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MULTIPLICITY RESULTS FOR FOURTH-ORDER BOUNDARY-VALUE PROBLEM AT RESONANCE WITH VARIABLE COEFFICIENTS

LING XU

Abstract. This paper studies the multiplicity of solutions for the fourth-order boundary value problem at resonance with variable coefficients

$$u^{(4)} + \beta(t)u'' - \lambda_1 u = g(t, u) + h(t), \quad t \in (0, 1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $\beta \in C[0, 1]$ with $\beta(t) < \pi^2$ on $[0, 1]$, $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is bounded continuous function, $h \in L^2(0, 1)$ and $\lambda_1 > 0$ is the first eigenvalue of the associated linear homogeneous boundary value problem

$$u^{(4)} + \beta(t)u'' - \lambda u = 0, \quad t \in (0, 1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

The proof of our main result is based on the connectivity properties of the solution sets of parameterized families of compact vector fields.

1. Introduction

A beam is one of the basic structures for engineering construction, so it is quite important to study beam equations in theory and practice. Generally, the deformations of an elastic beam can be described by the fourth-order ordinary differential equation. According to the different suspensory conditions for two ends, there are different fourth-order ordinary differential equations. Especially, the deformations of an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by the fourth-order two-point ordinary differential equation boundary value problem as follows

$$u^{(4)} = f(t, u, u''), \quad t \in (0, 1), \tag{1.1}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous on $[3, 4]$. Owing to its importance in physics, the existence of solutions and positive solutions to this problem have been studied by many authors. See [3, 4, 5, 6, 7, 8, 10]. Our ideas arise from [5, 6, 8]. Liu and Li [5, 6] studied the existence and multiplicity of solutions for the fourth-order

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boundary value problem with parameters

$$u^{(4)} + \eta u'' - \xi u = \lambda f(t, u), \quad t \in (0, 1), \tag{1.3}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.4}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, ξ, η and $\lambda \in \mathbb{R}$ are parameters. However, there are few papers concerning the existence of solutions for the fourth-order boundary value problem with variable coefficients. Now, a question that naturally arises is: are there any similar results happening to be the fourth-order boundary value problem with variable coefficients? In 2006, Ma [8] investigated the existence of nodal solutions of the fourth-order two-point boundary value problem at nonresonance with variable coefficient

$$u^{(4)} + \beta(t)u'' = a(t)f(u), \quad t \in (0, 1), \tag{1.5}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.6}$$

where $\beta \in C[0, 1]$ with $\beta(t) < \pi^2$ on $[0, 1]$, $a \in C[0, 1]$ with $a \geq 0$ on $[0, 1]$ and $a(t) \not\equiv 0$ on any subinterval of $[0, 1]$, $f \in C(\mathbb{R})$ satisfies $f(u)u > 0$ for all $u \neq 0$. But so far, very few multiplicity results were established for the fourth-order boundary value problem at resonance with variable coefficients.

In this paper, we consider nonexistence, existence and multiplicity of solutions for the fourth-order boundary value problem at resonance with variable coefficients

$$u^{(4)} + \beta(t)u'' - \lambda_1 u = g(t, u) + h(t), \quad t \in (0, 1), \tag{1.7}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0 \tag{1.8}$$

under the following assumptions:

- (H1) $\beta \in C[0, 1]$ with $\beta(t) < \pi^2$ on $[0, 1]$;
- (H2) $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is bounded continuous function; i.e., there exists a constant $M > 0$ such that

$$|g(t, u| \leq M, \quad t \in [0, 1], u \in \mathbb{R}.$$

Moreover, $\lambda_1 > 0$ is the first eigenvalue of the associated linear homogeneous boundary value problem

$$u^{(4)} + \beta(t)u'' - \lambda u = 0, \quad t \in (0, 1), \tag{1.9}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{1.10}$$

Therefore, the problem (1.7)-(1.8) is at resonance. From Ma [8], the problem (1.9)-(1.10) has an infinite sequence of positive eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$$

and to each eigenvalue λ_k there corresponds an essential unique eigenfunction $\varphi_k(t)$ which has exactly $k - 1$ simple zeros in (0, 1) and is positive near 0; 0 and 1 are also simple zeros of $\varphi_k(t)$. In particular, the first eigenvalue λ_1 there corresponds the eigenfunction $\varphi_1(t) > 0$ on $(0, 1)$.

The proof of our main result is based upon the connectivity properties of the solution sets of parameterized families of compact vector fields. It is a direct consequence of Mawhin [9, Lemma 2.3].

Theorem 1.1 ([9]). *Let E be a Banach space and let $C \subset E$ be a nonempty, bounded, closed convex subset. Suppose that $T : [a,b] \times C \to C$ is completely continuous, then the set*

$$S = \big\{(\lambda, x) \in [a,b] \times C : T(\lambda, x) = x\big\}$$

contains a closed connected subset Σ which connects $\{a\} \times C$ to $\{b\} \times C$.

The rest of this paper is organized as follows. In Section 2, we give some notations and statements. In Section 3, we establish the main result and provide the proof. In addition, we give an example to explain our result.

2. Preliminaries

We shall use the following terms and notation. We need the Banach spaces $C[0,1]$, $C^1[0,1]$, $C^3[0,1]$, $L^2(0,1)$ equipped with the usual norms and the Sobolev space $W^{k,2}(0,1)$ consisting of functions $u : [0,1] \to \mathbb{R}$ such that $u, \dots, u^{(k-1)}$ are absolutely continuous on $[0,1]$, and $u^{(k)} \in L^2(0,1)$ for $k = 2, 4$ equipped with the usual norm. In particular, let the Banach space $C[0,1]$ be equipped with the norm $\|u\|_\infty = \max t \in [0,1] |u(t)|$. Denote by H the Banach space $L^2(0,1)$ with the norm $\|u\|_{L^2} = \big(\int_0^1 |u(s)|^2 \, \mathrm{d}s\big)^{1/2}$.

Define a linear operator $L : D(L) \subset H \to H$ by setting

$$D(L) = \big\{u \in W^{4,2}(0,1) : u(0) = u(1) = u''(0) = u''(1) = 0\big\}$$

and for $u \in D(L)$,

$$Lu = u'''' + \beta(t)u'' - \lambda_1 u.$$

Then

$$\ker(L) = \{u \in H : u(t) = c\varphi_1(t), c \in \mathbb{R}\},$$

$$\mathrm{Im}(L) = \big\{u \in H : \int_0^1 u(t)\varphi_1(t) \, \mathrm{d}t = 0\big\}.$$

It follows that L is a Fredholm operator of index zero. Define continuous projectors

$$P : H \to \ker(L), \quad (Pu)(t) = \Gamma_0\Big(\int_0^1 u(s)\varphi_1(s) \, \mathrm{d}s\Big)\varphi_1(t), \tag{2.1}$$

$$Q : H \to \mathrm{Im}(L), (Qu)(t) = u(t) - \Gamma_0\Big(\int_0^1 u(s)\varphi_1(s) \, \mathrm{d}s\Big)\varphi_1(t), \tag{2.2}$$

where $\Gamma_0 = 1/\int_0^1 \varphi_1^2(t) \, \mathrm{d}t > 0$. It is easy to know $\ker(L) \cap \mathrm{Im}(L) = \{0\}$, let $V = \ker(L)$, $V^\perp = \mathrm{Im}(L)$, hence $H = V \oplus V^\perp$. Let $L_P := L|_{D(L) \cap V^\perp}$, then L_P is a one to one operator from $D(L) \cap V^\perp$ to $V^\perp$. Define $K_P = L_P^{-1}$.

For every $u \in D(L)$, we have the unique decomposition

$$u(t) = \rho\varphi_1(t) + w(t), \quad t \in [0,1],$$

where $\rho \in \mathbb{R}$ and $w \in V^\perp$.

Similarly, for every $h \in H$, we also have the unique decomposition

$$h(t) = \tau\varphi_1(t) + e(t), \quad t \in [0,1],$$

where $\tau \in \mathbb{R}$ and $e \in V^\perp$.

Let $N : H \to H$ be the nonlinear operator defined by

$$(Nu)(t) = g(t, u(t)), \quad t \in [0,1],$$

then N is uniformly bounded and continuous [3]. With these considerations, the problem (1.7)-(1.8) can be written in the form of the following equation in H:

$$Lu = Nu + h, \quad u \in D(L), \tag{2.3}$$

and by a solution of (1.7)-(1.8) it is meant a solution of (2.3). Now, (2.3) is equivalent to the following system of equations

$$w(t) = K_P[QN(\rho\varphi_1(t) + w(t)) + e(t)], \tag{2.4}$$

$$PN(\rho\varphi_1(t) + w(t)) + \tau\varphi_1(t) = 0, \quad (\rho \in \mathbb{R}, w \in D(L) \cap V^{\perp}). \tag{2.5}$$

Denote by $S \subset \mathbb{R} \times V^{\perp}$ the solution set

$$S := \big\{(\rho, w) \in \mathbb{R} \times V^{\perp} : w \in D(L), (\rho, w) \text{ satisfies } (2.4)\big\}. \tag{2.6}$$

of equation (2.4). Clearly $S = \cup_{\rho\in\mathbb{R}}(\{\rho\} \times F_\rho)$, where $F_\rho = \big\{w \in V^{\perp} : w = T_\rho(w)\big\}$ and $T_\rho(w) = K_P[QN(\rho\varphi_1(t) + w(t)) + e(t)]$.

Combining (2.4) with (H2), we obtain there exists a constant $\widetilde{M} > 0$, independent of ρ, such that

$$\|w\|_{L^2} \le \widetilde{M}, \quad \text{for all } w \in V^{\perp}. \tag{2.7}$$

From the compactness of K_P and the continuity and uniform boundedness of N it follows that each T_ρ is compact and maps into the ball $\overline{B}_\rho(0) = \big\{w \in V^{\perp} : \|w\|_{L^2} \le \widetilde{M}\big\}$. Therefore, by Schauder's fixed point theorem, each F_ρ is nonempty so that $\operatorname{Proj}_{\mathbb{R}} S = \mathbb{R}$ and, in fact $S \subset \mathbb{R} \times \overline{B}_\rho(0)$. Now, system (2.4)-(2.5) is equivalent to solving the equation $\Phi(\rho, w) = \tau$ in S where the mapping $\Phi : \mathbb{R}\times(D(L)\cap V^{\perp}) \to \mathbb{R}$, is given by

$$\Phi(\rho, w) = -\Gamma_0 \int_0^1 g(t, \rho\varphi_1(t) + w(t))\varphi_1(t)\,\mathrm{d}t. \tag{2.8}$$

It is clear that Φ is continuous and bounded.

In addition, we define W as the projection of S over $V^{\perp}$, that is

$$W := \big\{w \in V^{\perp} : (\rho, w) \in S \text{ for some } \rho \in \mathbb{R}\big\}. \tag{2.9}$$

3. Main results

The main result is the following:

Theorem 3.1. *Let* (H1), (H2) *hold, and* $e \in L^2(0, 1)$. *Assume that*

(i) *there exists a constant* $u^* > 0$ *such that*

$$g(t, u^*) > 0, \quad t \in [0, 1].$$

(ii) $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$ug(t, u) \ge 0, \quad t \in [0, 1],\ u \in \mathbb{R}.$$

(iii) $\lim_{u\to\pm\infty} g(t, u) = 0$ *uniformly for* $t \in [0, 1]$.

Then, each $\Lambda_h \subset \mathbb{R}$ *is a closed bounded set and contains a closed interval* Λ_h^*: $\operatorname{int}\Lambda_h^* \ne \emptyset$, *and problem* (1.7)-(1.8) *has*

(a) *no solution if* $\tau \notin \Lambda_h$;
(b) *at least one solution if* $\tau \in \Lambda_h$;
(c) *at least two solutions if* $\tau \in \Lambda_h^* \subset \Lambda_h$.

We shall start with some preliminary results.

Lemma 3.2. *For each* $w \in W$, $\|w\|_\infty \le \|w'\|_\infty \le \|w''\|_\infty \le \|w'''\|_\infty$.

Proof. Since $w \in W$, we have $w \in D(L) \cap V^{\perp}$, then

$$w(0) = w(1) = w''(0) = w''(1) = 0.$$

(1) From $w''(0) = 0$, we have $w''(t) = \int_0^t w'''(s)\,\mathrm{d}s$, $t \in [0,1]$, and so $|w''(t)| \le \int_0^1 |w'''(s)|\,\mathrm{d}s \le \|w'''\|_\infty$. Thus $\|w''\|_\infty \le \|w'''\|_\infty$.
(2) By $w(0) = w(1)$, there is a $\xi \in (0,1)$ such that $w'(\xi) = 0$, and so $-w'(t) = \int_t^\xi w''(s)\,\mathrm{d}s$, $t \in [0,\xi]$. Hence $|w'(t)| \le \int_t^\xi |w''(s)|\,\mathrm{d}s \le \int_0^1 |w''(s)|\,\mathrm{d}s \le \|w''\|_\infty$. Similarly, for all $t \in [\xi,1]$, $|w'(t)| \le \|w''\|_\infty$. Thus, $\|w'\|_\infty \le \|w''\|_\infty$.
(3) Because of $w(0) = 0$, similar to (1), we obtain $\|w\|_\infty \le \|w'\|_\infty$. □

Lemma 3.3. *Let* (H1), (H2) *hold. Then W is a bounded set in $C^3[0,1]$.*

Proof. For every $w \in W$, by the definition of W, there exists $(\rho, w) \in S$ such that

$$w^{(4)} + \beta(t)w'' - \lambda_1 w = QN(\rho\varphi_1 + w) + e. \tag{3.1}$$

Let $z = w''$, $f(t) = \lambda_1 w + QN(\rho\varphi_1 + w) + e$, this together with the boundary condition, (3.1) is equivalent to the boundary value problem

$$z'' + \beta(t)z = f(t), \quad t \in (0,1), \tag{3.2}$$

$$z(0) = z(1) = 0. \tag{3.3}$$

Combining (H2) with (2.7), there exists a constant $M_1 > 0$ such that

$$\|f(t)\|_{L^2} \le M_1.$$

Moreover, by (H1) and the Sobolev imbedding theorem

$$z \in W^{2,2}(0,1) \hookrightarrow\hookrightarrow C^1[0,1].$$

Then there exists a constant $M_2 > 0$ such that

$$\|z'\|_\infty \le M_2.$$

Hence $\|w'''\|_\infty \le M_2$. From Lemma 3.2, we obtain

$$\|w\|_\infty \le \|w'\|_\infty \le \|w''\|_\infty \le \|w'''\|_\infty \le M_2.$$

Therefore, W is a bounded set in $C^3[0,1]$. □

Lemma 3.4. *There exists $\alpha = \alpha(W) > 0$ such that*

$$\rho\varphi_1(t) + w(t) \ge 0, \quad -\rho\varphi_1(t) + w(t) \le 0 \tag{3.4}$$

for all $t \in [0,1], \rho \ge \alpha$ and $w \in W$.

Proof. Since $\varphi_1(t) > 0$ on $(0,1)$, and $\varphi_1(0) = \varphi_1(1) = 0$, we have $\varphi_1'(0) > 0$, $\varphi_1'(1) < 0$. This combines with the fact that W is a bounded set in $C^1[0,1]$, we know there exists $\alpha > 0$ such that

$$|w(t)| \le \alpha\varphi_1(t), \quad t \in [0,1];$$

that is,

$$\rho\varphi_1(t) + w(t) \ge 0, \quad -\rho\varphi_1(t) + w(t) \le 0$$

for all $t \in [0,1]$, $\rho \ge \alpha$ and $w \in W$. □

Lemma 3.5. *Let (i), (ii) and* (H2) *hold. Then there exists $\alpha_1 = \alpha_1(W) > 0$ such that $\Phi(\rho, w) < 0$ and $\Phi(-\rho, w) \ge 0$ for all $\rho \ge \alpha_1$ and $w \in W$.*

Proof. Let $\alpha = \alpha(W) > 0$ be given by Lemma 3.4 and take $\alpha_0 > 0$ so that $\alpha_0 \max_{t\in[0,1]} \varphi_1 - M_3 > u^*$, where $M_3 = \sup_{w\in W} \|w\|_\infty$. Letting $\alpha_1 = \alpha + \alpha_0$ we have

$$\rho\varphi_1(t) + w(t) \geq \alpha_0\varphi_1(t) \geq 0, \quad -\rho\varphi_1(t) + w(t) \leq -\alpha_0\varphi_1(t) \leq 0$$

for all $t \in [0,1]$, $\rho \geq \alpha_1$ and $w \in W$. Hence

$$\Phi(\rho, w) = -\Gamma_0 \int_0^1 g(t, \rho\varphi_1(t) + w(t))\varphi_1(t)\,\mathrm{d}t \leq 0, \tag{3.5}$$

$$\Phi(-\rho, w) = -\Gamma_0 \int_0^1 g(t, -\rho\varphi_1(t) + w(t))\varphi_1(t)\,\mathrm{d}t \geq 0 \tag{3.6}$$

for all $\rho \geq \alpha_1$ and $w \in W$.

Now, we will show that (3.5) has strict inequality. Since for each $\rho \geq \alpha_1$ and $w \in W$, the function $\rho\varphi_1(t) + w(t) = 0$ if $t = 0$. Moreover, we have from the definition of α_1 that

$$\rho\varphi_1(t_0) + w(t_0) \geq \alpha_0\varphi_1(t_0) - M_3 > u^*,$$

where $\varphi_1(t_0) = \max_{t\in[0,1]} \varphi_1(t)$. Therefore, there is a $t^* \in (0, t_0)$ such that $\rho\varphi_1(t^*) + w(t^*) = u^*$, Hence

$$g(t^*, \rho\varphi_1(t^*) + w(t^*)) = g(t^*, u^*) > 0.$$

$\square$

Proof of Theorem 3.1. As we already observed in Section 2, the problem (1.7)-(1.8) is equivalent to the system (2.4)-(2.5) which in turn is equivalent to solving the equation $\Phi(\rho, w) = \tau$ in S, where $\Phi : \mathbb{R} \times (D(L) \cap V^\perp) \to \mathbb{R}$,

$$\Phi(\rho, w) = -\Gamma_0 \int_0^1 g(t, \rho\varphi_1(t) + w(t))\varphi_1(t)\,\mathrm{d}t$$

is a bounded continuous function. So the problem (1.7)-(1.8) has at least one solution if $\tau \in \Phi(S)$.

From Lemma 3.5, there exists $\alpha_1 > 0$ such that

$$\Phi(\rho, w) < 0, \quad (\rho, w) \in S,\ \rho \geq \alpha_1, \tag{3.7}$$

$$\Phi(\rho, w) \geq 0, \quad (\rho, w) \in S, \rho \leq -\alpha_1, \tag{3.8}$$

which imply $0 \in \Phi(S)$.

Combining (iii) with Lemma 3.4, we have

$$\lim_{|\rho|\to\infty, (\rho,w)\in S} \Phi(\rho, w) = 0. \tag{3.9}$$

We claim that $\Lambda_h = \Phi(S)$. Obviously, Λ_h is bounded. Now, we show that $\Lambda_h = \Phi(S)$ is closed.

Let $\tau = \lim_{n\to\infty} \Phi(\rho_n, w_n)$ with $(\rho_n, w_n) \in S$. We assume $\tau \neq 0$ since we already know that $0 \in \Lambda_h$. From (2.7) it follows that

$$w_{n_j} \to w \quad \text{in } H \tag{3.10}$$

and $w_{n_j} \to w$ a.e. in $(0,1)$ for some subsequence $\{w_{n_j}\}$ of $\{w_n\}$. On the other hand, we must have $\{\rho_n\}$ is bounded sequence, otherwise from (3.9), $\tau = 0$. Therefore, we may assume that $\{\rho_{n_j}\}$ is convergent; i.e., $\rho_{n_j} \to \rho$, which together with (3.10) gives $(\rho_{n_j}, w_{n_j}) \to (\rho, w)$ in $\mathbb{R} \times V^\perp$. Hence, since S is closed and Φ is continuous, we obtain $(\rho, w) \in S$ and $\tau = \Phi(\rho, w)$.

Next, we will prove that (1.7)-(1.8) has at least two solutions if $\tau \in \Lambda_h^* \subset \Lambda_h$.

From (3.7), (3.8) and the fact that, by Theorem 1.1, $S \subset \mathbb{R} \times \overline{B}_\rho(0)$ contains a closed connected subset $C_{-\alpha_1,\alpha_1}$ which joins $\{-\alpha_1\} \times \overline{B}_\rho(0)$ to $\{\alpha_1\} \times \overline{B}_\rho(0)$ we conclude that Λ_h contains a interval $[-\beta, 0], \beta > 0$ is a constant.

Let $S_\sigma = S \cap (\{\sigma\} \times \overline{B}_\rho(0)), \sigma \in \mathbb{R}$. Now, taking $\alpha_1 > 0$ as above we consider $\nu_1 = \min \Phi(C_{-\alpha_1,\alpha_1})$ and $\nu_2 = \max \Phi(C_{-\alpha_1,\alpha_1})$. Clearly $\nu_1 < 0 \le \nu_2$. From (3.7) and (3.9), there exists $\gamma > \alpha_1$ such that $\nu_1 < \Phi(\gamma, w) < 0$ for all $w \in W$. Setting $m_\gamma = \min \Phi(S_\gamma), M_\gamma = \max \Phi(S_\gamma)$, we have $\nu_1 < m_\gamma \le M_\gamma < 0 \le \nu_2$. We further choose $\delta > \gamma$ such that $M_\gamma < \Phi(\delta, w) < 0$ for all $w \in W$, hence

$$\nu_1 < m_\gamma \le M_\gamma < m_\delta \le M_\delta < 0 \le \nu_2, \tag{3.11}$$

where $m_\delta = \min \Phi(S_\delta), M_\delta = \max \Phi(S_\delta)$. Finally, considering a closed connected subset $C_{\gamma,\delta}$ which joins S_γ to S_δ and letting $\nu_1^* = \min \Phi(C_{\gamma,\delta}), \nu_2^* = \max \Phi(C_{\gamma,\delta})$ we obtain

$$\begin{gathered} \nu_1 < \nu_1^* \le \max \Phi(C_{\gamma,\delta} \cap S_\gamma) \le M_\gamma, \\ m_\delta \le \min \Phi(C_{\gamma,\delta} \cap S_\delta) \le \nu_2^* < 0 \le \nu_2, \end{gathered}$$

hence

$$\nu_1 < \nu_1^* < \nu_2^* < 0 \le \nu_2$$

since $M_\gamma < m_\delta$. In conclusion, setting $\Lambda_h^* = [\nu_1^*, \nu_2^*] = \Phi(C_{\gamma,\delta})$ we have $\Lambda_h^* \subset [\nu_1, \nu_2] = \Phi(C_{-\alpha_1,\alpha_1})$ with $C_{\gamma,\delta}$ and $C_{-\alpha_1,\alpha_1}$ disjoint by construction, that is, for each $\tau \in \Lambda_h^*$, the problem (1.7)-(1.8) has at least two solutions; i.e., one in $C_{\gamma,\delta}$ and the other in $C_{-\alpha_1,\alpha_1}$. $\square$

Example. Consider the boundary value problem

$$u^{(4)} + (\sin\pi t)u'' - \lambda_1 u = g(t, u) + \cos t, \quad t \in (0, 1), \tag{3.12}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{3.13}$$

where $g(t, u) : [0, 1] \times \mathbb{R} \to \mathbb{R}$,

$$g(t, u) = \begin{cases} u, & |u| \le 1; \\ 1/u, & |u| \ge 1. \end{cases}$$

It is easy to check that (H1), (H2) and all the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a closed bounded set $\Lambda_h \subset \mathbb{R}$, which contains a closed interval Λ_h^*: $\operatorname{int} \Lambda_h^* \ne \emptyset$ such that the problem (3.12)-(3.13) has:

(a) no solution if $\tau \notin \Lambda_h$;
(b) at least one solution if $\tau \in \Lambda_h$;
(c) at least two solutions if $\tau \in \Lambda_h^* \subset \Lambda_h$.

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College of Mathematics and Information Science, Northwest Normal University, Lanzhou, Gansu 730070, China

E-mail address: `xuling_216@yahoo.cn`