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PROBLEMS WITHOUT INITIAL CONDITIONS FOR DEGENERATE IMPLICIT EVOLUTION EQUATIONS

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Abstract. We study some sufficient conditions for the existence and uniqueness of a solution to a problem without initial conditions for degenerate implicit evolution equations. We also establish a condition of Bohr's and Stepanov's almost periodicity of solutions for this problem.

1. Introduction

Problems for an implicit evolution equation of the form

$$\big(\mathcal{B}u(t)\big)' + \mathcal{A}\big(t, u(t)\big) = f(t), \quad t \in S, \tag{1.1}$$

where $\mathcal{A}(t,\cdot)$ and $\mathcal{B}$ are operators from a Banach space V to its dual V', S is an interval in $\mathbb{R}$, sometimes known as Sobolev equation (see, e.g., [1, 11]), has been studied extensively by many authors. See, for example, [1]-[14] and references therein. Note that in the case where $\mathcal{B}$ is linear and $\mathcal{A}$ is linear or nonlinear, the monographs by Showalter [12, 14] give many sufficient conditions to existence and uniqueness of solutions of the Cauchy problem for equation (1.1).

More recently in the papers [6, 7] the Cauchy problem for the inclusion of the form (1.1) was considered as $\mathcal{A}$ may be set-valued. The existence of almost periodic solutions of abstract differential equations of the type (1.1) (when $\mathcal{B} = I$) has been studied in several works; see for example [5, 8, 10, 15]. A problem without initial conditions for the equation of the form (1.1) (when $\mathcal{B} = I$ and $\mathcal{A}$ is almost linear) was investigated in [13, 14] in the class of integrable functions on $(-\infty, T)$, $T \in \mathbb{R}$. In [2] the similar problem was considered (when $\mathcal{B} = I$ and $\mathcal{A}$ is nonlinear) in the class of locally integrable functions on $(-\infty, T]$.

In this paper, we generalize the results of [2] and [10] for the case of degenerate implicit equation (1.1), that is, when $\mathcal{B}$ may vanish on non-zero vectors. We obtain sufficient conditions to existence (Theorems 3.3, 3.5) and uniqueness (Theorem 3.1) of solutions of a problem without initial conditions for (1.1) independent of an additional assumption on the behavior of the solution and data-in at $-\infty$. We also establish the existence of periodic (Theorem 3.8) and almost periodic by Bohr and Stepanov (Theorem 3.13) solutions of (1.1).

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We shall introduce here some of the notions that we shall use hereafter. We denote by $\|\cdot\|_X$ the norm (seminorm) of the norm (seminorm) space X and by $(\cdot,\cdot)_Y$ the scalar product in the Hilbert space Y. By X' we denote the dual space of X. The duality pairing between X and X' is denoted by $\langle\cdot,\cdot\rangle_X$. By $L^q_{\rm loc}(S;X)$, where $q\in[1,+\infty)$ and S is an unbounded connected subset of $\mathbb{R}$, we denote the space of (equivalence classes of) measurable functions in S, with values in X such that its restrictions on any compact $K\subset S$ belong to $L^q(K;X)$. We denote by $\mathscr{D}'(S;X)$ the space of X_w valued distributions on $\operatorname{int}S$, which we regard extended on all S by zero. It is known that the space $L^q_{\rm loc}(S;X)$ can be identified with some subspace of $\mathscr{D}'(S;X)$. For $v\in L^q_{\rm loc}(S;X)$, we denote by v' the derivative in the sense of $\mathscr{D}'(S;X)$ [4]. Throughout the paper the symbol $\hookrightarrow$ means a continuous imbedding.

Our paper is organized as follows. Section 2 is devoted to some preliminary facts needed in the sequel. In Section 3 we state a problem and formulate main results. We prove our main results in Section 4. The last section is devoted to a simple example of applications of our results.

2. Preliminary results

Let V be a separable reflexive Banach space. Assume that $\mathcal{B}:V\to V'$ is a linear, continuous, symmetric (i.e., $\langle\mathcal{B}v_1,v_2\rangle_V=\langle\mathcal{B}v_2,v_1\rangle_V\quad\forall\,v_1,\,v_2\in V$) and monotone (i.e., $\langle\mathcal{B}v,v\rangle_V\geqslant 0\quad\forall\,v\in V$) operator. Then $\langle\mathcal{B}\cdot,\cdot\rangle_V$ is a semiscalar product and $\|\cdot\|_{V_\mathcal{B}}:=\langle\mathcal{B}\cdot,\cdot\rangle_V^{1/2}$ is a seminorm on V. We denote the completion of the seminorm space $\{V\,,\,\|\cdot\|_{V_\mathcal{B}}\}$ by $V_\mathcal{B}$ and the dual Hilbert space by $V'_\mathcal{B}$. Note that $V\hookrightarrow V_\mathcal{B}$ is dense. By restriction of functionals we have $V'_\mathcal{B}\hookrightarrow V'$. The operator $\mathcal{B}$ has a unique continuous linear extension $\mathcal{B}:V_\mathcal{B}\to V'_\mathcal{B}$. The scalar product on $V'_\mathcal{B}$ satisfies

$$(w,\mathcal{B}v)_{V'_\mathcal{B}}=\langle w,v\rangle_V,\quad w\in V'_\mathcal{B},\quad v\in V.$$

Hence, taking $w=\mathcal{B}v$,

$$\|\mathcal{B}v\|_{V'_\mathcal{B}}=\|v\|_{V_\mathcal{B}},\quad v\in V_\mathcal{B}. \tag{2.1}$$

We define the norm on the range of $\mathcal{B}:V\to V'$ by

$$\|w\|_W:=\inf\{\|v\|_V:\ v\in V,\,\mathcal{B}v=w\},\quad w\in\operatorname{Rg}\mathcal{B}.$$

The normed linear space $W=\{\operatorname{Rg}\mathcal{B},\,\|\cdot\|_W\}$ is a reflexive Banach space. Note that $W\hookrightarrow V'_\mathcal{B}$. These results are due to the books by Showalter [12, 14].

Throughout the rest of this paper $S:=\mathbb{R}$ or $S:=(-\infty,T]$, where $T<+\infty$, unless the contrary is explicitly stated.

Lemma 2.1. *Let $v\in L^p_{\rm loc}(S;V)$, $(\mathcal{B}v)'\in L^{p'}_{\rm loc}(S;V')$, where $p\in[2;+\infty)$ and $p'=p/(p-1)$. Then $v\in C(S;V_\mathcal{B})$ and the function $\|v(\cdot)\|_{V_\mathcal{B}}$ is absolutely continuous on each closed subinterval of S. Furthermore,*

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{V_\mathcal{B}}=\big\langle\big(\mathcal{B}v(t)\big)',v(t)\big\rangle_V\quad\text{for a.e. } t\in S. \tag{2.2}$$

Proof. Let $t_1,\,t_2\in S$ be any numbers such that $t_1<t_2$. In view of the assumptions we have $v\in L^p(t_1,t_2;V)$ and $(\mathcal{B}v)'\in L^{p'}(t_1,t_2;V')$. With the same proof as that of [14, Proposition 1.2, p. 106] we obtain $v\in C\big([t_1,t_2];V_\mathcal{B}\big)$, the function $t\mapsto\|v(t)\|_{V_\mathcal{B}}$ is absolutely continuous on $[t_1,t_2]$ and (2.2) holds for a.e. $t\in[t_1,t_2]$. Since $t_1,\,t_2\in S$ are arbitrary, the conclusion of Lemma 2.1 follows. $\square$

Lemma 2.2. *Let $1 < p < +\infty$. Assume that the inclusion $V \hookrightarrow V_{\mathcal{B}}$ is compact and define*
$$U_p := \{u \in L^p_{\rm loc}(S;V) : \ (\mathcal{B}v)' \in L^{p'}_{\rm loc}(S;V')\}.$$
Then the imbedding $U_p \hookrightarrow L^p_{\rm loc}(S;V_{\mathcal{B}})$ is compact.

Proof. Let us first prove that $W \hookrightarrow V'_{\mathcal{B}}$ is compact. To do this, assume that $\{w_n\}_{n=1}^{+\infty} \subset W$ is any bounded sequence. The definition of the space W implies for each $n \in \mathbb{N}$ the existence of $v_n \in V$ such that $w_n = \mathcal{B}v_n$ and $\|v_n\|_V < \|w_n\|_W + 1$. Since $\{w_n\}_{n=1}^{+\infty}$ is bounded in W, it follows that $\{v_n\}_{n=1}^{+\infty}$ is bounded in V. Then, the compactness of the imbedding $V \hookrightarrow V_{\mathcal{B}}$ implies the existence of a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$ which is strongly convergent in the space $V_{\mathcal{B}}$. Since the operator $\mathcal{B} : V_{\mathcal{B}} \to V'_{\mathcal{B}}$ is continuous, it follows that $\{\mathcal{B}v_{n_k}\}_{k=1}^{+\infty}$ is strongly convergent in $V'_{\mathcal{B}}$. But $w_{n_k} = \mathcal{B}v_{n_k}$, $k \in \mathbb{N}$. Thus the sequence $\{w_{n_k}\}_{k=1}^{+\infty}$ is strongly convergent in $V'_{\mathcal{B}}$. Hence the imbedding $W \hookrightarrow V'_{\mathcal{B}}$ is compact.

Now we show the compactness of the imbedding $U_p \hookrightarrow L^p_{\rm loc}(S;V_{\mathcal{B}})$. Let $\{u_n\}_{n=1}^{+\infty}$ be any bounded sequence in U_p; that is, for every t_1, $t_2 \in S$, $t_1 < t_2$, the sequences of restrictions to (t_1,t_2) of the elements of $\{u_n\}_{n=1}^{+\infty}$ and $\{(\mathcal{B}u_n)'\}_{n=1}^{+\infty}$ are bounded sequences in $L^p(t_1,t_2;V)$ and $L^{p'}(t_1,t_2;V')$ respectively. Let t_1, $t_2 \in S$ with $t_1 < t_2$. Since the operator $\mathcal{B} : V \to W$ is linear and continuous, we have that $\mathcal{B} : L^p(t_1,t_2;V) \to L^p(t_1,t_2;W)$ is also linear and continuous (see, e.g., [14]). Thereby, the sequence $\{\mathcal{B}u_n\}_{n=1}^{+\infty}$ is bounded in $L^p(t_1,t_2;W)$. The compactness of the imbedding $W \hookrightarrow V'_{\mathcal{B}}$, and Lions-Aubin's theorem (see, e.g., [9] or [14, p. 106]), imply the existence of a subsequence $\{\mathcal{B}u_{n_k}\}_{k=1}^{+\infty}$ of $\{\mathcal{B}u_n\}_{n=1}^{+\infty}$, which is strongly convergent in $L^p(t_1,t_2;V'_{\mathcal{B}})$. From (2.1) it follows that $\{u_{n_k}\}_{k=1}^{+\infty}$ is strongly convergent in $L^p(t_1,t_2;V_{\mathcal{B}})$. Thus Lemma 2.2 is proved. $\square$

Lemma 2.3 ([2, Lemma 1.1])**.** *Let z be a nonnegative absolutely continuous function on each closed subinterval of S and*
$$z'(t) + \beta(t)\chi\big(z(t)\big) \leqslant 0 \quad \textit{for a.e. } t \in S,$$
where $\beta \in L^1_{\rm loc}(S)$, $\beta(t) \geqslant 0$ for a.e. $t \in S$, $\int_{-\infty} \beta(t)\,dt = +\infty$, $\chi \in C\big([0,+\infty)\big)$, $\chi(0) = 0$, $\chi(\tau) > 0$ for $\tau > 0$ and $\int^{+\infty} \frac{d\tau}{\chi(\tau)} < +\infty$. Then $z(\cdot) \equiv 0$.

Lemma 2.4 ([3], p. 60)**.** *Let $y \in C(S)$, $z \in L^1_{\rm loc}(S)$ be such that*
$$y(t_2) - y(t_1) + \int_{t_1}^{t_2} z(t)\,dt \leqslant 0$$
for any t_1, $t_2 \in S$. Then
$$y(t_2)\theta(t_2) - y(t_1)\theta(t_1) - \int_{t_1}^{t_2} y(t)\theta'(t)\,dt + \int_{t_1}^{t_2} z(t)\theta(t)\,dt \leqslant 0$$
for any $\theta \in C^1(S)$ and t_1, $t_2 \in S$.

3. Statement of the problem and main results

Throughout this section S, V, $V_{\mathcal{B}}$ and $\mathcal{B}$ are the same as in Section 2 and $p \in (1,+\infty)$. Assume that a family of operators $\mathcal{A}(t,\cdot) : V \to V'$, $t \in S$, is given such that

(i) for each measurable function $v : S \to V$ the function $w(\cdot) = \mathcal{A}\big(\cdot, v(\cdot)\big)$ is measurable on S;
(ii) $\mathcal{A}\big(\cdot, v(\cdot)\big) \in L^{p'}_{\rm loc}(S; V')$ whenever $v \in L^p_{\rm loc}(S; V)$, where $p' = p/(p-1)$.

Consider the problem: for every $f \in L^{p'}_{\rm loc}(S; V')$, find a function u in $L^p_{\rm loc}(S; V) \cap C(S; V_{\mathcal{B}})$ such that

$$\big(\mathcal{B}u(t)\big)' + \mathcal{A}\big(t, u(t)\big) = f(t) \quad \text{in } \mathscr{D}'(S; V'). \tag{3.1}$$

We call this problem a *Problem without initial conditions for degenerate implicit evolution equation* (3.1) or Problem (3.1) for short.

Theorem 3.1 (Uniqueness). *Assume that $p > 2$ and*

(iii) *for a.e. $t \in S$ and each v, $w \in V$, $v \neq w$,*

$$\langle \mathcal{A}(t, v) - \mathcal{A}(t, w), v - w\rangle_V > \gamma(t)\varphi\big(\|v - w\|^2_{V_{\mathcal{B}}}\big),$$

where $\gamma \in L^1_{\rm loc}(S)$, $\gamma(t) \geqslant 0$ for a.e. $t \in S$, $\int_{-\infty}^{a} \gamma(\tau)\, d\tau = +\infty$ for some $a \in S$, $\varphi \in C\big([0, +\infty)\big)$, $\varphi(0) = 0$, $\varphi(\tau) > 0$ for $\tau > 0$ and $\int_1^{+\infty} \frac{d\tau}{\varphi(\tau)} < +\infty$.

Then there is at most one solution of Problem (3.1)*.*

Remark 3.2. *Clearly, conditions of Theorem 3.1 are satisfied by the functions $\gamma(t) \equiv \gamma_0$, $t \in S$, and $\varphi(\tau) = \tau^{\mu}$, $\tau \geqslant 0$, where $\gamma_0 > 0$ and $\mu > 1$ are some constants.*

Theorem 3.3 (Existence). *Let $p > 2$ and suppose the embedding $V \hookrightarrow V_{\mathcal{B}}$ is compact. Assume that*

(iv) *there exist $\alpha_1 \in L^{\infty}_{\rm loc}(S)$ and $\alpha_2 \in L^{p'}_{\rm loc}(S)$, $p' = p/(p-1)$, such that*

$$\|\mathcal{A}(t, v)\|_{V'} \leqslant \alpha_1(t)\|v\|_V^{p-1} + \alpha_2(t), \quad v \in V, \text{ a.e. } t \in S;$$

(v) *$\langle \mathcal{A}(t, v_1) - \mathcal{A}(t, v_2), v_1 - v_2\rangle_V \geqslant 0$ for all $v_1, v_2 \in V$, a.e. $t \in S$;*
(vi) *there exist $\beta_1 \in L^{\infty}_{\rm loc}(S)$, $\operatorname{ess\,inf}_{t\in[a,b]} \beta_1(t) > 0$ for any $[a, b] \subset S$, and $\beta_2 \in L^1_{\rm loc}(S)$ such that*

$$\langle \mathcal{A}(t, v), v\rangle_V \geqslant \beta_1(t)\|v\|^p_V - \beta_2(t), \quad v \in V, \text{a.e. } t \in S;$$

(vii) *for almost every $t \in S$ and every vectors v_1, $v_2 \in V$ the real-valued function $s \mapsto \langle \mathcal{A}(t, v_1 + sv_2), v_2\rangle_V$ is continuous on $\mathbb{R}$.*

Then Problem (3.1) *has at least one solution and each its solution for any numbers t_1, $t_2 \in S$ $(t_1 < t_2)$, $\delta > 0$, satisfies the estimate*

$$\begin{aligned}
&\max_{t\in[t_1,t_2]} \|u(t)\|^2_{V_{\mathcal{B}}} + \overline{\beta}(t_1 - \delta, t_2) \int_{t_1}^{t_2} \|u(t)\|^p_V\, dt \\
&\leqslant C_1\big(\delta \cdot \overline{\beta}(t_1 - \delta, t_2)\big)^{\frac{2}{2-p}} + C_2\big(\overline{\beta}(t_1 - \delta, t_2)\big)^{\frac{1}{1-p}} \int_{t_1-\delta}^{t_2} \|f(t)\|^{p'}_{V'}\, dt \\
&\quad + 2\int_{t_1-\delta}^{t_2} \beta_2(t)\, dt,
\end{aligned} \tag{3.2}$$

where $\overline{\beta}(t_1 - \delta, t_2) = \operatorname{ess\,inf}_{t\in[t_1-\delta,t_2]} \beta_1(t)$, C_1, C_2 are positive constants depending only on $\mathcal{B}$ and p.

Remark 3.4. The family of operators $\mathcal{A}(t,\cdot)$ satisfies condition (i) in the context of conditions (v) and (vii) if we assume that the function $w(\cdot)=\mathcal{A}\big(\cdot,v\big)$ is weakly measurable on S for each $v\in V$ (see, e.g., [4, 14]). Condition (ii) is an immediate consequence of conditions (i) and (iv).

Theorem 3.5 (Existence and uniqueness)**.** *Assume that $p>2$ and the family of operators $\mathcal{A}(t,\cdot):V\to V'$, $t\in S$, satisfies conditions* (iv), (vi), (vii) *and*

(viii) *there exists $K_1>0$ such that for each v, $w\in V$, $v\neq w$,*
$$\langle\mathcal{A}(t,v)-\mathcal{A}(t,w),v-w\rangle_V>K_1\|v-w\|^q_{V_{\mathcal{B}}},\quad \text{a.e. } t\in S,$$
where $q\in(2;p]$ is some number.

Then there exists a unique solution of Problem (3.1)*. Moreover, if u is a solution of Problem* (3.1)*, then for any numbers t_1, $t_2\in S$ $(t_1<t_2)$ and $\delta>0$ we have the estimate*
$$\begin{aligned}
&\max_{t\in[t_1,t_2]}\|u(t)\|^2_{V_{\mathcal{B}}}+\int_{t_1}^{t_2}\beta_1(t)\|u(t)\|^p_V\,dt\\
&\leqslant C_3\big(\delta\cdot K_1\big)^{\frac{2}{2-q}}+C_4\int_{t_1-\delta}^{t_2}\beta_1^{\frac{1}{1-p}}(t)\left(\|f(t)\|^{p'}_{V'}+\|\mathcal{A}(t,0)\|^{p'}_{V'}\right)dt \\
&\quad+2\int_{t_1-\delta}^{t_2}\beta_2(t)\,dt,
\end{aligned}\tag{3.3}$$
where C_3, C_4 are some positive constants depending only on $\mathcal{B}$ and p.

Remark 3.6. Clearly condition (viii) is satisfied in the context of the condition

(ix) there exists $K_2>0$ such that for every v, $w\in V$,
$$\langle\mathcal{A}(t,v)-\mathcal{A}(t,w),v-w\rangle_V\geqslant K_2\|v-w\|^p_V,\quad \text{a.e. } t\in S.$$

Corollary 3.7. *Let $S=\mathbb{R}$. Suppose that the hypotheses of Theorem 3.5 hold and there exists a constant $C_5\geqslant 0$ such that*
$$\sup_{\tau\in\mathbb{R}}\int_\tau^{\tau+1}\Big(\beta_1^{\frac{1}{1-p}}(t)\big(\|f(t)\|^{p'}_{V'}+\|\mathcal{A}(t,0)\|^{p'}_{V'}\big)+\beta_2(t)\Big)\,dt\leqslant C_5.$$
Then the solution u for Problem (3.1) *satisfies*
$$\sup_{\tau\in\mathbb{R}}\|u(\tau)\|_{V_{\mathcal{B}}}+\sup_{\tau\in\mathbb{R}}\int_\tau^{\tau+1}\beta_1(t)\|u(t)\|^p_V\,dt\leqslant C_6,\tag{3.4}$$
where $C_6\geqslant 0$ is a constant depending only on p, q, K_1 and C_5.

Theorem 3.8. *Let $S=\mathbb{R}$ and the assumptions of Theorem 3.5 hold. Suppose that there exists a number $\sigma>0$ such that $\mathcal{A}(t+\sigma,v)=\mathcal{A}(t,v)$ and $f(t+\sigma)=f(t)$ for any $v\in V$ and a.e. $t\in\mathbb{R}$. Then Problem* (3.1) *has a unique solution. Moreover, this solution is σ-periodic* (*that is, $u(t+\sigma)=u(t)$ for a.e. $t\in\mathbb{R}$*) *and satisfies the estimate*
$$\begin{aligned}
&\max_{t\in[0,\sigma]}\|u(t)\|^2_{V_{\mathcal{B}}}+\int_0^\sigma\|u(t)\|^p_V\,dt\\
&\leqslant C_7\max\Big\{\int_0^\sigma\big(\|f(t)\|^{p'}_{V'}+\beta_2(t)\big)\,dt,\ \Big(\int_0^\sigma\big(\|f(t)\|^{p'}_{V'}+\beta_2(t)\big)\,dt\Big)^{2/p}\Big\},
\end{aligned}\tag{3.5}$$
where C_7 is some positive constant depending only on p, σ, $\mathcal{B}$ and $\operatorname{ess\,inf}_{t\in[0,\sigma]}\beta_1(t)$.

Following [8] and [10] we recall some definitions.

Definition 3.9. A subset $Q \subset \mathbb{R}$ is called *relatively dense* if there exists $l > 0$ such that $[a, a+l] \cap Q \neq \varnothing$ for all $a \in \mathbb{R}$.

Let X be a complete seminorm space with the seminorm $\|\cdot\|_X$ or a complete metric space with the metric $d_X(\cdot,\cdot)$. By $BC(\mathbb{R}; X)$ we denote the space of all bounded continuous functions $g : \mathbb{R} \to X$. For any $g \in C(\mathbb{R}; X)$ and $\varepsilon > 0$ define

$$F_\varepsilon(g) := \big\{\sigma \in \mathbb{R} : \sup_{t\in\mathbb{R}} \|g(t+\sigma) - g(t)\|_X < \varepsilon\big\}$$

if X is the seminorm space, and

$$F_\varepsilon(g) := \big\{\sigma \in \mathbb{R} : \sup_{t\in\mathbb{R}} d_X\big(g(t+\sigma), g(t)\big) < \varepsilon\big\}$$

if X is the metric space.

Definition 3.10. A function $g \in C(\mathbb{R}; X)$ is said to be *Bohr almost periodic* if for any $\varepsilon > 0$ the set $F_\varepsilon(g)$ is relatively dense in $\mathbb{R}$.

Denote by $CAP(\mathbb{R}; X)$ the set of all Bohr almost periodic functions $\mathbb{R} \to X$. Note that $CAP(\mathbb{R}; X) \subset BC(\mathbb{R}; X)$.

Let $\{Y, \|\cdot\|_Y\}$ be a Banach space and $q \in [1, +\infty)$. The Banach space of Stepanov bounded on $\mathbb{R}$ functions, with the exponent q, is the space $BS^q(\mathbb{R}; Y)$ which consists of all functions $g \in L^q_{\rm loc}(\mathbb{R}; Y)$ having finite norm

$$\|g\|^q_{S^q} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g(t)\|^q_Y \, dt.$$

Definition 3.11. The *Bochner transform* $g^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $g(t)$, $t \in \mathbb{R}$, with values in Y, is defined by

$$g^b(t,s) := g(t+s).$$

Definition 3.12. A function $g \in L^q_{\rm loc}(\mathbb{R}; Y)$ is called a *Stepanov almost periodic function, with the exponent* q, if $g^b \in CAP\big(\mathbb{R}; L^q(0,1;Y)\big)$.

The space of all Stepanov almost periodic functions with values in Y is denoted by $S^q(\mathbb{R}; Y)$. Clearly the following inclusion holds $S^q(\mathbb{R}; Y) \subset BS^q(\mathbb{R}; Y)$.

Denote by $Y_{p,V}$ the space of all operators $A : V \to V'$ such that

$$\|A(v)\|_{V'} \leqslant C_A(\|v\|^{p-1}_V + 1) \quad \forall\, v \in V,$$

where $C_A > 0$ is some constant depending on A. The space $Y_{p,V}$ is a complete metric space with respect to the metric

$$d_{p,V}(A_1, A_2) := \sup_{v\in V} \frac{\|A_1(v) - A_2(v)\|_{V'}}{\|v\|^{p-1}_V + 1}, \quad A_1, A_2 \in Y_{p,V}.$$

Theorem 3.13. *Let $S = \mathbb{R}$ and $p > 2$. Assume that the family of operators $\mathcal{A}(t,\cdot) : V \to V'$, $t \in \mathbb{R}$, belongs to the space $CAP(\mathbb{R}; Y_{p,V})$, satisfies conditions* (iv), (vii), (ix) *and $f \in S^{p'}(\mathbb{R}; V')$. Then Problem* (3.1) *has a unique solution and this solution belongs to the space $CAP(\mathbb{R}; V_{\mathcal{B}}) \cap S^p(\mathbb{R}; V)$.*

4. Proof main results

We now turn to the proof of Theorems 3.1-3.13 and Corollary 3.7.

Proof of Theorem 3.1. Suppose that u_1 and u_2 are two solutions of Problem (3.1), and write $w := u_1 - u_2$. By taking the difference between (3.1) for $u = u_1$ and (3.1) for $u = u_2$ we get

$$\big(\mathcal{B}w(t)\big)' + \mathcal{A}\big(t, u_1(t)\big) - \mathcal{A}\big(t, u_2(t)\big) = 0 \quad \text{in} \quad \mathscr{D}'(S; V'). \tag{4.1}$$

This and condition (ii) give us $(\mathcal{B}w)' \in L^{p'}_{\rm loc}(S; V')$, so using Lemma 2.1 we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{V_\mathcal{B}} = \big\langle \big(\mathcal{B}w(t)\big)', w(t)\big\rangle_V \quad \text{for a.e. } t \in S. \tag{4.2}$$

Multiplying (4.1) by w we get

$$\big\langle \big(\mathcal{B}w(t)\big)', w(t)\big\rangle_V + \big\langle \mathcal{A}\big(t, u_1(t)\big) - \mathcal{A}\big(t, u_2(t)\big), u_1(t) - u_2(t)\big\rangle_V = 0 \tag{4.3}$$

for a.e. $t \in S$. From (4.2) and (4.3) we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{V_B} + \big\langle \mathcal{A}\big(t, u_1(t)\big) - \mathcal{A}\big(t, u_2(t)\big), u_1(t) - u_2(t)\big\rangle_V = 0 \quad \text{a.e. on } S. \tag{4.4}$$

From (4.4) and (iii) we have

$$\frac{1}{2}\frac{dy(t)}{dt} + \gamma(t)\varphi\big(y(t)\big) \leqslant 0 \quad \text{for a.e. } t \in S, \tag{4.5}$$

where $y(t) = \|u_1(t) - u_2(t)\|^2_{V_\mathcal{B}}$. Further, from (4.5) we obtain $y \equiv 0$ on S by Lemma 2.3. This and (4.4) imply

$$\big\langle \mathcal{A}\big(t, u_1(t)\big) - \mathcal{A}\big(t, u_2(t)\big), u_1(t) - u_2(t)\big\rangle_V = 0 \quad \text{a.e. on } S. \tag{4.6}$$

From (4.6) and (iii) we get $u_1(t) = u_2(t)$ for a.e. $t \in S$. Theorem 3.1 is proved. $\square$

Proof of Theorem 3.3. First we obtain a priori estimate (3.2) for any solution of Problem (3.1). Let u be a solution of Problem (3.1). Hence, using Lemma 2.1, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{V_\mathcal{B}} = \big\langle \big(\mathcal{B}u(t)\big)', u(t)\big\rangle_V \tag{4.7}$$

for a.e. $t \in S$. Take $\theta_1 \in C^1(\mathbb{R})$ with the following properties: $\theta_1(t) = 0$ if $t \in (-\infty, -1]$, $\theta_1(t) = \exp(\frac{t^2}{t^2-1})$ if $t \in (-1, 0)$, $\theta_1(t) = 1$ if $t \in [0, +\infty)$. It is clear that

$$\sup_{t\in(-1,+\infty)} \frac{\theta_1'(t)}{\theta_1^\nu(t)} < C_8(\nu), \tag{4.8}$$

where $0 < \nu < 1$, $C_8(\nu) > 0$ is a constant depending only on ν.

Let t_1, $t_2 \in S$ $(t_1 < t_2)$, $\delta > 0$ be any numbers. We define the function $\theta(t) := \theta_1(\frac{t-t_1}{\delta})$ for each $t \in S$. It is clear that $\theta u \in L^p_{\rm loc}(S; V)$. Multiply equation (3.1) by θu and integrate from $t_1 - \delta$ to $\tau \in [t_1, t_2]$ with respect to t:

$$\begin{aligned} &\int_{t_1-\delta}^{\tau} \Big\{\theta(t)\big\langle \big(\mathcal{B}u(t)\big)', u(t)\big\rangle_V + \theta(t)\big\langle \mathcal{A}\big(t, u(t)\big), u(t)\big\rangle_V\Big\}\, dt \\ &= \int_{t_1-\delta}^{\tau} \theta(t)\langle f(t), u(t)\rangle_V \, dt. \end{aligned} \tag{4.9}$$

Substituting (4.7) into (4.9) yields

$$\begin{aligned}&\int_{t_1-\delta}^{\tau}\theta(t)\frac{d}{dt}\|u(t)\|^2_{V_{\mathcal{B}}}\,dt+2\int_{t_1-\delta}^{\tau}\theta(t)\big\langle\mathcal{A}\big(t,u(t)\big),u(t)\big\rangle_V\,dt\\&=2\int_{t_1-\delta}^{\tau}\theta(t)\langle f(t),u(t)\rangle_V\,dt.\end{aligned}\tag{4.10}$$

Integrating by parts the first term of the left hand side of equality (4.10) we obtain

$$\begin{aligned}&\|u(\tau)\|^2_{V_{\mathcal{B}}}+2\int_{t_1-\delta}^{\tau}\theta(t)\big\langle\mathcal{A}\big(t,u(t)\big),u(t)\big\rangle_V\,dt\\&=\int_{t_1-\delta}^{\tau}\theta'(t)\|u(t)\|^2_{V_{\mathcal{B}}}\,dt+2\int_{t_1-\delta}^{\tau}\theta(t)\langle f(t),u(t)\rangle_V\,dt.\end{aligned}\tag{4.11}$$

Let us estimate the first term of the right hand side of (4.11) using (4.8), the continuity of the imbedding V in $V_{\mathcal{B}}$ and Young's inequality:

$$\begin{aligned}\int_{t_1-\delta}^{\tau}\theta'(t)\|u(t)\|^2_{V_{\mathcal{B}}}\,dt&\leqslant C_9\int_{t_1-\delta}^{\tau}\theta'(t)\|u(t)\|^2_V\,dt\\&\leqslant C_9\int_{t_1-\delta}^{\tau}\frac{\theta'(t)}{\theta^{2/p}(t)}\theta^{2/p}(t)\|u(t)\|^2_V\,dt\\&\leqslant\varepsilon\int_{t_1-\delta}^{\tau}\theta(t)\|u(t)\|^p_V\,dt\\&\quad+C_{10}\varepsilon^{-\frac{p}{p-2}}\int_{t_1-\delta}^{t_2}\big(\theta'(t)\theta^{-2/p}(t)\big)^{\frac{p}{p-2}}\,dt\\&\leqslant\varepsilon\int_{t_1-\delta}^{\tau}\theta(t)\|u(t)\|^p_V\,dt+C_{11}(\delta\cdot\varepsilon)^{-\frac{p}{p-2}},\end{aligned}\tag{4.12}$$

where $\varepsilon>0$ is any number, C_9, C_{10} C_{11} are positive constants depending only on p and $\mathcal{B}$.

Now we estimate the second term of the right hand side of (4.11) using Young's inequality

$$\begin{aligned}&2\int_{t_1-\delta}^{\tau}\theta(t)\langle f(t),u(t)\rangle_V\,dt\\&\leqslant\eta\int_{t_1-\delta}^{\tau}\theta(t)\|u(t)\|^p_V\,dt+C_{12}\eta^{\frac{1}{1-p}}\int_{t_1-\delta}^{\tau}\theta(t)\|f(t)\|^{p'}_{V'}\,dt,\end{aligned}\tag{4.13}$$

where $\eta>0$ is any number and $C_{12}>0$ is a constant depending only on p. Next let us estimate the second term of the left hand side of (4.11) using (vi)

$$\begin{aligned}2\int_{t_1-\delta}^{\tau}\theta(t)\big\langle\mathcal{A}\big(t,u(t)\big),u(t)\big\rangle_V\,dt&\geqslant2\int_{t_1-\delta}^{\tau}\theta(t)\beta_1(t)\|u(t)\|^p_V\,dt-2\int_{t_1-\delta}^{\tau}\theta(t)\beta_2(t)\,dt\\&\geqslant2\overline{\beta}(t_1-\delta,\tau)\int_{t_1-\delta}^{\tau}\theta(t)\|u(t)\|^p_V\,dt\\&\quad-2\int_{t_1-\delta}^{\tau}\theta(t)\beta_2(t)\,dt.\end{aligned}\tag{4.14}$$

From (4.11), using (4.12)-(4.14) and taking $\varepsilon = \eta = \frac{1}{2}\overline{\beta}(t_1-\delta,\tau)$, we get

$$\begin{aligned} &\|u(\tau)\|^2_{V_{\mathcal{B}}} + \overline{\beta}(t_1-\delta,\tau)\int_{t_1-\delta}^{\tau} \theta(t)\|u(t)\|^p_V\,dt \\ &\leqslant C_{13}\big(\delta\cdot\overline{\beta}(t_1-\delta,\tau)\big)^{\frac{2}{2-p}} + C_{14}\big(\overline{\beta}(t_1-\delta,\tau)\big)^{\frac{1}{1-p}}\int_{t_1-\delta}^{\tau}\theta(t)\|f(t)\|^{p'}_{V'}\,dt \\ &\quad + 2\int_{t_1-\delta}^{\tau}\theta(t)\beta_2(t)\,dt, \end{aligned} \tag{4.15}$$

where $\delta > 0$ is any number, C_{13} and C_{14} are some positive constants depending only on $\mathcal{B}$ and p. Since $\tau \in [t_1, t_2]$ is arbitrary, we see that (4.15) implies (3.2).

Second, we construct a sequence of functions approximating a solution for Problem (3.1). We assume without loss of generality that $T > 0$ if $S = (-\infty, T]$. Define $S_k := S \cap \{t \in \mathbb{R} :\ t \geqslant -k\}$, $k \in \mathbb{N}$. Let us for each $k \in \mathbb{N}$ consider the problem of finding $\hat{u}_k \in L^p_{\rm loc}(S_k; V)$, $\mathcal{B}\hat{u}_k \in C(S_k; V'_{\mathcal{B}})$ such that

$$\big(\mathcal{B}\hat{u}_k(t)\big)' + \mathcal{A}\big(t, \hat{u}_k(t)\big) = f(t) \quad \text{in } \mathscr{D}'(S_k; V') \tag{4.16a}$$

$$\lim_{t\to -k}\mathcal{B}\hat{u}_k(t) = 0 \qquad \text{in } V'_{\mathcal{B}}. \tag{4.16b}$$

The existence and uniqueness of a solution $\hat{u}_k$ of problem (4.16) follow from results of [14, Corollary III.6.3]. Let us extend $\hat{u}_k$ to $(-\infty, -k]$ by zero and denote this extension by u_k. It is clear that u_k is a solution of the problem without initial conditions

$$\big(\mathcal{B}u_k(t)\big)' + \mathcal{A}\big(t, u_k(t)\big) = f_k(t) \quad \text{in} \quad \mathscr{D}'(S; V'), \tag{4.17}$$

where $f_k(t) = f(t)$ on S_k and $f_k(t) = \mathcal{A}(t, 0)$ on $(-\infty, -k]$.

For each $k \in \mathbb{N}$ the solution of problem (4.17) satisfies estimate (3.2), where f is replaced by f_k. Thus from this estimate and the definition of f_k we get

$$\int_{t_1}^{t_2} \|u_k(t)\|^p_V\,dt \leqslant C_{15}(t_1, t_2) \tag{4.18}$$

for any numbers t_1, $t_2 \in S$, where $C_{15}(t_1, t_2) > 0$ is a constant dependent on t_1 and t_2 but independent on k. From this estimate and (iv) we obtain

$$\int_{t_1}^{t_2} \big\|\mathcal{A}\big(t, u_k(t)\big)\big\|^{p'}_{V'}\,dt \leqslant C_{16}(t_1, t_2), \tag{4.19}$$

where $C_{16}(t_1, t_2) > 0$ is a constant independent on k. From estimates (4.18) and (4.19) (see, e.g., [9, 14]) the existence of the subsequence of $\big\{u_k\big\}_{k=1}^{+\infty}$ follows, which we hereafter denote by $\big\{u_k\big\}_{k=1}^{+\infty}$, such that

$$u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} u(\cdot) \quad \text{weakly in } L^p_{\rm loc}(S; V), \tag{4.20}$$

$$\mathcal{A}\big(\cdot, u_k(\cdot)\big) \overset{k\to+\infty}{\longrightarrow} \chi(\cdot) \quad \text{weakly in } L^{p'}_{\rm loc}(S; V'). \tag{4.21}$$

Since the operator $\mathcal{B} : V \to V'$ is linear and continuous, it follows that its realization $\mathcal{B} : L^p_{\rm loc}(S; V) \to L^p_{\rm loc}(S; V')$ is also linear and continuous, and hence weakly continuous. From this and (4.20) we have

$$\mathcal{B}u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} \mathcal{B}u(\cdot) \quad \text{weakly in } L^p_{\rm loc}(S; V'). \tag{4.22}$$

Finally we show that u is a solution for Problem (3.1). To see this, let us pass to the limit as $k \to +\infty$ in (4.17) and use (4.21), (4.22):

$$\big(\mathcal{B}u(t)\big)' + \chi(t) = f(t) \quad \text{in} \quad \mathscr{D}'(S; V'). \tag{4.23}$$

From (4.23) we have $(\mathcal{B}u)' \in L^{p'}_{\rm loc}(S; V')$, so by Lemma 2.1 we get $u \in C(S; V_B)$. It remains to prove only that

$$\chi(t) = \mathcal{A}\big(t, u(t)\big) \quad \text{in } V' \text{ for a.e. } t \in S. \tag{4.24}$$

We will establish (4.24) using the monotonicity method of Browder and Minty.

Let us define

$$E_k = \int_S \psi(t)\big\langle \mathcal{A}\big(t, u_k(t)\big) - \mathcal{A}\big(t, v(t)\big), u_k(t) - v(t)\big\rangle_V \, dt, \quad k \in \mathbb{N},$$

for any $\psi \geqslant 0$ from $\mathscr{D}(S)$ and v from $L^p_{\rm loc}(S; V)$. From (v) it follows that $E_k \geqslant 0$.

Multiplying (4.17) by ψu_k, $k \in \mathbb{N}$, and integrating over S with respect to t, we obtain

$$\begin{aligned} &\int_S \Big\{ \psi(t)\big\langle \big(\mathcal{B}u_k(t)\big)', u_k(t)\big\rangle_V + \psi(t)\big\langle \mathcal{A}\big(t, u_k(t)\big), u_k(t)\big\rangle_V \Big\} \, dt \\ &= \int_S \psi(t)\langle f_k(t), u_k(t)\rangle_V \, dt. \end{aligned} \tag{4.25}$$

Then from (4.25), using (4.7) where u is replaced by u_k and the definition of f_k, after integrating by parts, we have

$$\begin{aligned} &\int_S \psi(t)\big\langle \mathcal{A}\big(t, u_k(t)\big), u_k(t)\big\rangle_V \, dt \\ &= \frac{1}{2}\int_S \psi'(t)\|u_k(t)\|^2_{V_\mathcal{B}} \, dt + \int_S \psi(t)\langle f(t), u_k(t)\rangle_V \, dt. \end{aligned} \tag{4.26}$$

Let t_1, t_2 be any real numbers such that $\operatorname{supp}\psi' \subset [t_1, t_2] \subset S$. From (4.20) we obtain

$$u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} u(\cdot) \quad \text{weakly in } L^p(t_1, t_2; V).$$

Hence, using the compactness of the imbedding $V \hookrightarrow V_\mathcal{B}$ and Lemma 2.2, by dropping to a subsequence and reindexing, we get

$$u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} u(\cdot) \quad \text{strongly in } L^p(t_1, t_2; V_\mathcal{B}).$$

This and $p > 2$ imply

$$u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} u(\cdot) \quad \text{strongly in } L^2(t_1, t_2; V_\mathcal{B}). \tag{4.27}$$

From (4.27) we have

$$\int_S \psi'(t)\|u_k(t)\|^2_{V_\mathcal{B}} \, dt \overset{k\to+\infty}{\longrightarrow} \int_S \psi'(t)\|u(t)\|^2_{V_\mathcal{B}} \, dt. \tag{4.28}$$

Passing to the limit as $k \to +\infty$ in (4.26) and using (4.20), (4.28), we obtain

$$\begin{aligned} &\int_S \psi(t)\big\langle \mathcal{A}\big(t, u_k(t)\big), u_k(t)\big\rangle_V \, dt \\ &\overset{k\to+\infty}{\longrightarrow} \frac{1}{2}\int_S \psi'(t)\|u(t)\|^2_{V_\mathcal{B}} \, dt + \int_S \psi(t)\langle f(t), u(t)\rangle_V \, dt. \end{aligned} \tag{4.29}$$

Now multiply equality (4.23) by ψu_k and integrate over S with respect to t. We get

$$\int_S \psi(t)\langle\chi(t),u(t)\rangle_V = \frac{1}{2}\int_S \psi'(t)\|u(t)\|^2_{V_{\mathcal{B}}}\,dt + \int_S \psi(t)\langle f(t),u(t)\rangle_V\,dt. \tag{4.30}$$

From (4.29) and (4.30) we have

$$\int_S \psi(t)\big\langle \mathcal{A}\big(t,u_k(t)\big),u_k(t)\big\rangle_V\,dt \overset{k\to+\infty}{\longrightarrow} \int_S \psi(t)\langle\chi(t),u(t)\rangle_V\,dt. \tag{4.31}$$

Using (4.20), (4.21) and (4.31), we deduce

$$0 \leqslant \lim_{k\to\infty} E_k = \int_S \psi(t)\big\langle \chi(t)-\mathcal{A}\big(t,v(t)\big),u(t)-v(t)\big\rangle_V\,dt. \tag{4.32}$$

Setting $v=u-sw$ in (4.32), where $s>0$ and $w\in L^p_{\rm loc}(S;V)$ is any function, we obtain

$$\int_S \psi(t)\big\langle \chi(t)-\mathcal{A}\big(t,u(t)-sw(t)\big),w(t)\big\rangle_V\,dt \geqslant 0. \tag{4.33}$$

Passing to limit as $s\to 0$ in (4.33) and using (vii), we get

$$\int_S \psi(t)\big\langle \chi(t)-\mathcal{A}\big(t,u(t)\big),w(t)\big\rangle_V\,dt \geqslant 0. \tag{4.34}$$

Since $\psi\geqslant 0$ and w are arbitrary functions from $\mathscr{D}(S)$ and $L^p_{\rm loc}(S;V)$ respectively, we deduce from (4.34) equality (4.24), as desired. This completes the proof. $\square$

Proof of Theorem 3.5. The uniqueness of a solution for Problem (3.1) follows directly from condition (viii) and Theorem 3.1 by taking $\gamma(t)\equiv K_1$, $t\in S$, $\varphi(\tau)=\tau^{q/2}$, $\tau\in[0,+\infty)$ (see Remark 3.2).

Estimate (3.3) follows from (4.11) in the same manner as we establish (3.2) by using (4.12), where p and $\|\cdot\|_V$ are replaced by q and $\|\cdot\|_{V_{\mathcal{B}}}$ respectively, (4.13), (4.14) and

$$\begin{aligned}&\int_{t_1-\delta}^{\tau}\theta(t)\big\langle\mathcal{A}\big(t,u(t)\big),u(t)\big\rangle_V\,dt\\ &\geqslant K_1\int_{t_1-\delta}^{\tau}\theta(t)\|u(t)\|^q_{V_{\mathcal{B}}}\,dt+\int_{t_1-\delta}^{\tau}\theta(t)\big\langle\mathcal{A}(t,0),u(t)\big\rangle_V\,dt.\end{aligned}$$

The last inequality is an immediate consequence of (viii).

Now we prove the existence of a solution for Problem (3.1). By the same argument used in the proof of Theorem 3.3 it is sufficient to show that the sequence $\{u_k\}_{k=1}^{+\infty}$, where u_k ($k\in\mathbb{N}$) is a solution of problem (4.17), satisfies

$$u_k(\cdot) \overset{k\to+\infty}{\longrightarrow} u(\cdot) \quad \text{strongly in } L^p(t_1,t_2;V_{\mathcal{B}}) \tag{4.35}$$

for any t_1, $t_2\in S$. Multiplying (4.17) by v, where $v\in L^p_{\rm loc}(S;V)$ is any function, and integrating from t_1 to t_2 with respect to t, where t_1, $t_2\in S$, ($t_1<t_2$) are any numbers, we obtain

$$\int_{t_1}^{t_2}\big\langle\big(\mathcal{B}u_k(t)\big)',v(t)\big\rangle_V dt+\int_{t_1}^{t_2}\big\langle\mathcal{A}\big(t,u_k(t)\big),v(t)\big\rangle_V dt=\int_{t_1}^{t_2}\langle f_k(t),v(t)\rangle_V dt. \tag{4.36}$$

Let l, $m \in \mathbb{N}$ be any numbers. Taking the difference between (4.36) for $k = l$ and (4.36) for $k = m$, and then setting $v = u_l - u_m$, we get

$$\begin{aligned} &\int_{t_1}^{t_2} \big\langle \big(\mathcal{B}w_{lm}(t)\big)', w_{lm}(t)\big\rangle_V \,dt + \int_{t_1}^{t_2} \big\langle \mathcal{A}\big(t, u_l(t)\big) - \mathcal{A}\big(t, u_m(t)\big), w_{lm}(t)\big\rangle_V \,dt \\ &= \int_{t_1}^{t_2} \langle f_l(t) - f_m(t), w_{lm}(t)\rangle_V \,dt, \end{aligned} \tag{4.37}$$

where $w_{lm} := u_l - u_m$. Since $f_l(t) = f_m(t)$ for a.e. $t \in [t_1, t_2]$ whenever l, $m > -t_1$, it follows from (4.37), using Lemma 2.1 and condition (viii), that

$$\|w_{lm}(t_2)\|^2_{V_{\mathcal{B}}} - \|w_{lm}(t_1)\|^2_{V_{\mathcal{B}}} + 2K_1 \int_{t_1}^{t_2} \|w_{lm}(t)\|^q_{V_{\mathcal{B}}} \,dt \leqslant 0.$$

From here and Lemma 2.4 in the same manner as was obtained (3.2) we show that for any natural numbers l, $m > -t_1 + \delta$

$$\max_{t\in[t_1,t_2]} \|w_{lm}(t)\|^2_{V_{\mathcal{B}}} \equiv \max_{t\in[t_1,t_2]} \|u_l(t) - u_m(t)\|^2_{V_{\mathcal{B}}} \leqslant C_{17}\delta^{\frac{2}{2-q}}, \tag{4.38}$$

where $\delta > 0$ is any number, C_{17} is some positive constant depending only on K_1, $\mathcal{B}$ and p.

Thus from (4.38) it follows that $\{u_k\}_{k=1}^{+\infty}$ is a Cauchy sequence in $C\big([t_1, t_2]; V_{\mathcal{B}}\big)$, and therefore is a Cauchy sequence in $L^p(t_1, t_2; V_{\mathcal{B}})$. Consequently, we conclude from (4.20) and completeness of $L^p(t_1, t_2; V_{\mathcal{B}})$ that (4.35) holds, so the proof is complete. □

We remark that the Proof of Corollary 3.7 follows from estimate (3.3).

Proof of Theorem 3.8. Existence and uniqueness of a solution u for Problem (3.1) follows from Theorem 3.5. Note that the function $u(t+\sigma)$, $t \in \mathbb{R}$, is also a solution of this problem. The uniqueness of a solution for Problem (3.1) implies $u(t+\sigma) = u(t)$ for a.e. $t \in \mathbb{R}$. Thus a solution of Problem (3.1) is σ-periodic.

Now we prove estimate (3.5). Let u be a σ-periodic solution for Problem (3.1). Multiplying equation (3.1) by u, using (4.7) and integrating from $t_1 \in \mathbb{R}$ to $t_2 \in \mathbb{R}$ ($t_1 < t_2$) with respect to t, we obtain

$$\frac{1}{2}\int_{t_1}^{t_2} \frac{d}{dt}\|u(t)\|^2_{V_{\mathcal{B}}} \,dt + \int_{t_1}^{t_2} \big\langle \mathcal{A}\big(t, u(t)\big), u(t)\big\rangle_V \,dt = \int_{t_1}^{t_2} \langle f(t), u(t)\rangle_V \,dt. \tag{4.39}$$

From (4.39), using (vi) and Young's inequality for the right hand side of (4.39), we get

$$\begin{aligned} &\|u(t_2)\|^2_{V_{\mathcal{B}}} - \|u(t_1)\|^2_{V_{\mathcal{B}}} + \int_{t_1}^{t_2} \beta_1(t)\|u(t)\|^p_V \,dt \\ &\leqslant C_{18} \int_{t_1}^{t_2} \Big(\beta_1^{-\frac{1}{p-1}}(t)\, \|f(t)\|^{p'}_{V'} + \beta_2(t)\Big) \,dt, \end{aligned} \tag{4.40}$$

where $C_{18} > 0$ is a constant depending on p. Set $t_1 = 0$ and $t_2 = \sigma$. Since u is a σ-periodic, from (4.40) it follows that

$$\int_0^{\sigma} \|u(t)\|^p_V \,dt \leqslant C_{19} \int_0^{\sigma} \Big(\|f(t)\|^{p'}_{V'} + \beta_2(t)\Big) \,dt, \tag{4.41}$$

where $C_{19} > 0$ is a constant depending on p and $\operatorname{ess\,inf}_{t\in[0,\sigma]} \beta_1(t)$.

Let us take $\theta \in C^1(\mathbb{R})$ with the following properties: $\theta(t) = 0$ if $t \in (-\infty, -\sigma]$, $\theta(t) = \exp(-\frac{t^2}{(t+\sigma)^2})$ if $t \in (-\sigma, 0)$, $\theta(t) = 1$ if $t \in [0, +\infty)$. From (4.40), setting $t_1 = -\sigma$, $t_2 = \tau \in [0;\sigma]$ and using Lemma 2.4, we obtain

$$\begin{aligned} &\|u(\tau)\|^2_{V_{\mathcal{B}}} + \int_0^\tau \beta_1(t)\|u(t)\|^p_V \,dt \\ &\leqslant \int_{-\sigma}^0 \theta'(t)\|u(t)\|^2_{V_{\mathcal{B}}} \,dt + C_{18} \int_{-\sigma}^\sigma \Big(\beta_1^{-\frac{1}{p-1}}(t)\, \|f(t)\|^{p'}_{V'} + \beta_2(t)\Big) \,dt. \end{aligned} \tag{4.42}$$

Now we estimate the first term of the right hand side of (4.42). Since the imbedding $V \hookrightarrow V_{\mathcal{B}}$ is continuous, from (4.41) we see that

$$\begin{aligned} \int_{-\sigma}^0 \theta'(t)\|u(t)\|^2_{V_{\mathcal{B}}} \,dt &\leqslant C_{20} \int_0^\sigma \|u(t)\|^2_V \,dt \\ &\leqslant C_{21}\Big(\int_0^\sigma \|u(t)\|^p_V \,dt\Big)^{2/p} \\ &\leqslant C_{22}\Big(\int_0^\sigma \Big(\|f(t)\|^{p'}_{V'} + \beta_2(t)\Big) \,dt\Big)^{2/p}, \end{aligned} \tag{4.43}$$

where C_{20}, C_{21} and C_{22} are constants depending on p, σ, $\mathcal{B}$ and $\operatorname{ess\,inf}_{t\in[0,\sigma]} \beta_1(t)$. Thus estimate (3.5) follows from (4.41)-(4.43). $\square$

Proof of Theorem 3.13. Note that Theorem 3.5 implies the existence and uniqueness of a solution u for Problem (3.1). Define $u_\sigma(t) := u(t+\sigma)$, $w_\sigma(t) := u(t+\sigma) - u(t)$, $f_\sigma(t) := f(t+\sigma)$ and $\mathcal{A}_\sigma(t,\cdot) := \mathcal{A}(t+\sigma,\cdot)$, $t \in \mathbb{R}$, for any $\sigma \neq 0$. Clearly u_σ is a solution for Problem (3.1) with $\mathcal{A}$ replaced by $\mathcal{A}_\sigma$ and f replaced by f_σ.

Taking the difference between (3.1) for $u = u_\sigma$ and (3.1) for u we obtain

$$\big(\mathcal{B}w_\sigma(t)\big)' + \mathcal{A}_\sigma\big(t, u_\sigma(t)\big) - \mathcal{A}\big(t, u(t)\big) = f_\sigma(t) - f(t) \quad \text{in } \mathscr{D}'(\mathbb{R}; V'). \tag{4.44}$$

Let $\theta_1 \in C^1(\mathbb{R})$ be the same as in proof of Theorem 3.3 and $\tau \in \mathbb{R}$, $\delta > 0$ be any numbers. Multiplying (4.44) by θw_σ, where $\theta(t) = \theta_1(\frac{t-\tau}{\delta})$, $t \in \mathbb{R}$, and integrating from $\tau - \delta$ to $\tau + 1$ with respect to t we get

$$\begin{aligned} &\int_{\tau-\delta}^{\tau+1} \theta(t)\frac{d}{dt}\|w_\sigma(t)\|^2_{V_{\mathcal{B}}} \,dt + 2\int_{\tau-\delta}^{\tau+1} \theta(t)\big\langle \mathcal{A}\big(t, u_\sigma(t)\big) - \mathcal{A}\big(t, u(t)\big), w_\sigma(t)\big\rangle_V \,dt \\ &= 2\int_{\tau-\delta}^{\tau+1} \theta(t)\big\langle \mathcal{A}\big(t, u_\sigma(t)\big) - \mathcal{A}_\sigma\big(t, u_\sigma(t)\big), w_\sigma(t)\big\rangle_V \,dt \\ &\quad + 2\int_{\tau-\delta}^{\tau+1} \theta(t)\langle f_\sigma(t) - f(t), w_\sigma(t)\rangle_V \,dt. \end{aligned} \tag{4.45}$$

From (4.45), using (ix) and the estimates similar to (4.12), (4.13), in the same way as was shown (3.2), we obtain

$$
\begin{aligned}
&\|w_\sigma(\tau+1)\|^2_{V_{\mathcal{B}}} + \int_0^1 \|w_\sigma(s+\tau)\|^p_V \, ds \\
&= \|w_\sigma(\tau+1)\|^2_{V_{\mathcal{B}}} + \int_\tau^{\tau+1} \|w_\sigma(t)\|^p_V \, dt \\
&\leqslant C_{23}\, \delta^{\frac{2}{2-p}} + C_{24} \int_{\tau-\delta}^{\tau+1} \big\|\mathcal{A}_\sigma\big(t,u_\sigma(t)\big) - \mathcal{A}\big(t,u_\sigma(t)\big)\big\|^{p'}_{V'} \, dt \\
&\quad + C_{24} \int_{\tau-\delta}^{\tau+1} \|f_\sigma(t)-f(t)\|^{p'}_{V'} \, dt
\end{aligned} \tag{4.46}
$$

for any $\tau \in \mathbb{R}$ and $\delta > 0$, where C_{23}, C_{24} are some positive constants depending only on $\mathcal{B}$, K_2 and p.

Let $\varepsilon > 0$ be any number. Fix $\delta \in \mathbb{N}$ large enough that

$$
C_{23}\delta^{\frac{2}{2-p}} < \frac{\varepsilon}{2}. \tag{4.47}
$$

Since $\mathcal{A} \in BC(\mathbb{R}; Y_{p,V})$, it follows that

$$
\begin{aligned}
\sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|\mathcal{A}(t,0)\|^{p'}_{V'} \, dt &\leqslant \sup_{t\in\mathbb{R}} \|\mathcal{A}(t,0)\|^{p'}_{V'} \\
&\leqslant \sup_{t\in\mathbb{R}} \Big(\sup_{v\in V} \frac{\|\mathcal{A}(t,v)\|_{V'}}{\|v\|^{p-1}_V + 1} \Big)^{p'} \\
&= \sup_{t\in\mathbb{R}} \Big(d_{p,V}\big(\mathcal{A}(t,\cdot),0\big)\Big)^{p'} \leqslant C_{25},
\end{aligned} \tag{4.48}
$$

where C_{25} is some positive constant. Thus (4.48), the assumptions of the theorem and Corollary 3.7 imply

$$
\sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|u_\sigma(t)\|^p_V \, dt \leqslant C_{26}, \tag{4.49}
$$

where $C_{26} \geqslant 0$ is some constant independent on σ. From (4.49) it follows that

$$
\begin{aligned}
&\int_{\tau-\delta}^{\tau+1} \big\|\mathcal{A}_\sigma\big(t,u_\sigma(t)\big) - \mathcal{A}\big(t,u_\sigma(t)\big)\big\|^{p'}_{V'} \, dt \\
&\leqslant \sup_{t\in\mathbb{R}} \sup_{v\in V} \frac{\|\mathcal{A}_\sigma(t,v) - \mathcal{A}(t,v)\|^{p'}_{V'}}{\|v\|^p_V + 1} \sum_{i=0}^{\delta} \int_{\tau-i}^{\tau+1-i} \big(\|u_\sigma(t)\|^p_V + 1\big) \, dt \\
&\leqslant C_{27} \Big(\sup_{t\in\mathbb{R}} d_{p,V}\big(\mathcal{A}_\sigma(t,\cdot), \mathcal{A}(t,\cdot)\big)\Big)^{p'},
\end{aligned} \tag{4.50}
$$

where C_{27} is positive constant depending only on p, δ and C_{26}. Since $f \in S^{p'}(\mathbb{R}; V')$, it follows that

$$\begin{aligned} \int_{\tau-\delta}^{\tau+1} \|f_\sigma(t) - f(t)\|_{V'}^{p'}\, dt &= \sum_{i=0}^{\delta} \int_{\tau-i}^{\tau+1-i} \|f_\sigma(t) - f(t)\|_{V'}^{p'}\, dt \\ &\leqslant (\delta+1) \sup_{s\in\mathbb{R}} \int_s^{s+1} \|f_\sigma(t) - f(t)\|_{V'}^{p'}\, dt \\ &= (\delta+1)\, \|f_\sigma - f\|_{S^{p'}}^{p'}. \end{aligned} \tag{4.51}$$

Take $\varepsilon_0 > 0$ such that

$$C_{24}\big(C_{27} + (\delta+1)\big){\varepsilon_0}^{p'} < \frac{\varepsilon}{2}. \tag{4.52}$$

Define

$$U_\varepsilon := \big\{\sigma : \sup_{\tau\in\mathbb{R}} \|w_\sigma(\tau)\|_{V_\mathcal{B}}^2 + \sup_{\tau\in\mathbb{R}} \int_0^1 \|w_\sigma(t+\tau)\|_V^p\, dt < \varepsilon\big\}$$

for any $\varepsilon > 0$.

Since $f^b \in CAP\big(\mathbb{R}; L^{p'}(0,1;V')\big)$ and $\mathcal{A} \in CAP(\mathbb{R}; Y_{p,V})$, we see that the set $G_{\varepsilon_0} := \big\{\sigma \in \mathbb{R} : \|f_\sigma - f\|_{S^{p'}} + \sup_{t\in\mathbb{R}} d_{p,V}\big(\mathcal{A}_\sigma(t,\cdot), \mathcal{A}(t,\cdot)\big) < \varepsilon_0\big\}$ is relatively dense in $\mathbb{R}$ (see, e.g., [8, Property I.VII]). Then from (4.46), (4.47) and (4.50)-(4.52) it follows that $\sigma \in U_\varepsilon$ whenever $\sigma \in G_{\varepsilon_0}$. Thus the proof is complete. $\square$

5. Example

Let Ω, Ω_1 be bounded domains in $\mathbb{R}^n$, $n \in \mathbb{N}$, such that $\Omega_1 \subset \Omega$, $\Omega_0 := \Omega \setminus \Omega_1$, $\partial\Omega$ be a C^1 manifold, $S := \mathbb{R}$, and $2 < p < +\infty$. Set $V := W_0^{1,p}(\Omega)$, then $V' = W^{-1,p'}(\Omega)$, where $p' = p/(p-1)$. Define the operators $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle \mathcal{A}(u), v\rangle_{W_0^{1,p}(\Omega)} := \int_\Omega \sum_{i=1}^n \Big|\frac{\partial u(x)}{\partial x_i}\Big|^{p-2} \frac{\partial u(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_i}\, dx, \quad u, v \in W_0^{1,p}(\Omega),$$

and $\mathcal{B} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle \mathcal{B}(u), v\rangle_{W_0^{1,p}(\Omega)} := \int_{\Omega_1} u(x)v(x)\, dx, \quad u, v \in W_0^{1,p}(\Omega).$$

Then $V_\mathcal{B} \cong \{L^2(\Omega), \|\cdot\|_{V_\mathcal{B}}\}$ and $V'_\mathcal{B} = L^2(\Omega_1)$, which we identify as the subspace of $L^2(\Omega)$ whose elements are zero a.e. on Ω_0 (see, e.g., [12, 14]).

Let $f \in L^{p'}_{\rm loc}\big(\mathbb{R}; L^{p'}(\Omega)\big)$. Then the operators $\mathcal{A}$, $\mathcal{B}$ and f satisfy the hypothesis of Theorem 3.5 (see, e.g., [2, 14]). Thus there exists a unique generalized solution $u \in L^p_{\rm loc}\big(\mathbb{R}; W_0^{1,p}(\Omega)\big) \cap C(\mathbb{R}; V_\mathcal{B})$ of the problem without initial conditions

$$\frac{\partial}{\partial t} u(x,t) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u(x,t)}{\partial x_i}\Big|^{p-2} \frac{\partial u(x,t)}{\partial x_i}\Big) = f(x,t), \quad (x,t) \in \Omega_1 \times \mathbb{R}, \tag{5.1a}$$

$$- \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u(x,t)}{\partial x_i}\Big|^{p-2} \frac{\partial u(x,t)}{\partial x_i}\Big) = f(x,t), \quad (x,t) \in \Omega_0 \times \mathbb{R}, \tag{5.1b}$$

$$u(s,t) = 0, \qquad (s,t) \in \partial\Omega \times \mathbb{R}. \tag{5.1c}$$

Furthermore, if the set

$$\left\{\sigma : \sup_{\tau\in\mathbb{R}} \int_{\tau}^{\tau+1} \int_{\Omega} |f(x,t+\sigma) - f(x,t)|^{p'}\,dx\,dt < \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$; that is, if $f \in S^{p'}\big(\mathbb{R}; L^{p'}(\Omega)\big)$, then Theorem 3.13 implies that the solution u for problem (5.1) is almost periodic by Stepanov as an element of $BS^p\big(\mathbb{R}; W_0^{1,p}(\Omega)\big)$ and by Bohr as an element of $BC(\mathbb{R}; V_{\mathcal{B}})$.

Note that more general examples can be obtained similarly as in [12] and [14] by a corresponding choice of the operators $\mathcal{A}$ and $\mathcal{B}$.

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