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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We show that solutions of a nonlinear parabolic equation of second order with nonlinear boundary conditions approach zero as t approaches infinity. Also, under additional assumptions, the solutions behave as a function determined here.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$\frac{\partial\varphi(u)}{\partial t} - Lu + f(x,t,u) = 0 \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

$$\frac{\partial u}{\partial N} + g(x,t,u) = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \overline{\Omega}, \tag{1.3}$$

where

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i}, \quad \frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} \cos(\nu, x_i) a_{ij}(x)\frac{\partial u}{\partial x_j}.$$

Here the coefficients $a_{ij}(x) \in C(\Omega)$ satisfy the inequality

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C|\xi|^2 \quad \text{for } \xi\in\mathbb{R}^n,\ \xi\neq 0,\ C>0,$$

$a_{ij}(x) = a_{ji}(x)$, ν is the exterior normal unit vector on $\partial\Omega$, $f_{x,t}(s) = f(x,t,s)$ and $g_{x,t}(s) = g(x,t,s)$ are positive, increasing and convex functions for $s \geq 0$ with $f_{x,t}(0) = f'_{x,t}(0) = g_{x,t}(0) = g'_{x,t}(0) = 0$. For positive values of s, $\varphi(s)$ is a positive and concave function. Throughout this paper, we assume the following condition:

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(H0) There exist functions $f_*(s)$, $g_*(s)$ of class $C^1([0,\infty))$, positive for positive values of s such that for any $\alpha(t)$ tending to zero as $t\to\infty$,
$$\lim_{t\to\infty}\frac{f(x,t,\alpha(t))}{f_*(\alpha(t))}=a(x),\quad \lim_{t\to\infty}\frac{g(x,t,\alpha(t))}{g_*(\alpha(t))}=b(x),$$
$$\frac{f_*}{\varphi'}(0)=\frac{g_*}{\varphi'}(0)=(\frac{f_*}{\varphi'})'(0)=(\frac{g_*}{\varphi'})'(0)=0,$$
where $a(x)$ is a bounded nonnegative function in Ω and $b(x)$ is a bounded nonnegative function on $\partial\Omega$.

Existence of positive classical solutions, local in time, was proved by Ladyzenskaya, Solonnikov and Ural'ceva in [9]. In this paper, we are dealing with the asymptotic behavior as $t\to\infty$ of positive solutions of (1.1)–(1.3). The asymptotic behavior of solutions for parabolic equations has been the subject of study of many authors (see, for instance [1, 2, 3, 4, 6, 7, 10]. In particular, Kondratiev and Oleinik [6] considered the problem
$$\frac{\partial u}{\partial t}-Lu+a|u|^{p-1}u=0\quad\text{in } \Omega\times(0,\infty), \tag{1.4}$$
$$\frac{\partial u}{\partial N}=0\quad\text{on } \partial\Omega\times(0,\infty), \tag{1.5}$$
$$u(x,0)=u_0(x)\quad\text{in } \overline{\Omega}, \tag{1.6}$$
where $p>1$, and a is a positive constant. They proved that if u is a positive solution of Problem (1.4)–(1.6), then
$$\lim_{t\to\infty}t^{\frac{1}{p-1}}u(x,t)=\Big(\frac{p-1}{|\Omega|}\int_\Omega av_1(x)dx\Big)^{\frac{-1}{p-1}} \tag{1.7}$$
uniformly in $x\in\Omega$, where $v_1(x)$ is a positive solution of the boundary value problem
$$\begin{gathered}L^*(v)=0\quad\text{in }\Omega\\ \frac{\partial v}{\partial N}=\sum_{i=1}^n a_i(x)\cos(\nu,x_i)v\quad\text{on }\partial\Omega,\end{gathered} \tag{1.8}$$
with
$$L^*(v)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial v}{\partial x_j})-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}(a_i(x)v).$$
Notice that Problem (1.8) is the adjoint of the Neumann problem for the operator L. The same result with $v_1(x)=1$, $a=a(x)$ has been also obtained in [2] and [7] in the case where $a(x)$ is a bounded function in Ω and $a_i(x)=0$ $(i=1,\dots,n)$ (i.e. the operator L is self-adjoint). In [4], the second author has shown similar results about the asymptotic behavior of solutions for another particular case of Problem (1.1)–(1.3) which corresponds to this last for $a_i(x)=0$ $(i=1,\dots,n)$, $\varphi(u)=u$, $f(x,t,u)=a(x)f_*(u)$, $g(x,t,u)=b(x)g_*(u)$. Our aim in this paper is to generalize the above results, describing the asymptotic behavior of solutions for Problem (1.1)–(1.3). Our paper is written in the following manner. Under some conditions, we obtain in the next section the asymptotic behavior of positive solutions for Problem (1.1)–(1.3).

Introduce the function class Z_p defined as follows: $u \in Z_p$ if u is continuous in $\overline{G}$, $\frac{\partial u}{\partial x_i} \in G'$ and $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x_i \partial x_j} \in G$, where $G = \Omega \times (0,\infty)$, $G' = \overline{\Omega} \times (0,\infty)$, and $\overline{G}$ is the closure of G.

2. Asymptotic behavior

In this section, we show that under some assumptions, any positive solution $u \in Z_p$ of Problem (1.1)–(1.3) tends to zero as $t \to \infty$ uniformly in $x \in \Omega$. We also describe its asymptotic behavior as $t \to \infty$. The following lemma will be useful later.

Lemma 2.1. *Let $u, v \in Z_p$ satisfying the following inequalities*

$$\begin{gathered}
\frac{\partial \varphi(u)}{\partial t} - Lu + f(x,t,u) > \frac{\partial \varphi(v)}{\partial t} - Lv + f(x,t,v) \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial u}{\partial N} + g(x,t,u) > \frac{\partial v}{\partial N} + g(x,t,v) \quad \text{on } \partial\Omega \times (0,\infty), \\
u(x,0) > v(x,0) \quad \text{in } \overline{\Omega}.
\end{gathered}$$

Then we have $u(x,t) > v(x,t)$ in $\Omega \times (0,\infty)$.

Proof. The function $w(x,t) = u(x,t) - v(x,t)$ is continuous in $\overline{\Omega} \times [0,\infty)$. Then its minimum value m is attained at a point $(x_0,t_0) \in \overline{\Omega} \times [0,\infty]$. If $t_0 = 0$, then $m > 0$. If $0 < t_0 \leq \infty$, suppose that there exists t_1 such that $0 < t_1 \leq t_0$ with $u(x,t) > v(x,t)$ for $0 \leq t < t_1$ but $u(x_1,t_1) = v(x_1,t_1)$ for some $x_1 \in \overline{\Omega}$.
If $x_1 \in \Omega$ then we have

$$\frac{\partial \varphi(u) - \varphi(v)}{\partial t}(x_1,t_1) \leq 0, \quad Lw(x_1,t_1) \geq 0, \quad f(u(x_1,t_1)) = f(v(x_1,t_1)).$$

Consequently, we have a contradiction because

$$\frac{\partial \varphi(u) - \varphi(v)}{\partial t}(x_1,t_1) - Lw(x_1,t_1) + [f(x_1,t_1,u(x_1,t_1)) - f(x_1,t_1,v(x_1,t_1))] > 0.$$

Finally if $x_1 \in \partial\Omega$, then $\frac{\partial w}{\partial N}(x_1,t_1) \leq 0$. We have again an absurdity because of the fact that

$$\frac{\partial w}{\partial N}(x_1,t_1) + [g(x_1,t_1,u(x_1,t_1)) - g(x_1,t_1,v(x_1,t_1))] > 0.$$

Therefore we have $m > 0$. $\square$

For the limit of $f_*(t)/g_*(t)$ as $t \to 0$, we have the following possibilities:

(P1) $\lim_{t\to 0} \frac{f_*(t)}{g_*(t)} = 0$,
(P2) $\lim_{t\to 0} \frac{f_*(t)}{g_*(t)} = \infty$,
(P3) $\lim_{t\to 0} \frac{f_*(t)}{g_*(t)} = C_*$, where C_* is a positive constant.

Let ε_f and ε_g be such that:

(H1) $\varepsilon_f = 0$, $\varepsilon_g = 1$ if (P1) is satisfied;
(H2) $\varepsilon_f = 1$, $\varepsilon_g = 0$ if (P2) is satisfied;
(H3) $\varepsilon_f = \sqrt{\frac{C_*}{1+C_*}}$, $\varepsilon_g = \sqrt{\frac{C_*}{1+C_*}}$ if (P3) is satisfied.

Assumption (P1) is always used with the coefficients ε_f, ε_g defined in (H1)–(H3). The function

$$h(t) = \varepsilon_f f_*(t) + \varepsilon_g g_*(t) \tag{2.1}$$

is crucial for the study of asymptotic behavior of solutions. Let

$$G(s) = \int_s^1 \frac{\varphi'(t)dt}{h(t)} \tag{2.2}$$

and let $H(s)$ be the inverse function of $G(s)$. In this notation the initial-value problem

$$\varphi'(\beta(t))\beta'(t) = -\lambda h(\beta(t)), \quad \beta(0) = 1 \quad (\lambda > 0) \tag{2.3}$$

has the unique solution $\beta(t) = H(\lambda t)$. It follows from $\frac{h}{\varphi'}(0) = (\frac{h}{\varphi'})'(0) = 0$ that $0 < \frac{h(t)}{\varphi'(t)} < t$ for $0 < t < \delta$ ($\delta > 0$) and hence

$$G(0) = \infty, \quad G(1) = 0 \quad \textit{and} \quad H(0) = 1, \quad H(\infty) = 0, \tag{2.4}$$

which implies that $\beta(\infty) = 0$. The function $\beta(t)$ will be used later in the construction of supersolutions and subsolutions of (1.1)–(1.3) to obtain the asymptotic behavior of solutions.

Remark 2.2. If (P1)–(P3) are satisfied, then

$$\lim_{t\to\infty}\{-\varepsilon_f a(x) + \frac{f(x,t,\beta(t))}{h(\beta(t))}\} = 0,$$
$$\lim_{t\to\infty}\{-\varepsilon_g b(x) + \frac{g(x,t,\beta(t))}{h(\beta(t))}\} = 0.$$

In the following theorems, we suppose that (P1) or (P2) or (P3) is satisfied. Consider the boundary-value problem

$$-\lambda - L\psi = -\varepsilon_f a(x) + \delta, \quad \frac{\partial\psi}{\partial N} = -\varepsilon_g b(x) + \delta. \tag{2.5}$$

This problem has a solution if and only if

$$\delta\Big(\int_\Omega v_0(x)dx + \int_{\partial\Omega} v_0(x)ds\Big) = I(a,b) - \lambda\int_\Omega v_0(x)dx, \tag{2.6}$$

where $v_0(x)$ is a solution of Problem (1.8) and

$$I(a,b) = \varepsilon_g\int_{\partial\Omega} b(x)v_0(x)ds + \varepsilon_f\int_\Omega a(x)v_0(x)dx, \tag{2.7}$$

(see, for instance [6]). Thus in this paper, for problem (2.5), we suppose that for given $\lambda > 0$, δ satisfies (2.6), which implies that problem (2.5) has a solution ψ. Without loss of generality, we may suppose that $\psi > 0$. Indeed, when ψ is a solution of (2.5), we see that $\psi + C$ is also a solution of (2.5) for any constant $C > 0$. The function ψ will be used later to construct supersolutions and subsolutions of (1.1)–(1.3) for getting the asymptotic behavior of solutions. The function $v_0(x)$ does not change sign in Ω. We shall suppose that $v_0(x) > 0$ in Ω. If $a_i(x) = 0$, then the operator$L$ is self-adjoint and $v_0(x) = 1$.

Theorem 2.3. (i) *Suppose that $I(a,b) > 0$ and $\lim_{s\to 0}\frac{h(s)\varphi''(s)}{\varphi'(s)} = 0$. If $u \in Z_p$ is a positive solution of* (1.1)–(1.3)*, then*

$$\lim_{t\to\infty} u(x,t) = 0$$

uniformly in $x \in \overline{\Omega}$.

(ii) *Moreover if there exists a positive constant c_2 such that*

$$\lim_{s\to\infty}\frac{sh(H(s))}{H(s)\varphi'(H(s))}\le c_2,$$

we have $u(x,t)=H(c_{fg}t)(1+o(1))$ as $t\to\infty$, where $c_{fg}=\frac{I(a,b)}{\int_\Omega v_0(x)dx}$.

Proof. (i) Put $w(x,t)=\beta(t)+\psi(x)h(\beta(t))$, where $\beta(t)$ and $\psi(x)$ are solutions of (2.3) and (2.5) respectively for $\lambda\le\frac{I(a,b)}{2\int_\Omega v_0(x)dx}$, which implies that $\delta>0$. A straightforward computation reveals that

$$\begin{aligned}
&\frac{\partial\varphi(w)}{\partial t}-Lw+f(x,t,w)\\
&=h(\beta(t))(-\lambda-L\psi)-\lambda h(\beta(t))h'(\beta(t))\psi(x)+f(x,t,\beta(t))+\psi(x)h(\beta(t))f'_{x,t}(y)\\
&\quad-\lambda\psi(x)\frac{h^2(\beta(t))\varphi''(z)}{\varphi'(\beta(t))}-\lambda\psi^2(x)\frac{h^2(\beta(t))h'(\beta(t))\varphi''(z)}{\varphi'(\beta(t))},
\end{aligned}$$

$$\frac{\partial w}{\partial N}+g(x,t,w)=h(\beta(t))\frac{\partial\psi}{\partial N}+g(x,t,\beta(t))+\psi(x)h(\beta(t))g'_{x,t}(l),$$

with $\{l,y,z\}\in[\beta(t),\beta(t)+\psi(x)h(\beta(t))]$. It follows from (2.5) that

$$\begin{aligned}
&\frac{\partial\varphi(w)}{\partial t}-Lw+f(x,t,w)\\
&=(\delta-\varepsilon_f a(x))h(\beta(t))-\lambda h(\beta(t))h'(\beta(t))\psi(x)+f(x,t,\beta(t))+\psi(x)h(\beta(t))f'_{x,t}(y)\\
&\quad-\lambda\psi(x)\frac{h^2(\beta(t))\varphi''(z)}{\varphi'(\beta(t))}-\lambda\psi^2(x)\frac{h^2(\beta(t))h'(\beta(t))\varphi''(z)}{\varphi'(\beta(t))},
\end{aligned}$$

$$\frac{\partial w}{\partial N}+g(x,t,w)=(\delta-\varepsilon_g b(x))h(\beta(t))+g(x,t,\beta(t))+\psi(x)h(\beta(t))g'_{x,t}(l).$$

Since $f'_{x,\infty}(0)=g'_{x,\infty}(0)=0$, $\lim_{s\to0}\frac{h(s)\varphi''(s)}{\varphi'(s)}=0$, using Remark 2.1, there exists $t_1\ge0$ such that

$$\begin{gathered}
\frac{\partial\varphi(w)}{\partial t}-Lw+f(x,t,w)>0\quad\text{in }\Omega\times(t_1,\infty),\\
\frac{\partial w}{\partial N}+g(x,t,w)>0\quad\text{on }\partial\Omega\times(t_1,\infty).
\end{gathered}$$

Let $k>1$ be large enough that

$$u(x,t_1)<kw(x,t_1)\quad\text{in }\overline{\Omega}.$$

Since $f_{x,t}(s)$ and $g_{x,t}(s)$ are convex with $f_{x,t}(0)=g_{x,t}(0)$, $\varphi(s)$ is concave and $w_t\le0$, we get

$$\begin{gathered}
\frac{\partial\varphi(kw)}{\partial t}-Lkw+f(x,t,kw)>0\quad\text{in }\Omega\times(t_1,\infty),\\
\frac{\partial kw}{\partial N}+g(x,t,kw)>0\quad\text{on }\partial\Omega\times(t_1,\infty).
\end{gathered}$$

It follows from Comparison Lemma 2.1 that

$$u(x,t_1+t)<kw(x,t_1+t)\quad\text{in }\Omega\times(0,\infty).$$

Since $\lim_{t\to\infty}w(x,t)=0$ uniformly in $x\in\overline{\Omega}$, we have the result. $\square$

The proof of Theorem 2.3 (ii) is based on the following lemmas:

Lemma 2.4. *Under the hypotheses of Theorem 2.3 (i), if $u \in Z_p$ is a positive solution of problem* (1.1)–(1.3)*, then for any $\varepsilon > 0$ small enough, there exist τ and T such that*

$$u(x, t+\tau) \le \beta_1(t+T) + \psi_1(x)h(\beta_1(t+T)),$$

where $\beta_1(t)$ and $\psi_1(x) > 0$ are solutions of (2.3) *and* (2.5) *respectively for $\lambda = c_{fg} - \frac{\varepsilon}{2}$.*

Proof. Put

$$w_1(x,t) = \beta_1(t) + \psi_1(x)h(\beta_1(t)).$$

Since $c_{fg} = I(a,b)/\int_\Omega v_0(x)dx$, it follows that

$$\delta = \frac{\varepsilon \int_\Omega v_0(x)dx}{2(\int_\Omega v_0(x)dx + \int_{\partial\Omega} v_0(x)dx)},$$

which implies that for any $\varepsilon > 0$ small enough $\delta > 0$ and as in the proof of Theorem 2.3 (i), there exists $T \ge 0$ such that

$$\begin{gathered}
\frac{\partial \varphi(w_1)}{\partial t} - Lw_1 + f(x,t,w_1) > 0 \quad \text{in } \Omega \times (T,\infty), \\
\frac{\partial w_1}{\partial N} + g(x,t,w_1) > 0 \quad \text{on } \partial\Omega \times (T,\infty).
\end{gathered}$$

Since $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in \overline{\Omega}$, there exists a $\tau > T$ such that

$$u(x,\tau) < w_1(x,T) \quad \text{in } \overline{\Omega}.$$

Set $z_1(x,t) = w_1(x, T-\tau+t)$ in $\overline{\Omega} \times (\tau,\infty)$. We have

$$\begin{gathered}
z_1(x,\tau) = w_1(x,T) > u(x,\tau) \quad \text{in } \overline{\Omega}, \\
\frac{\partial \varphi(z_1)}{\partial t} = \frac{\partial \varphi(w_1)}{\partial t} \quad \text{in } \Omega \times (\tau,\infty), \\
Lz_1 = Lw_1 \quad \text{in } \Omega \times (\tau,\infty), \\
\frac{\partial z_1}{\partial N} = \frac{\partial w_1}{\partial N} \quad \text{on } \partial\Omega \times (\tau,\infty).
\end{gathered}$$

Therefore,

$$\begin{gathered}
\frac{\partial \varphi(z_1)}{\partial t} - Lz_1 + f(x,t,z_1) > 0 \quad \text{in } \Omega \times (\tau,\infty), \\
\frac{\partial z_1}{\partial N} + g(x,t,z_1) > 0 \quad \text{on } \partial\Omega \times (\tau,\infty), \\
z_1(x,\tau) > u(x,\tau) \quad \text{in } \overline{\Omega}.
\end{gathered}$$

It follows from Comparison Lemma 2.1 that

$$u(x,t+\tau) \le w_1(x,t+T) = \beta_1(t+T) + \psi_1(x)h(\beta_1(t+T)),$$

which yields the result. □

Lemma 2.5. *Under the hypotheses of Theorem 2.3 (i), if $u \in Z_p$ is a positive solution of* (1.1)–(1.3)*, then for any $\varepsilon > 0$ small enough, there exists T_2 such that*

$$u(x,t+\tau) \ge \beta_2(t+T_2) + \psi_2(x)h(\beta_1(t+T_2)),$$

where $\beta_2(t)$ and $\psi_2(x) > 0$ are solutions of (2.3) *and* (2.5) *respectively for $\lambda = c_{fg} + \frac{\varepsilon}{2}$.*

Proof. Put
$$w_2(x,t)=\beta_2(t)+\psi_1(x)h(\beta_2(t)).$$
Since $c_{fg}=\frac{I(a,b)}{\int_\Omega v_0(x)dx}$, it follows that
$$\delta=\frac{-\varepsilon\int_\Omega v_0(x)dx}{2(\int_\Omega v_0(x)dx+\int_{\partial\Omega} v_0(x)dx)},$$
which implies that for any $\varepsilon>0$ small enough $\delta<0$. As in the proof of Theorem 2.3 (i), w_2 satisfies
$$\begin{aligned}
&\frac{\partial\varphi(w_2)}{\partial t}-Lw_2+f(x,t,w_2)\\
&=(\delta-\varepsilon_f a(x))h(\beta_2(t))\\
&\quad-(c_{fg}+\frac{\varepsilon}{2})h(\beta_2(t))h'(\beta_2(t))\psi(x)+f(x,t,\beta_2(t))+\psi(x)h(\beta_2(t))f'_{x,t}(y_2),\\
&\quad-(c_{fg}+\frac{\varepsilon}{2})\psi(x)\frac{h^2(\beta(t))\varphi''(z_2)}{\varphi'(\beta(t))}-(c_{fg}+\frac{\varepsilon}{2})\psi^2(x)\frac{h^2(\beta(t))h'(\beta(t))\varphi''(z_2)}{\varphi'(\beta(t))},
\end{aligned}$$
$$\frac{\partial w_2}{\partial N}+g(x,t,w_2)=(\delta-\varepsilon_g b(x))h(\beta_2(t))+g(x,t,\beta_2(t))+\psi(x)h(\beta_2(t))g'_{x,t}(l_2).$$
with $\{y_2,z_2,l_2\}\in[\beta_2(t),\beta_2(t)+\psi_2(x)h(\beta_2(t))]$. Since $f'_{x,\infty}(0)=g'_{x,\infty}(0)=0$, $\lim_{s\to0}\frac{h(s)\varphi''(s)}{\varphi'(s)}=0$, using Remark 2.1, for any $\varepsilon>0$ small enough, there exists $T_1>0$ such that
$$\begin{gathered}
\frac{\partial\varphi(w_2)}{\partial t}-Lw_2+f(x,t,w_2)<0\quad\text{in }\Omega\times(T_1,\infty),\\
\frac{\partial w_2}{\partial N}+g(x,t,w_2)<0\quad\text{on }\partial\Omega\times(T_1,\infty).
\end{gathered}$$
Since $\lim_{t\to\infty}w_2(x,t)=0$ uniformly for $x\in\overline{\Omega}$, then there exists a $T_2>T_1$ such that
$$u(x,\tau)>w_2(x,T_2)\quad\text{in }\overline{\Omega}.$$
Set
$$z_2(x,t)=w_2(x,T_2-\tau+t)\quad\text{in }\overline{\Omega}\times(\tau,\infty).$$
We get
$$\begin{gathered}
z_2(x,\tau)=w_2(x,T_2)<u(x,\tau)\quad\text{in }\overline{\Omega},\\
\frac{\partial\varphi(z_2)}{\partial t}=\frac{\partial\varphi(w_2)}{\partial t}\quad\text{in }\Omega\times(\tau,\infty),\\
Lz_2=Lw_2\quad\text{in }\Omega\times(\tau,\infty),\\
\frac{\partial z_2}{\partial N}=\frac{\partial w_2}{\partial N}\quad\text{on }\partial\Omega\times(\tau,\infty).
\end{gathered}$$
Hence, we find that
$$\begin{gathered}
\frac{\partial\varphi(z_2)}{\partial t}-Lz_2+f(x,t,z_2)<0\quad\text{in }\Omega\times(T_2,\infty),\\
\frac{\partial z_2}{\partial N}+g(x,t,z_2)<0\quad\text{on }\partial\Omega\times(T_2,\infty),\\
z_2(x,\tau)<u(x,\tau)\quad\text{in }\overline{\Omega}.
\end{gathered}$$

It follows from Comparison Lemma 2.1 that

$$u(x,t+\tau) \le w_2(x,t+T) = \beta_2(t+T) + \psi_2(x)h(\beta_2(t+T)),$$

which gives the result. $\square$

Lemma 2.6. *Let $\beta(t,\lambda)$ be a solution of Problem* (2.3). *Then*

(i) *for $\gamma > 0$,*

$$\lim_{t\to\infty} \frac{\beta(t+\gamma,\lambda)}{\beta(t,\lambda)} = 1\,.$$

(ii) *if $\lim_{s\to\infty} \frac{sh(H(s))}{H(s)\varphi'(H(s))} \le c_2$ and $\alpha > 0$, then*

$$1 \ge \lim_{t\to\infty} \sup \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} \ge \lim_{t\to\infty} \inf \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} \ge 1 - \frac{c_2\alpha}{\lambda}, \tag{2.8}$$

$$1 \le \lim_{t\to\infty} \inf \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \le \lim_{t\to\infty} \sup \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \le 1 + \frac{2c_2\alpha}{\lambda}, \tag{2.9}$$

for α small enough.

Proof. (i) Since $\beta_\lambda(t) = \beta(t,\lambda)$ is decreasing and convex,

$$\beta(t,\lambda) - \gamma\lambda\frac{h(\beta(t,\lambda))}{\varphi'(\beta(t,\lambda))} \le \beta(t+\gamma,\lambda) \le \beta(t,\lambda),$$

which implies $\lim_{t\to\infty} \frac{\beta(\gamma+t,\lambda)}{\beta(t,\lambda)} = 1$ because $\lim_{s\to 0} \frac{h(s)}{s\varphi'(s)} = 0$.

(ii) We have

$$1 \ge \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} = \frac{H(\lambda t+\alpha)}{H(\lambda t)} \ge \frac{H(\lambda t) - \alpha t\frac{h(H(\lambda t))}{\varphi'(H(\lambda t))}}{H(\lambda t)}.$$

Since $\lim_{s\to\infty} \frac{h(H(s))}{H(s)\varphi'(H(s))} \le c_2$, we obtain (2.8). We also get by means of (2.8) the following inequalities:

$$1 \le \lim_{t\to\infty} \inf \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \le \lim_{t\to\infty} \sup \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \le \frac{1}{1-\frac{c_2\alpha}{\lambda-\alpha}} \le 1 + \frac{2c_2\alpha}{\lambda},$$

which yields (2.9). $\square$

Proof of Theorem 2.3 (ii). From Lemmas 2.4, 2.5 and 2.6, for any $\varepsilon > 0$ small enough, we have

$$1 - k_1\varepsilon \le \lim_{t\to\infty} \inf \frac{u(x,t)}{\beta(t)} \le \lim_{t\to\infty} \sup \frac{u(x,t)}{\beta(t)} \le 1 + k_2\varepsilon$$

where k_1 and k_2 are two positive constants. Consequently

$$u(x,t) = \beta(t)(1+o(1)) \quad as \quad t\to\infty,$$

which gives the result. $\square$

Remark 2.7. Let $\varphi(u) = u^m$, $f(x,t,u) = a_1(x,t)u^p$, $g(x,t,u) = b_1(x,t)u^q$ with $0 < m \le 1$, $\inf\{p,q\} > 1$. Assume that $\lim_{t\to\infty} a_1(x,t) = a(x)$, $\lim_{t\to\infty} b_1(x,t) = b(x)$,

$$\varepsilon_q \int_{\partial\Omega} b(x)ds + \varepsilon_p \int_\Omega a(x)dx > 0\,,$$

where $\varepsilon_p = 0$, $\varepsilon_q = 1$ if $p > q$, $\varepsilon_p = 1$, $\varepsilon_q = 0$ if $p < q$ and $\varepsilon_p = 1$, $\varepsilon_q = 1$ if $p = q$. If $u \in Z_p$ is a positive solution of Problem (1.1)–(1.3), then u tends to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$. Moreover

$$\begin{aligned}&\lim_{t\to\infty} \frac{u(x,t)}{t^{-\frac{1}{\inf\{p,q\}-m}}}\\ &= \Big(\frac{\inf\{p,q\}-m}{m\int_\Omega v_0(x)dx}[\varepsilon_q \int_{\partial\Omega} v_0(x)b(x)ds + \varepsilon_p \int_\Omega v_0(x)a(x)dx]\Big)^{\frac{1}{m-\inf\{p,q\}}}.\end{aligned}$$

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