2^*/n},$$

where $v_{h_0} = \int_M dv_{h_0}$ stands for the volume of M on h_0. Aubin [2] and Schoen [8] proved that for any compact Riemannian manifold non-conformal to $(\mathbb{S}^n, g_0)$, one has

$$\mu_g < \frac{1}{K(n,2)^2}, \tag{2.1}$$

so that

$$\mu_g^{-1} < \Big(\frac{1}{K(n,2)^2}\Big)^{2^*/n} \mu_g^{-2^*/2}\,.$$

Combining these relations, one obtains

$$\frac{1}{K(n,2)^2 \mu_g} < \Big(\frac{4(n-1)}{n-2}\frac{1}{K(n,2)^2 R}\Big)^{n/(n-2)} v_{h_0}^{-2^*/n}\,.$$

Now, let $a \in C^\infty(M)$ be a function satisfying

$$0 < \max_M a(x) < \frac{1}{K(n,2)^2 \mu_g} \min_M a(x)\,. \tag{2.2}$$

Using the function a, we find

$$\begin{aligned}\max_M a(x) &< \Big(\frac{4(n-1)}{n-2}\frac{1}{K(n,2)^2R}\Big)^{n/(n-2)} v_{h_0}^{-\frac{2^*}{n}} \min_M a(x)\\ &\leq \Big(\frac{4(n-1)}{n-2}\frac{1}{K(n,2)^2R}\Big)^{n/(n-2)} v_{h_0}^{-\frac{2^*}{n}} a(x)\end{aligned}$$

for all $x \in M$, so that

$$\big(\max_M a(x)\big)^{(n-2)/n} < \frac{4(n-1)}{n-2}\frac{1}{K(n,2)^2Rv_{h_0}}\Big(\int_M a(x)\,dv_{h_0}\Big)^{(n-2)/n},$$

or equivalently,

$$\frac{n-2}{4(n-1)}Rv_{h_0} < \frac{1}{K(n,2)^2}\Big(\frac{\int_M a(x)\,dv_{h_0}}{\max_M a(x)}\Big)^{(n-2)/n}. \tag{2.3}$$

Consider now the functional $J_{h_0}(u)$ on $H_1^2(M)$ defined by

$$J_{h_0}(u) = \int_M |\nabla_{h_0}u|^2\,dv_{h_0} + \frac{n-2}{4(n-1)}R\int_M u^2\,dv_{h_0}\ .$$

The next step is to minimize $J_{h_0}(u)$ on the set

$$\mathcal{H} = \{u \in H_1^2(M) : \int_M a(x)|u|^{2^*}\,dv_{h_0} = 1\}\,.$$

Note that $\mathcal{H}$ is non-empty since $\overline{u} = \big(\int_M a(x)\,dv_{h_0}\big)^{-\frac{1}{2^*}} \in \mathcal{H}$. In addition,

$$\inf_{\mathcal{H}} J_{h_0}(u) \leq J_{h_0}(\overline{u}) = \Big(\int_M a(x)\,dv_{h_0}\Big)^{\frac{2-n}{n}}\ \frac{n-2}{4(n-1)}Rv_{h_0}\,.$$

So, by (2.3),

$$\inf_{\mathcal{H}} J_{h_0}(u) < \frac{1}{K(n,2)^2\big(\max_M a(x)\big)^{(n-2)/n}}\,.$$

By a classical result due to Aubin [2], it follows that

$$\frac{4(n-1)}{n-2}\Delta_{h_0}v + Rv = a(x)v^{2^*-1}$$

admits a positive solution $v_0 \in C^\infty(M)$, where $\Delta_{h_0}u = -\mathrm{div}_{h_0}(\nabla_{h_0}u)$ stands for the Laplacian on the metric h_0. Setting $h = v_0^{2^*-2}h_0$, one has $\mathrm{Scal}_h = a$. By (2.1), there exist an infinitely many functions a satisfying (2.2). Therefore, by the above-described process, we may construct an infinitely conformal metrics h satisfying the conclusion of Theorem 1.1.

Existence of extremal functions for $(J^2_{h,opt})$. Proceeding, by contradiction, suppose that $(J^2_{h,opt})$ admits no extremal function. Then, by Djadli and Druet [3],

$$B_0(2,h) = \frac{n-2}{4(n-1)}K(n,2)^2\max_M a(x)\,.$$

Let $u_0 \in C^\infty(M)$ be a positive solution of the Yamabe problem

$$\Delta_h u + \frac{n-2}{4(n-1)}a(x)u = \mu_h u^{2^*-1}\,. \tag{2.4}$$

Clearly, $\|u_0\|_{L^{2^*}(M)} = 1$ and $\mu_h = \mu_g$, since h is conformal to g. Then

$$\frac{1}{\mu_g}\int_M |\nabla_h u_0|^2 dv_h + \frac{1}{\mu_g}\frac{n-2}{4(n-1)}\int_M a(x)u_0^2 dv_h = 1\,.$$

Using the inequalities

$$0 < \max_M a(x) < \frac{1}{K(n,2)^2\mu_g}\min_M a(x)$$

and

$$0 < \mu_g < \frac{1}{K(n,2)^2}$$

on the left hand-side above, one obtains

$$K(n,2)^2\int_M |\nabla_h u_0|^2 dv_h + \frac{n-2}{4(n-1)}K(n,2)^2\max_M a(x)\int_M u_0^2 dv_h < 1\,.$$

But, this contradicts the value of $B_0(2,h)$ given above.

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