Electronic Journal of Differential Equations, Vol. 2008(2008), No. 06, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF GLOBAL SOLUTIONS FOR A PREDATOR-PREY MODEL WITH CROSS-DIFFUSION

SHENGHU XU

ABSTRACT. In this article, we prove the existence of global classical solutions for a prey-predator model when the space dimension $n < 10$. Under certain conditions on the coefficients of the reaction functions, the convergence of solutions is established for the system with large diffusion by constructing a Lyapunov function.

1. INTRODUCTION

To investigate the spatial segregation under the self and cross population pressure, Shigesada, Kawasaki and Teramoto [1] proposed a competition model in 1979. Then there have been established many results in the literatures; see for example [2, 3, 4, 5, 6, 7, 8, 9]. For the cross-diffusion systems with prey-predator type reaction functions, there are a few results mainly on the steady-state problems with the elliptic systems, see [10, 11, 12, 13, 14].

In this paper, we study the following cross-diffusion system, with prey-predator type reactions,

$$\begin{gathered} u_t - \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] = u(a_1 - b_1 u - c_1 v) \quad \text{in } \Omega \times [0, \infty), \\ v_t - \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] = v(a_2 + b_2 u - c_2 v) \quad \text{in } \Omega \times [0, \infty), \\ \partial_\eta u = \partial_\eta v = 0 \quad \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, η is the outward unit normal vector of the boundary $\partial\Omega$, and $\partial_\eta = \partial/\partial_\eta$. α_{ij} are given nonnegative constants for $i, j = 1, 2$. And $d_i, b_i, c_i (i = 1, 2)$ and a_1 are positive constants only a_2 may be non-positive.

In system (1.1), u and v are nonnegative functions which represent the population densities of the prey and predator species, respectively, d_1 and d_2 are the random diffusion rates of the two species, α_{11} and α_{22} are self-diffusion rates, and α_{12} and α_{21} are the so-called cross-diffusion rates. When $\alpha_{ij} = 0$ $(i, j = 1, 2)$, the system is the well-known Lotka-Volterra prey-predator model. For more details on the biological background, see references [1, 18].

2000 *Mathematics Subject Classification.* 35K57, 35B35, 92D25.
Key words and phrases. Cross-diffusion; global solution; gradient estimates; stability.

Submitted October 13, 2007. Published January 12, 2008.

Local existence (in time) of solutions to (1.1) was established by Amann in a series of important papers [15, 16, 17]. His result can be summarized as follows:

Theorem 1.1. *Suppose that u_0, v_0 are in $W_p^1(\Omega)$ for some $p > n$. Then* (1.1) *has a unique non-negative smooth solution $u(x,t), v(x,t)$ in*
$$C([0,T), W_p^1(\Omega)) \bigcap C^\infty((0,T), C^\infty(\Omega))$$
with maximal existence time T. Moreover, if the solution (u,v) satisfies the estimate
$$\sup_{0\le t\le T} \|u(.,t)\|_{W_p^1(\Omega)} < \infty \quad \text{and} \quad \sup_{0\le t\le T} \|v(.,t)\|_{W_p^1(\Omega)} < \infty,$$
then $T = \infty$.

However, little is known about global existence of solutions to (1.1). In 2006, Shim [18] proved the existence of global solutions to (1.1) in two cases: Case(A) $n = 1$, $d_1 = d_2$ and $\alpha_{11} = \alpha_{22} = 0$; Case(B) $n = 1$, $0 < \alpha_{21} < 8\alpha_{11}$ and $0 < \alpha_{12} < 8\alpha_{22}$.

In [19] the author considered the case when $\alpha_{11}, \alpha_{12}, \alpha_{22} > 0$ and $\alpha_{21} = 0$ for the system (1.1), and established the existence of global solutions with $n = 1$.

We shall prove the existence of global solutions to the following system (namely, the system (1.1) for $\alpha_{12} = 0$)
$$\begin{gathered}
u_t - \Delta[(d_1 + \alpha_{11}u)u] = u(a_1 - b_1u - c_1v) \quad \text{in } \Omega\times[0,\infty),\\
v_t - \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] = v(a_2 + b_2u - c_2v) \quad \text{in } \Omega\times[0,\infty),\\
\partial_\eta u = \partial_\eta v = 0 \quad \text{on } \partial\Omega\times[0,\infty),\\
u(x,0) = u_0(x) \ge 0, \quad v(x,0) = v_0(x) \ge 0 \quad \text{in } \Omega.
\end{gathered} \tag{1.2}$$

This paper draws on ideas from two papers [6] and [9] which deal with cross-diffusion system with competition type reactions. Duo to the difference in the reaction functions. Therefore, in order to obtain the L^p-estimate of v, we have to estimate the term uv^p. We also obtain result on the asymptotic stability of the global solution to (1.2) if the diffusion coefficients are large enough by an important Lemma 5.1 from [21]. We summarize our results in the following theorems:

Theorem 1.2. *Let $\alpha_{22} > 0$ and assume that $u_0 \ge 0, v_0 \ge 0$ satisfy zero Neumann boundary condition and belong to $C^{2+\lambda}(\overline{\Omega})$ for some $\lambda > 0$. Then* (1.2) *possesses a unique non-negative solution $u, v \in C^{2+\lambda, \frac{2+\lambda}{2}}(\overline{\Omega}\times[0,\infty))$ if either* (i) $\alpha_{11} = 0$ *or* (ii) $\alpha_{11} > 0$ *and $n < 10$.*

Theorem 1.3. *Assume that all conditions in Theorem 1.2 are satisfied. Assume further that*
$$-\frac{a_1b_2}{b_1c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}, \tag{1.3}$$
$$4\rho\overline{u}\overline{v}d_1d_2 > m^2(\overline{v}\alpha_{21})^2. \tag{1.4}$$
Then (1.2) *has the unique positive equilibrium point $(\overline{u}, \overline{v})$ which is global asymptotic stable, where m is the positive constant in Lemma 2.1 (independent of d_1, d_2), $\rho = (b_2c_1 + 2b_1c_2)b_2^{-2}$ and*
$$(\overline{u}, \overline{v}) = \Big(\frac{a_1c_2 - a_2c_1}{b_1c_2 + b_2c_1}, \frac{a_2b_1 + a_1b_2}{b_1c_2 + b_2c_1}\Big).$$

The paper is organized as follows. In section 2, we collect some well known results and prove three new lemmas that are needed in section 3 and section 4. In section 3, we establish L^r-estimates of the solution v of (1.2) and in section 4 we give a proof of Theorem 1.2. In section 5, we give a proof of Theorem 1.3.

2. Preliminaries

We list here some notation.

$$
\begin{gathered}
Q_T = \Omega \times [0,T), \\
\|u\|_{L^{p,q}(Q_T)} = \Big(\int_0^T \big(\int_\Omega |u(x,t)|^p dx\big)^{\frac{q}{p}} dt\Big)^{1/q}, L^p(Q_T) := L^{p,p}(Q_T), \\
\|u\|_{W_P^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)}, \\
\|u\|_{V_2(Q_T)} := \sup_{0\le t\le T} \|u(.,t)\|_{L^2(\Omega)} + \|\nabla u(x,t)\|_{L^2(Q_T)}.
\end{gathered}
$$

Firstly, we present some useful lemmas.

Lemma 2.1. *Let u, v be a solution of (1.2) in $[0,T)$. Then $0 \le u \le m$ and $v \ge 0$ in Q_T, where $m = \max\{\frac{a_1}{b_1}, \|u_0\|_{L^\infty(\Omega)}\}$.*

Proof. The first equation in (1.2) is expressed as

$$
u_t = (d_1 + 2\alpha_{11} u)\Delta u + 2\alpha_{11}\nabla u \cdot \nabla u + u(a_1 - b_1 u - c_1 v), \tag{2.1}
$$

and the second equation is written as

$$
v_t = (d_2 + \alpha_{21} u + 2\alpha_{22} v)\Delta v + 2(\alpha_{21}\nabla u + \alpha_{22}\nabla v)\nabla v + v(\alpha_{21}\Delta u + a_2 + b_2 u - c_2 v). \tag{2.2}
$$

Then application of the maximum principle for (2.1) and (2.2) yields the nonnegative of u and v. Applying the maximum principle to (2.1) again one can also show the boundedness of u. $\square$

Lemma 2.2. *There exists a positive $C_1(T)$ such that*

$$
\begin{gathered}
\sup_{0\le t\le T} \|u(.,t)\|_{L^1(\Omega)} < C_1(T), \quad \sup_{0\le t\le T} \|v(.,t)\|_{L^1(\Omega)} < C_1(T), \\
\|u\|_{L^2(Q_T)} < C_1(T), \quad \|v\|_{L^2(Q_T)} < C_1(T).
\end{gathered}
$$

Proof. Integrating the first equation in (1.2) over the domain Ω, we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u\,dx &= a_1\int_\Omega u\,dx - b_1\int_\Omega u^2 dx - c_1\int_\Omega uv dx \\
&\le a_1\int_\Omega u\,dx - b_1\int_\Omega u^2 dx \\
&\le a_1\int_\Omega u\,dx - \frac{b_1}{|\Omega|}\Big(\int_\Omega u\,dx\Big)^2,
\end{aligned} \tag{2.3}
$$

where we used Hölder's inequality. Then we have $\|u(.,t)\|_{L^1(\Omega)} \le M_1'$, where $M_1' = \max\{\|u_0\|_{L^1(\Omega)}, \frac{a_1}{b_1}|\Omega|\}$. Furthermore,

$$
\sup_{0\le t\le T} \|u(.,t)\|_{L^1(\Omega)} < C_1(T). \tag{2.4}
$$

Since

$$
\frac{d}{dt}\int_\Omega u\,dx \le a_1\int_\Omega u\,dx - b_1\int_\Omega u^2 dx. \tag{2.5}
$$

Integrating (2.5) from 0 to T, we have

$$\|u\|_{L^2(Q_T)}^2 \leq \frac{a_1}{b_1} M_1' |Q_T| + \|u_0\|_{L^1(\Omega)}.$$

Therefore,

$$\|u\|_{L^2(Q_T)} \leq C_1(T). \tag{2.6}$$

Now, integrating the second equation in the system (1.2) over the domain Ω we have

$$\frac{d}{dt}\int_\Omega v dx = a_2 \int_\Omega v dx + b_2 \int_\Omega uv dx - c_2 \int_\Omega v^2 dx. \tag{2.7}$$

Multiplying (2.3) by $\frac{b_2}{c_1}$ and adding it to (2.7), we have

$$\frac{d}{dt}\int_\Omega \big(\frac{b_2}{c_1}u + v\big) dx \leq \frac{a_1 b_2}{c_1}\int_\Omega u\, dx + |a_2| \int_\Omega v\, dx - \frac{b_1 b_2}{c_1}\int_\Omega u^2 dx - c_2 \int_\Omega v^2 dx. \tag{2.8}$$

Then

$$\begin{aligned}
&\min\{1, \frac{b_2}{c_1}\}\frac{d}{dt}\int_\Omega \big(u+v\big)dx \\
&\leq \max\{\frac{a_1 b_2}{c_1}, |a_2|\}\int_\Omega (u+v)dx - \min\{\frac{b_1 b_2}{c_1}, c_2\}\int_\Omega (u^2+v^2)dx \\
&\leq \max\{\frac{a_1 b_2}{c_1}, |a_2|\}\int_\Omega (u+v)dx - \frac{1}{2}\min\{\frac{b_1 b_2}{c_1}, c_2\}\big[\int_\Omega (u+v)dx\big]^2.
\end{aligned}$$

Therefore, $\|v(.,t)\|_{L^1(\Omega)} \leq M_2'$, where $M_2' = \max\{\frac{A_1}{A_2}, \|u_0 + v_0\|_{L^1(\Omega)}\}$,

$$A_1 = \frac{\max\{\frac{a_1 b_2}{c_1}, |a_2|\}}{\min\{1, \frac{b_2}{c_1}\}}, \quad A_2 = \frac{\min\{\frac{b_1 b_2}{c_1}, c_2\}}{2\min\{1, \frac{b_2}{c_1}\}}.$$

Then

$$\sup_{0\leq t\leq T}\|v(.,t)\|_{L^1(\Omega)} < C_1(T). \tag{2.9}$$

Integrating (2.8) from 0 to T, we have

$$c_2 \int_{Q_T} v^2\, dx\, dt \leq \frac{a_1 b_2}{c_1}\int_0^T M_1' dt + |a_2| \int_0^T M_2' dt + \frac{b_2}{c_1}\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)},$$

which implies $\|v\|_{L^2(Q_T)} \leq C_1(T)$. □

Lemma 2.3. *Let $w_1 = (d_1 + \alpha_{11} u)u$ .Then there exists a constant $C_2(T)$, depending on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$ such that*

$$\|w_1\|_{W_2^{2,1}(Q_T)} \leq C_2(T). \tag{2.10}$$

Furthermore, $\nabla w_1 \in V_2(Q_T)$.

Proof. Note that w_1 satisfies the equation

$$w_{1t} = (d_1 + 2\alpha_{11}u)\Delta w_1 + n_1 + n_2 v, \tag{2.11}$$

where $n_1 = u(d_1 + 2\alpha_{11}u)(a_1 - b_1 u), n_2 = -c_1(d_1 + 2\alpha_{11}u)u$ depend on u and are bounded functions because of Lemma 2.1. Multiplying the above equation by $-\Delta w_1$ and integration by parts over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w_1|^2 dx = -\int_\Omega (d_1 + 2\alpha_{11}u)(\Delta w_1)^2 dx - \int_\Omega (n_1 + n_2 v)\Delta w_1 dx. \tag{2.12}$$

Integrating (2.12) from 0 to t, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla w_1(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla w_1(x,0)|^2 dx \\
&= -\int_{Q_t} (d_1 + 2\alpha_{11}u)(\Delta w_1)^2\, dx\, dt - \int_{Q_t} (n_1 + n_2 v)\Delta w_1\, dx\, dt \\
&\le -d_1 \int_{Q_t} |\Delta w_1|^2\, dx\, dt + \int_{Q_t} (n_1 + n_2 v)\cdot |\Delta w_1| dx\, dt\,.
\end{aligned}$$

By Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla w_1(x,t)|^2 dx + d_1 \int_{Q_T} |\Delta w_1|^2\, dx\, dt \\
&\le (\|n_1\|_{L^2(Q_T)} + \|n_2 v\|_{L^2(Q_T)}) \cdot \|\Delta w_1\|_{L^2(Q_T)} + \frac{1}{2}\int_\Omega |\nabla w_1(x,0)|^2 dx \\
&\le m_1(1 + \|v\|_{L^2(Q_T)}) \cdot \|\Delta w_1\|_{L^2(Q_T)} + \frac{1}{2}\int_\Omega |\nabla w_1(x,0)|^2 dx \\
&\le m_1(1 + C_1(T)) \cdot \|\Delta w_1\|_{L^2(Q_T)} + \frac{1}{2}\int_\Omega |\nabla w_1(x,0)|^2 dx \\
&\le \frac{d_1}{2}\|\Delta w_1\|^2_{L^2(Q_T)} + \frac{m_1^2(1 + C_1(T))^2}{2d_1} + \frac{1}{2}\int_\Omega |\nabla w_1(x,0)|^2 dx.
\end{aligned}$$

Therefore,

$$\sup_{0\le t\le T} \int_\Omega |\nabla w_1|^2(x,t) dx + d_1 \int_{Q_T} |\Delta w_1|^2\, dx\, dt \le m_2,$$

where m_2 depends on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$. This implies $\nabla w_1 \in V_2(Q_T)$. Since $w_1 \in L^2(Q_T)$ we have from the elliptic regularity estimate [2, Lemma 2.3]

$$\int_{Q_T} \|(w_1)_{x_i x_j}\|^2\, dx\, dt \le m_3 \quad \text{for } i,j = 1,\dots,n.$$

From (2.11), since n_1, n_2 and u are bounded and $v \in L^2(Q_T)$, we have $w_{1t} \in L^2(Q_T)$. Hence, $w_1 \in W_2^{2,1}(Q_T)$. □

Let $a(x,t,\xi)$ be continuous and (x,ξ)-differentiable for $(x,t,\xi) \in \mathrm{Q_T} \times \mathrm{R}$. Assume also that $a(x,t,\xi)$ satisfies the following conditions

- (i) There is $d > 0$ such that $a(x,t,\xi) \ge d$ and $a_\xi(x,t,\xi) \ge 0$ for all $(x,t) \in Q_T$ and ξ in R.
- (ii) There is a continuous function M on R such that $a(x,t,\xi) \le M(\xi)$ for all $(x,t) \in Q_T$.
- (iii) For any bounded measurable function g on Q_T, $|\nabla_x a(.,.,g(.,.))|$ is in the space $L^{2p}(Q_T)$.

Lemma 2.4. *Assume that $w \in W_p^{2,1}(Q_T)\bigcap C^{2,1}(\overline{\Omega}\times[0,T))$ is a bounded function satisfying*

$$w_t \le a(x,t,w)\Delta w + f(x,t) \quad in \quad Q_T$$

with boundary condition $\frac{\partial w}{\partial \nu} \le 0$ on ∂_{Q_T}, where $f \in L^p(Q_T)$. Then, ∇w is in $L^{2p}(Q_T)$.

The proof of the above lemma can be found in [9, Proposition 2.1].

Lemma 2.5. *Let $q>1, \widetilde{q}=2+\frac{4q}{n(q+1)}$, $\widetilde{\beta}$ in $(0,1)$ and let $C_T>0$ be any number which may depend on T. Then there is a constant M_1 depending on $q,n,\Omega,\widetilde{\beta}$ and C_T such that for any g in $C([0,T),W_2^1(\Omega))$ with $(\int_\Omega |g(.,t)|^{\widetilde{\beta}}dx)^{1/\widetilde{\beta}}\le C_T$ for all $t\in[0,T]$, we have the inequality*

$$\|g\|_{L^{\widetilde{q}}(Q_T)}\le M_1\big\{1+\big(\sup_{0\le t\le T}\|g(.,t)\|_{L^{2q/q+1}(\Omega)}\big)^{4q/n(q+1)\widetilde{q}}\|\nabla g\|^{2/\widetilde{q}}_{L^2(Q_T)}\big\}.$$

The proof of the above lemma can be found in [6, Lemmas 2.3, 2.4].

3. L^r-estimates for v

Lemma 3.1. *There exists a constant $C_3(T)$ such that $\|\nabla u\|_{L^4(Q_T)}\le C_3(T)$.*

Proof. Let $\delta=\alpha_{11}/d_1$, $w_1=(1+\delta u)u$. By Lemma 2.1, u is bounded. Therefore, w_1 is also bounded. By Lemma 2.3, we have $w_1\in W_2^{2,1}(Q_T)$. Moreover, w_1 satisfies

$$\begin{aligned}
w_{1t}&\le d_1(1+2\delta u)\Delta w_1+a_1u(1+2\delta u)\\
&=\sqrt{d_1^2+4\delta d_1w_1}\Delta w_1+a_1u(1+2\delta u).
\end{aligned}$$

By Lemma 2.4 with $p=2$, $a(x,t,\xi)=\sqrt{d_1^2+4\delta d_1\xi}$, $f(x,t)=a_1u(x,t)(1+2\delta u(x,t))$, we obtain the desired result. $\square$

Lemma 3.2. *Let $r>2$ and $p_r=\frac{2r}{r-2}$ be two positive numbers. Assume that $\alpha_{22}>0$ and assume also that there is a constant $M_{r,T}>0$ depending only on r,T,Ω and the coefficients of (1.2) such that*

$$\|\nabla u\|_{L^r(Q_T)}\le M_{r,T}.$$

Then for any $q>1$, there exists a constant $C(r,q,T)>0$ such that

$$\begin{aligned}
&\|v(.,t)\|^q_{L^q(\Omega)}+\|\nabla(v^{q/2})\|^2_{L^2(Q_t)}+\|\nabla(v^{(q+1)/2})\|^2_{L^2(Q_t)}\\
&\le C(r,q,T)\big(1+\|v\|^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\big).
\end{aligned}\tag{3.1}$$

Proof. For any constant $q>1$, multiplying the second equation of (1.2) by qv^{q-1} and using the integration by parts, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega v^q dx\\
&=q\int_\Omega v^{q-1}\nabla\cdot[(d_2+\alpha_{21}u+2\alpha_{22}v)\nabla v+\alpha_{21}v\nabla u]dx+q\int_\Omega v^q(a_2+b_2u-c_2v)dx\\
&=-q(q-1)\int_\Omega v^{q-2}(d_2+\alpha_{21}u+2\alpha_{22}v)|\nabla v|^2dx-\alpha_{21}(q-1)\int_\Omega\nabla(v^q)\cdot\nabla u\,dx\\
&\quad+q\int_\Omega v^q(a_2+b_2u-c_2v)dx\\
&\le -q(q-1)d_2\int_\Omega v^{q-2}|\nabla v|^2dx-2\alpha_{22}q(q-1)\int_\Omega v^{q-1}|\nabla v|^2dx\\
&\quad-\alpha_{21}(q-1)\int_\Omega\nabla(v^q)\cdot\nabla u\,dx+q\int_\Omega v^q(a_2+b_2u-c_2v)dx\\
&=-\frac{4(q-1)d_2}{q}\int_\Omega|\nabla(v^{\frac q2})|^2dx-\frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_\Omega|\nabla(v^{\frac{q+1}{2}})|^2dx
\end{aligned}$$

$$- \alpha_{21}(q-1)\int_\Omega \nabla(v^q)\cdot\nabla u\,dx + q\int_\Omega v^q(a_2+b_2u-c_2v)dx.$$

Integrating the above inequality from 0 to t, we have

$$\begin{aligned}
&\int_\Omega v^q(x,t)dx + \frac{4(q-1)d_2}{q}\int_{Q_t}|\nabla(v^{\frac{q}{2}})|^2dx\,dt + \frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(v^{\frac{q+1}{2}})|^2\,dx\,dt\\
&\le \int_\Omega v^q(x,0)dx - \alpha_{21}(q-1)\int_{Q_t}\nabla(v^q)\cdot\nabla u\,dx\,dt + q\int_{Q_t}v^q(a_2+b_2u-c_2v)\,dx\,dt.
\end{aligned} \tag{3.2}$$

By Hölder's inequality, we have

$$\begin{aligned}
&q\int_{Q_t}v^q(a_2+b_2u-c_2v)\,dx\,dt\\
&= a_2q\int_{Q_t}v^qdxdt - c_2q\int_{Q_t}v^{q+1}dxdt + b_2q\int_{Q_t}uv^qdxdt\\
&\le -c_2q\|v\|^{q+1}_{L^{q+1}(Q_t)} + |a_2|q|Q_T|^{\frac{1}{q+1}}\|v\|^q_{L^{q+1}(Q_t)} + b_2q\int_{Q_t}uv^qdxdt \qquad (3.3)\\
&\le -c_2q\|v\|^{q+1}_{L^{q+1}(Q_t)} + |a_2|q\big[\varepsilon\|v\|^{q+1}_{L^{q+1}(Q_t)} + \varepsilon^{-q}|Q_T|^{\frac{q}{q+1}}\big] + b_2q\int_{Q_t}uv^qdxdt\\
&\le B_1 + b_2q\int_{Q_t}uv^qdxdt,
\end{aligned}$$

where $\varepsilon = \frac{c_2}{|a_2|}$, B_1 depends on $T, q, |\Omega|$ and the coefficients of (1.2).

On the other hand, since that $\frac{1}{r}+\frac{1}{2}+\frac{1}{p_r}=1$, using the Hölder's inequality and Poincaré inequality, we have

$$\begin{aligned}
\int_{Q_t}uv^qdxdt &= \int_{Q_t}u\cdot v^{\frac{q-1}{2}}\cdot v^{\frac{q+1}{2}}\,dx\,dt\\
&\le \|v^{\frac{q-1}{2}}\|_{L^{p_r}(Q_t)}\cdot\|v^{\frac{q+1}{2}}\|_{L^2(Q_t)}\cdot\|u\|_{L^r(Q_t)}\\
&\le C_4m\|v\|^{(q-1)/2}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}.
\end{aligned} \tag{3.4}$$

The substitution (3.4) into (3.3) leads to

$$q\int_{Q_t}v^q(a_2+b_2u-c_2v)dxdt \le B_1 + C_5\|v\|^{(q-1)/2}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}. \tag{3.5}$$

Since that $\frac{1}{r}+\frac{1}{2}+\frac{1}{p_r}=1$ and ∇u is in $L^r(Q_T)$, using the Hölder's inequality, we have

$$\begin{aligned}
\Big|-\int_{Q_t}\nabla(v^q)\cdot\nabla u\;dx\,dt\Big| &= \frac{2q}{q+1}\Big|\int_{Q_t}v^{\frac{q-1}{2}}\cdot\nabla(v^{\frac{(q+1)}{2}})\cdot\nabla u\,dx\,dt\Big|\\
&\le \frac{2q}{q+1}\|v^{\frac{q-1}{2}}\|_{L^{p_r}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u\|_{L^r(Q_t)}\\
&\le \frac{2q}{q+1}\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u\|_{L^r(Q_t)}\\
&\le \frac{2q}{q+1}M_{r,T}\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}.
\end{aligned}$$

The substitution (3.5) and the above inequality into (3.2) leads to

$$
\begin{aligned}
&\int_\Omega v^q(x,t)dx + \frac{4(q-1)d_2}{q}\int_{Q_t}|\nabla(v^{\frac{q}{2}})|^2 dx\,dt + \frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(v^{\frac{q+1}{2}})|^2\,dx\,dt\\
&\le B_2 + C_6\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}\\
&\le B_2 + \frac{C_6}{4\varepsilon}\|v\|^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)} + C_6\varepsilon\|\nabla(v^{\frac{q+1}{2}})\|^2_{L^2(Q_t)},
\end{aligned}
\tag{3.6}
$$

where $B_2 > 0$ depending on q, T, Ω coefficients of (1.2) and initial datal v_0. For any $\varepsilon > 0$, from (3.6) and by choosing a sufficiently small ε, such that $C_6\varepsilon < \frac{8\alpha_{22}q(q-1)}{(q+1)^2}$, we get (3.1). This completes the proof of the lemma. □

For any number a, we denote $a_+ = \max\{a, 0\}$.

Proposition 3.3. *Let $\alpha_{22} > 0$.*

(i) *If $\alpha_{11} > 0$, then there is a constant $C_7(T) > 0$ such that*

$$\|v\|_{V_2(Q_T)} \le C_7(T).$$

Moreover, for any constant $r < \frac{4(n+1)}{(n-2)_+}$, there exists a positive constant C_T such that

$$\|v\|_{L^r(Q_T)} \le C_T.$$

(ii) *If $\alpha_{11} = 0$, then*

$$\|v\|_{L^r(Q_T)} \le C_T \quad \textit{for any} \quad r > 1.$$

Proof. (i) Set $w = v^{(q+1)/2}$ so that $v^q = w^{2q/(q+1)}$ and $v^{q+1} = w^2$. Then

$$
\begin{aligned}
E &\equiv \sup_{0\le t\le T}\int_\Omega v^q(x,t)dx + \int_{Q_T}|\nabla(v^{(q+1)/2})|^2\,dx\,dt\\
&= \sup_{0\le t\le T}\int_\Omega w^{2q/q+1}dx + \int_{Q_T}|\nabla w|^2\,dx\,dt.
\end{aligned}
$$

Let $r_0 = 4$, $p_0 = \frac{2r_0}{r_0-2}$. By Lemma 3.1, we see that ∇u is in $L^{r_0}(Q_T)$. So, from Lemma 3.2, we have

$$E + \|\nabla(v^{\frac{q}{2}})\|^2_{L^2(Q_T)} \le C(r_0,q,T)\Big(1 + \|w\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)}\Big), \tag{3.7}$$

where $C(r_0, q, T) > 0$ depending only T, Ω, initial data u_0, v_0 and the coefficients of (1.2). Since $q > 1$, if we restrict our q so that

$$(np_0 - 2n - 4)q \le 2n + np_0. \tag{3.8}$$

Then, $\frac{p_0(q-1)}{q+1} \le \widetilde{q}$, where $\widetilde{q} = 2 + \frac{4q}{n(q+1)}$. Therefore, by Hölder's inequality

$$\|w\|_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)} \le C_8(q,T)\|w\|_{L^{\widetilde{q}}(Q_T)}, \tag{3.9}$$

where $C_8(q,T) = |Q_T|^{\frac{q+1}{p_0(q-1)} - \frac{1}{\widetilde{q}}}$. Setting $\widetilde{\beta} = 2/(q+1) \in (0,1)$, by Lemma 2.2 we have

$$\|w(.,t)\|_{L^{\widetilde{\beta}}(\Omega)} = \|v(.,t)\|^{\frac{1}{\widetilde{\beta}}}_{L^1(\Omega)} \le (C_1(T))^{\frac{1}{\widetilde{\beta}}}, \quad \forall t \in [0,T). \tag{3.10}$$

Hence, by Lemma 2.5 and the definition of E, (3.10) yields

$$\|w\|_{L^{p_0(q-1)/q+1}(Q_T)} \le C_8(q,T)\|w\|_{L^{\widetilde{q}}(Q_T)} \le C_8(q,T)M_1\{1+E^{2/n\widetilde{q}}E^{\frac{1}{\widetilde{q}}}\}. \tag{3.11}$$

Then (3.7) together with the above inequality, we can find a constant $C_9(q,T)>0$ such that

$$E \le C_9(q,T)(1+E^{\mu}E^{\nu}) \tag{3.12}$$

with

$$\mu=\frac{4(q-1)}{n\widetilde{q}(q+1)},\quad \nu=\frac{2(q-1)}{\widetilde{q}(q+1)}.$$

Since

$$\mu+\nu=\frac{2(q-1)}{\widetilde{q}(q+1)}\big[\frac{2}{n}+1\big]<\frac{1}{\widetilde{q}}\big[\frac{4q}{n(q+2)}+2\big]=1,$$

it is easy to see from (3.12) that E is bounded. Therefore, from (3.11) and (3.12) we get $w\in L^{\widetilde{q}}(Q_T)$ which in turn implies that $v\in L^r(Q_T)$ with $r=\frac{\widetilde{q}(q+1)}{2}$ for any q satisfying (3.8). Now, looking at (3.8), if $n\le 2$, we have

$$np_0-2n-4=2(n-2)\le 0, \tag{3.13}$$

then (3.8) holds for all q. so for $n\le 2$, $v\in L^r(Q_T)$ for all $r>1$. Now, suppose that $n>2$, we see (3.8) is equivalent to

$$1<q<q_0:=\frac{2n+np_0}{(np_0-2n-4)}=\frac{3n}{n-2}.$$

Then, we have

$$\frac{\widetilde{q}(q+1)}{2}=q+1+\frac{2q}{n}\le \overline{r}_1:=q_0+1+\frac{2q_0}{n}=\frac{4(n+1)}{n-2}.$$

So, we see that v is in $L^r(Q_T)$ for all $1<r\le\overline{r}_1$. Since (3.8) holds true for $q=2$. So when $q=2$, we have E is finite. Therefore, from (3.7) and (3.11), we see that $\|v\|_{V_2(Q_T)}$ is bounded for any n, this completes the proof of Proposition 3.3 when $\alpha_{11}>0$ and $r<\frac{4(n+2)}{(n-2)_+}$.

Next, we consider the case $\alpha_{11}=0$. By Hölder's inequality, we have

$$\begin{aligned}
&q\int_{Q_t} v^q(a_2+b_2u-c_2v)\,dx\,dt\\
&=a_2q\int_{Q_t} v^q dxdt-c_2q\int_{Q_t} v^{q+1}dxdt+b_2q\int_{Q_t} uv^q dxdt\\
&\le -c_2q\|v\|^{q+1}_{L^{q+1}(Q_t)}+|a_2|q|Q_T|^{\frac{1}{q+1}}\|v\|^q_{L^{q+1}(Q_t)}\\
&\quad +b_2q\|v\|^q_{L^{q+1}(Q_t)}\cdot\|u\|_{L^{q+1}(Q_t)}\\
&\le -c_2q\|v\|^{q+1}_{L^{q+1}(Q_t)}+|a_2|q|Q_T|^{\frac{1}{q+1}}\|v\|^q_{L^{q+1}(Q_t)}+b_2qm\|v\|^q_{L^{q+1}(Q_t)}\\
&\le -c_2q\|v\|^{q+1}_{L^{q+1}(Q_t)}+q\varepsilon\|v\|^{q+1}_{L^{q+1}(Q_t)}+B_3\\
&\le B_3,
\end{aligned}\tag{3.14}$$

where $\varepsilon=c_2$ and $B_3>0$ which depends only on $T,q,|\Omega|$, $\|u_0\|_{L^\infty(\Omega)}$ and the coefficients of (1.2).

We can integrate by parts once to obtain from Lemma 2.1 and analogue of [20, Theorem 9.1, p. 341-342] for Neumann boundary condition [20, p.351]

$$\begin{aligned}
&\Big| -\int_{Q_t} \nabla(v^q)\cdot\nabla u \,dx\,dt\Big| \\
&= \Big| -\int_{Q_t} v^q \Delta u \,dx\,dt\Big| \\
&\le \|v\|^q_{L^{q+1}(Q_T)}\cdot\|\Delta u\|_{L^{q+1}(Q_T)} \\
&\le C_{10}\|v\|^q_{L^{q+1}(Q_T)}\Big(\|u(a_1-b_1u-c_1v)\|_{L^{q+1}(Q_T)}+\|u_0\|_{W^{2-\frac{2}{q+1}}_{q+1}(\Omega)}\Big) \\
&\le C_{11}\Big(1+\|v\|^{q+1}_{L^{q+1}(Q_T)}\Big).
\end{aligned} \tag{3.15}$$

The substitution of (3.14) and (3.15) into (3.2) leads to

$$\sup_{0\le t\le T}\|v^q(t)\|^q_{L^q(\Omega)}+\|\nabla(v^{(q+1)/2})\|^2_{L^2(Q_T)}\le C_{12}\big(1+\|v\|^{q+1}_{L^{q+1}(Q_T)}\big). \tag{3.16}$$

We introduce $w=v^{\frac{q+1}{2}}$, then (3.16) leads to

$$E\equiv\sup_{0\le t\le T}\|w(t)\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(\Omega)}+\|\nabla w\|^2_{L^2(Q_T)}\le C_{12}\big(1+\|w\|^2_{L^2(Q_T)}\big). \tag{3.17}$$

Recall that Lemma 2.2 implies $v\in L^2(Q_T)$, so $\|w\|_{L^{\frac{4}{q+1}}(Q_T)}\le C_{13}$. Since $\frac{4}{q+1}<2\le\widetilde{q}$. Then we see from Hölder's inequality

$$\|w\|^2_{L^2(Q_T)}\le\|w\|^{2(1-\lambda)}_{L^{\widetilde{q}}(Q_T)}\|w\|^{2\lambda}_{L^{\frac{4}{q+1}}(Q_T)}\le C_{13}^{2\lambda}\|w\|^{2(1-\lambda)}_{L^{\widetilde{q}}(Q_T)}, \tag{3.18}$$

where $\lambda=(\frac12-\frac{1}{\widetilde{q}})/(\frac{q+1}{4}-\frac{1}{\widetilde{q}})$. Setting $\widetilde{\beta}=2/(q+1)\in(0,1)$, we have $\|w(.,t)\|_{L^{\widetilde{\beta}}(\Omega)}=\|v(.,t)\|^{\frac{1}{\widetilde{\beta}}}_{L^1(\Omega)}\le C_1(T)^{\frac{1}{\widetilde{\beta}}}$ for all $t\in[0,T)$ by Lemma 2.2. Then it follow from (3.17), (3.18) and Lemma 2.5 that

$$E\le C_{14}(1+E^\alpha) \tag{3.19}$$

with

$$\alpha=\frac{2(1-\lambda)}{\widetilde{q}}\big(\frac{2}{n}+1\big)<1.$$

Thus (3.19) implies

$$\sup_{0\le t\le T}\|w(t)\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(\Omega)}\le E\le C_{15}$$

with some $C_{15}>0$, let $r=q>1$, so that $\sup_{0\le t\le T}\|v(t)\|_{L^r(\Omega)}\le C_T$ and the proof is complete. □

4. Proof of Theorem 1.2

The first step of the proof is to show v is in $L^r(Q_T)$ for any $r>1$.

Lemma 4.1. *Let $\alpha_{11}>0$ and suppose that there are $r_1>\max\{\frac{n+2}{2},3\}$ and a positive constant $C_{r_1,T}$ such that*

$$\|v\|_{L^{r_1}(Q_T)}\le C_{r_1,T}.$$

Then, v is in $L^r(Q_T)$ for any $r>1$.

Proof. The proof is almost identical to [9, Lemma 4.1], but for completeness we repeat it here. First, the equation for u can be written in the divergence form as

$$u_t = \nabla \cdot [(d_1 + 2\alpha_{11}u)\nabla u] + u(a_1 - b_1u - c_1v), \tag{4.1}$$

where $d_1 + 2\alpha_{11}u$ is bounded in $\overline{Q}_T$ by Lemma 2.1 and $u(a_1 - b_1u - c_1v)$ is in L^{r_1} with $r_1 > \frac{n+2}{2}$. Application of the Hölder continuity result in [20, Theorem 10.1, p. 204] to (4.1) yields

$$u \in C^{\beta, \frac{\beta}{2}}(\overline{Q}_T) \quad \text{with some } \beta > 0. \tag{4.2}$$

Moreover, we have

$$w_{1t} = (d_1 + 2\alpha_{11}u)\Delta w_1 + f_1, \tag{4.3}$$

where $w_1 = (d_1 + \alpha_{11}u)u$ is as in the proof of Lemma 2.3, $f_1 = (d_1 + 2\alpha_{11}u)u(a_1 - b_1u - c_1v)$. Since u is bounded and by the assumption of this Lemma, we see that f_1 is in $L^{r_1}(Q_T)$. From (4.2), Lemma 2.1 and Proposition 3.3, applying [20, Theorem 9.1, pp. 341-342] and its remark [20, P. 351], we have

$$w_1 \in W^{2,1}_{r_1}(Q_T). \tag{4.4}$$

This implies $\nabla u = \frac{1}{d_1 + 2\alpha_{11}u}\nabla w_1$ in $L^{r_1}(Q_T)$. Now, following the proof of Proposition 3.3 with r_1 instead of r_0 and $p_1 = \frac{2r_1}{r_1 - 2}$ instead of p_0, we see that either v is in $L^r(Q_T)$ for any $r > 1$ or else v is in $L^{r_2}(Q_T)$ with

$$r_2 := \frac{(n+1)r_1}{n+2-r_1}.$$

The later case happens if and only if $n + 2 - r_1 > 0$.

If v is in $L^{r_2}(Q_T)$, we see that f_1 is in $L^{r_2}(Q_T)$. Therefore, applying [20, Theorem 9.1, p. 341-342] and its remark [20, p. 351] again, we have ∇u in $L^{r_2}(Q_T)$. Then we go back and do the same argument again. Keep doing likes this we will get a sequence of numbers

$$r_{k+1} := \frac{(n+1)r_k}{n+2-r_k}. \tag{4.5}$$

We stop and get the conclusion that v is in $L^r(Q_T)$ for any $r > 1$ when

$$n + 2 - r_k \le 0. \tag{4.6}$$

Since $r_1 > 3$, from (4.5) we can prove by induction that $r_k > 3, k = 1, 2, \dots$. Then, we have

$$\frac{r_{k+1}}{r_k} = \frac{n+1}{n+2-r_k} \ge \frac{n+1}{n-1} > 1. \tag{4.7}$$

Thus, the sequence r_k is strictly increasing. Therefore, there must be some k such that (4.6) holds. we stop at this k and conclude that v is in $L^r(Q_T)$ for any $r > 1$, namely, there is a positive constant C_{16} such that $\|v\|_{L^r(Q_T)} \le C_{16}$, where $C_{16} > 0$ depending on q, T, Ω and the coefficients of the system (1.2) but not on r. □

So, from Proposition 3.3 and Lemma 4.1, we have the following lemma.

Lemma 4.2. *Let $\alpha_{22} > 0$ and suppose* (i) $\alpha_{11} = 0$ *or* (ii) $\alpha_{11} > 0$ *and* $n < 10$. *Then there exists M_2 such that*

$$\|v\|_{L^r(Q_T)} \le M_2 \quad \textit{for any } r > 1.$$

Moreover, for any $r > 1$, v is in $V_2(Q_T)$.

Proof of Theorem 1.2. We give the proof only in case $\alpha_{11} > 0$ because the proof for $\alpha_{11} = 0$ is essentially the same. By Lemma 4.2, v is bounded in $\overline{Q}_T$. From (4.3), we have

$$w_{1t} = (d_1 + 2\alpha_{11}u)\Delta w_1 + f_1,$$

where $f_1 = (d_1 + 2\alpha_{11}u)u(a_1 - b_1u - c_1v)$ is bounded in $\overline{Q}_T$ by Lemma 2.1 and Lemma 4.2, $(d_1 + 2\alpha_{11}u) \in C^{\beta,\frac{\beta}{2}}(Q_T)$ by (4.2). By [20, Theorem 9.1, p.341-342], we have

$$\|w_1\|_{W_r^{2,1}}(Q_T) < M_3 \quad \text{for } \frac{n+2}{2} < r < \frac{4(n+1)}{(n-2)_+}.$$

Hence it follows from [20, Lemma 3.3, p.80] that

$$w_1 \in C^{1+\beta^*,\frac{(1+\beta^*)}{2}}(\overline{Q}_T), \quad \forall\, 0 < \beta^* < 1. \tag{4.8}$$

Since $u = \frac{-d_1+\sqrt{d_1^2+4w_1\alpha_{11}}}{2\alpha_{11}}$, it follow from (4.8) that

$$u \in C^{1+\beta^*,\frac{(1+\beta^*)}{2}}(\overline{Q}_T), \quad \forall\, 0 < \beta^* < 1. \tag{4.9}$$

Next, we rewrite the equation for v in divergence form as

$$v_t = \nabla\cdot[(d_2 + \alpha_{21}u + 2\alpha_{22}v)\nabla v + \alpha_{21}v\nabla u] + f_2(x,t),$$

where $f_2(x,t) = v(a_2 + b_2u - c_2v)$, u, v and ∇u are all bounded functions because of Lemma 2.1, Lemma 4.2 and (4.9). By [20, Theorem 10.1, p.204], we have

$$v \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)\text{with some} 0 < \sigma < 1. \tag{4.10}$$

Now, we then return to the equation for u and write it as

$$u_t = (d_1 + 2\alpha_{11}u)\Delta u + f_3(x,t), \tag{4.11}$$

where $f_3(x,t) = 2\alpha_{11}|\nabla u|^2 + u(a_1 - b_1u - c_1v) \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$ by (4.9) and (4.10). Then the Schuader estimate in [20, Theorem 5.3, p.320-321] applied to (4.11) yields

$$u \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T) \quad \text{with } \sigma^* = \min\{\lambda,\sigma\}. \tag{4.12}$$

Let $w_2 = (d_2 + \alpha_{21}u + \alpha_{22}v)v$. Then w_2 satisfies

$$w_{2t} = (d_2 + \alpha_{21}u + 2\alpha_{22}v)\Delta w_2 + f_4(x,t), \tag{4.13}$$

where $f_4(x,t) = (d_2 + \alpha_{21}u + 2\alpha_{22}v)v(a_2 + b_2u - c_2v) + \alpha_{21}vu_t \in C^{\sigma^*,\frac{\sigma^*}{2}}(\overline{Q}_T)$ by (4.11) and (4.12), $d_2 + \alpha_{21}u + 2\alpha_{22}v \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$ by (4.9) and (4.10), by applying the Schuader estimate to the equation (4.13), we have

$$w_2 \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T). \tag{4.14}$$

Then

$$v = \frac{-(d_2 + \alpha_{21}u) + \sqrt{(d_2 + \alpha_{21}u)^2 + 4w_2\alpha_{22}}}{2\alpha_{22}} \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T). \tag{4.15}$$

Now repeat the procedure by making use of (4.12) and (4.15) in place of (4.9) and (4.10), we have

$$u, v \in C^{2+\lambda,\frac{2+\lambda}{2}}(\overline{Q}_T). \tag{4.16}$$

Finally, the estimates (4.12) and (4.15) imply that the hypotheses of Theorem 1.1 are satisfied. So that (u,v) exists globally in time. The proof of Theorem 1.2 is now complete. □

5. Stability

In this section, we discuss global asymptotic stability of positive equilibrium point $(\overline{u}, \overline{v})$ for (1.2), namely to prove Theorem 1.3.

Proof of Theorem 1.3. Define the Lyapunov function:

$$H(u,v) = \int_\Omega \big[(u - \bar{u} - \bar{u}\ln\frac{u}{\bar{u}}) + \rho(v - \bar{v} - \bar{v}\ln\frac{v}{\bar{v}})\big]dx,$$

where $\rho = (b_2c_1 + 2b_1c_2)b_2^{-2}$. Obviously, $H(u,v)$ is nonnegative and $H(u,v) = 0$ if and only if $(u,v) = (\overline{u}, \overline{v})$. By Theorem 1.2, $H(u,v)$ is well-posed for $t \geq 0$ if (u,v) is positive solution to system (1.2). The time derivative of $H(u,v)$ for system (1.2) satisfies

$$\begin{aligned}
&\frac{dH(u,v)}{dt} \\
&= \int_\Omega \big(\frac{u-\bar{u}}{u}u_t + \rho\frac{v-\bar{v}}{v}v_t\big)dx \\
&= \int_\Omega \{\frac{u-\bar{u}}{u}\nabla\cdot[(d_1 + 2\alpha_{11}u)\nabla u] + (u-\bar{u})(a_1 - b_1u - c_1v) \\
&\quad + \rho\frac{v-\bar{v}}{v}\nabla\cdot[(d_2 + \alpha_{21}u + 2\alpha_{22}v)\nabla v + \alpha_{21}v\nabla u] + \rho(v-\bar{v})(a_2 + b_2u - c_2v)\}dx \\
&= -\int_\Omega \big[\frac{(d_1 + 2\alpha_{11}u)\bar{u}}{u^2}|\nabla u|^2 + \frac{\rho\alpha_{21}\bar{v}}{v}\nabla u\cdot\nabla v + \frac{\rho(d_2 + \alpha_{21}u + 2\alpha_{22}v)\bar{v}}{v^2}|\nabla v|^2\big]dx \\
&\quad - \int_\Omega [b_1(u-\bar{u})^2 + (c_1 - \rho b_2)(u-\bar{u})(v-\bar{v}) + c_2\rho(v-\bar{v})^2]dx.
\end{aligned}$$

The second integrand in the above equality is positive definite by the choice of ρ. Meanwhile the first integrand is positive semi-definite if

$$4\rho\overline{uv}(d_1 + 2\alpha_{22}u)(d_2 + \alpha_{21}u + 2\alpha_{22}v) > u^2(\alpha_{21}\overline{v})^2. \tag{5.1}$$

By the Lemma 2.1 and Theorem 1.2, the condition (1.4) implies (5.1). Therefore, when all conditions in Theorem 1.3 hold, there exists positive constant δ depending on b_1, b_2, c_1 and c_2 such that

$$\frac{dH(u,v)}{dt} \leq -\delta\int_\Omega [(u-\bar{u})^2 + (v-\bar{v})^2]dx. \tag{5.2}$$

To obtain the uniform convergence of the solution to (1.2), we recall the following result which can be find in [21].

Lemma 5.1. *Let a and b positive constant. Assume that $\varphi, \psi \in C^1[a,+\infty)$, $\psi(t) \geq 0$, φ is bounded. If $\varphi'(t) \leq -b\psi(t)$ and $\psi'(t)$ is bounded in $[a,+\infty)$, then* $\lim_{t\to\infty}\psi(t) = 0$.

Using integration by parts, Hölder's inequality, Lemma 2.1, and Lemma 4.2, one can easily verify that $\frac{d}{dt}\int_\Omega[(u-\bar{u})^2 + (v-\bar{v})^2]dx$ is bounded from above. Then from Lemma 5.1 and (5.2), we have

$$\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot,t) - \overline{v}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$

Namely, (u,v) converges uniformly to $(\overline{u}, \overline{v})$. By the fact that $H(u,v)$ is decreasing for $t \geq 0$, it is obvious that $(\overline{u}, \overline{v})$ is global asymptotic stable, and the proof of Theorem 1.3 is complete. □

Acknowledgements. The author would like to thank professor Sheng-mao Fu for the encouragement and useful discussions, also the anonymous referee for the very careful reading of the original manuscript and the helpful suggestions.

References

[1] N. Shigesada, K. Kawasaki, E. Teramoto, *Spatial segregation of interacting species*, J. Theor. Biology, 79(1979), 83-99.
[2] Y. Lou, W. Ni, Y. Wu, *On the global existence of a cross-diffusion system*, Discrete Contin. Dynam. Systems, 4(1998), 193-203.
[3] Y. Li, C. Zhao, *Global existence of solutions to a cross-diffusion system in higher dimensional domains*, Discrete Contin. Dynam. Systems, 12(2005), 193-203.
[4] Seong-A Shim, *Uniform boundedness and convergence of solutions to the systems with cross-diffusion dominated by self-diffusion*, Nonlinear Analysis: RWA, 4(2003), 65-86.
[5] Y. S. Choi, R. Lui, Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete Contin. Dynam. Systems, 9(2003), 1193-1200.
[6] Y. S. Choi, R. Lui, Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete Contin. Dynam. Systems, 10(2004), 719-730.
[7] D. Le, T. Nguyen, *Global existence for a class of triangular parabolic system on domains of arbitary dimension*, Proceedings of AMS, 7(2005), 1985-1992.
[8] P. V. Tuoc, *Global eistence of solutions to Shigesada-Kawasaki-Teramoto cross-diffusion system on domains of arbitary dimensions*, Proceedings of AMS, 135(2007), 3933-3941.
[9] P. V. Tuoc, *On global existence of solutions to a cross-diffusion system*, IMA Preprint Series 2149.
[10] K. Kuto, Y. Yamada, *Multiple coexistence states for a prey-predator system with cross-diffusion*, J. Differential Equations, 197(2004), 315-348.
[11] E. Ahmed, A. S. Hegazi, A. S. Elgazzar, *On persistence and stability of some biological systems with cross diffusion*, Adv. Compelex Syst, 7(2004), 65-76.
[12] K. Kuto, *Stability of steady-state solutions to a prey-predator system with cross-diffusion*, J. Differential Equations, 197(2004), 293-314.
[13] K. Nakashima, Y. Yamada, *Positive steady ststes for prey-predator models with cross-diffusion*, Adv. Differential Equations, 1(1996), 1099-1122.
[14] K. Ryu, I. Ahn, *Positive steady-ststes for two interacting species models with linear self-cross diffusions*, Discrete Contin. Dynam. Systems, 9(2003), 1049-1061.
[15] H. Amann, *Dynamic theory of quasilinear parabolic equations: Abstract evolution equations*, Nonlinear Analysis, 12(1988), 859-919.
[16] H. Amann, *Dynamic theory of quasilinear parabolic equations: Reaction-diffusion*, Diff. Int. Eqs, 3(1990), 13-75.
[17] H. Amann, *Dynamic theory of quasilinear parabolic equations: Global existence*, Math. Z., 202(1989), 219-250.
[18] Seong-A Shim, *Long-time properties of prey-predator system with cross-diffusion*, Commum. Korean Math. Soc. 21(2006), 293-320.
[19] Seong-A Shim, *Global existence of solutions to the prey-predator system with a singel cross-diffusion*, Bull. Korean Math. Soc. 43(2006), 443-459.
[20] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, 23, AMS, 1968.
[21] M. X. Wang, *Nonlinear Parabolic Equation of Parabolic Type*. Science Press, Beijing, 1993 (in Chinese).

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
E-mail address: xuluck2001@163.com