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ON STABILITY AND OSCILLATION OF EQUATIONS WITH A DISTRIBUTED DELAY WHICH CAN BE REDUCED TO DIFFERENCE EQUATIONS

ELENA BRAVERMAN, SERGEY ZHUKOVSKIY

Abstract. For the equation with a distributed delay

$$x'(t) + ax(t) + \int_0^1 x(s + [t-1])dR(s) = 0$$

we obtain necessary and sufficient conditions of stability, exponential stability and oscillation. These results are applied to some particular cases, such as integro-differential equations and equations with a piecewise constant argument. Well known results for equations with a piecewise constant argument are obtained as special cases.

1. Introduction

The study of equations with a piecewise constant delay was initiated in 1984 by Cooke and Wiener [8] and was later continued in many other publications [1, 2, 11, 18, 33], some of these results are summarized in [14]. During the last two decades this topic has been extensively studied, see [3, 4, 5, 6, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36] and references therein. One of the reasons of such interest is a hybrid character of such equations [4]: they incorporate properties of continuous and discrete models. Moreover, a solution of an equation with a piecewise constant argument at certain points also satisfies some difference equation. Using this technique, the known results for delay equations were applied to delay (high order) difference equations, see, for example, [9, 15, 19]. Another reason for attention to equations with piecewise constant arguments is the following: such equations are semidiscretizations of delay equations and thus are useful in numerical applications [7, 12, 13, 16, 17].

We consider the equation with a distributed delay

$$x'(t) + ax(t) + \int_0^1 x(s + [t-1])dR(s) = 0, \tag{1.1}$$

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where $R(s) : [0,1] \to \mathbb{R}$ is a left-continuous function of bounded variation, $a \in \mathbb{R}$. For example, if $R(s)$ is differentiable, $R'(s) = b(s)$, then (1.1) is the integro-differential equation

$$x'(t) + ax(t) + \int_0^1 b(s)x(s + [t-1])ds = 0. \tag{1.2}$$

Let $R(s) = b\chi_{(\alpha,1]}(s)$, where $\chi_{(a,b]}$ is the characteristic function of the interval $(a,b]$, i.e., $\chi_{(a,b]}(x) = 1$, if $x \in (a,b]$ and $\chi_{(a,b]}(x) = 0$, otherwise. Then (1.1) has the form

$$x'(t) + ax(t) + bx(\alpha + [t-1]) = 0, \tag{1.3}$$

which involves equations

$$x'(t) + ax(t) + bx([t-1]) = 0 \tag{1.4}$$

and

$$x'(t) + ax(t) + bx([t]) = 0 \tag{1.5}$$

as particular cases when $\alpha = 0$ and $\alpha \to 1$, as well as equations where the piecewise constant argument refers to the fractional points.

In spite of its "continuous" form, equation (1.1) incorporates properties of both continuous and discrete systems, in the next section we will reduce its solution to the solution of a specially constructed difference equation.

The paper is organized as follows. In Section 2 we present relevant definitions and auxiliary results. In particular, we reduce (1.1) to a second order difference equation. Section 3 presents necessary and sufficient oscillation and stability conditions for (1.1). The general results are applied to some special cases of integro-differential equations and equations with piecewise constant arguments, which allows to deduce some known results. Finally, Section 4 involves discussion and outlines some open problems and possible generalizations of equation (1.1). Some long but straightforward proofs are presented in the Appendix.

2. Preliminaries and Solution Representation

We consider (1.1) with the initial condition

$$x(t) = \varphi(t), \quad t \in [-1, 0], \tag{2.1}$$

under the following assumptions:

- (A1) $R(s) : [0,1] \to \mathbb{R}$ is a left-continuous function of bounded variation which has a nonzero variation in $[0,1]$;
- (A2) $\varphi : [0,1] \to \mathbb{R}$ is a Borel measurable bounded function such that the Lebesgue Stiltjes integral $\int_0^1 \varphi(s-1)dR(s)$ exists (and is finite).

Definition 2.1. Function $x(t)$ is *a solution* of (1.1), (2.1) if it satisfies (1.1) almost everywhere for $t \geq 0$ and (2.1) for $t \in [-1, 0]$.

Denote

$$x_n = x(n), \quad x_{-1} = \int_0^1 \varphi(s-1)dR(s), \tag{2.2}$$

$$K_n = \int_0^1 x(s+n)dR(s), \quad K_{-1} = \int_0^1 \varphi(s-1)dR(s), \tag{2.3}$$

$$P(a) = \int_0^1 e^{-as}dR(s), \quad Q(a) = \int_0^1 \frac{1-e^{-as}}{a}dR(s). \tag{2.4}$$

First, let us reduce the solution of (1.1) at integer points to a solution of a second order difference equation. Let us notice that in the following we will understand expressions at $a=0$ as a limit; for example,

$$\frac{e^{ak}-1}{a}\Big|_{a=0} = \lim_{a\to 0}\frac{e^{ka}-1}{a} = k = \lim_{a\to 0}\frac{1-e^{-ka}}{a}.$$

Lemma 2.2. *(1) The solution of* (1.1)*, (2.1) between integer points is*

$$x(t) = x_n e^{a(n-t)} + \frac{e^{a(n-t)}-1}{a}K_{n-1}, \quad t \in [n, n+1), \tag{2.5}$$

with K_n, x_n defined by (2.2), (2.3), $n = 0, 1, 2, \dots$.

(2) The solution of (1.1)*, (2.1) at integer points satisfies the second order difference equation*

$$x_{n+2} - (e^{-a} - Q(a))x_{n+1} + \Big(\frac{1-e^{-a}}{a}P(a) - e^{-a}Q(a)\Big)x_n = 0, \quad n \ge -1. \tag{2.6}$$

Proof. The first part is checked by a straightforward computation and leads to

$$x_{n+1} = e^{-a}x_n - \Big(\frac{1-e^{-a}}{a}\Big)K_{n-1},$$

$$\begin{aligned} K_n &= \int_0^1 x(s+n)dR(s) = \int_0^1 \Big(x_n e^{a(n-s)} + \frac{e^{a(n-s)}-1}{a}K_{n-1}\Big)dR(s) \\ &= x_n \int_0^1 e^{-as}dR(s) - K_{n-1}\int_0^1 \frac{1-e^{-as}}{a}dR(s) = P(a)x_n - Q(a)K_{n-1}. \end{aligned}$$

Hence, if we denote $Y_n = (x_n, K_{n-1})^T$, then $Y_{n+1} = AY_n$, where

$$A = \begin{bmatrix} e^{-a} & -\frac{1-e^{-a}}{a} \\ P(a) & -Q(a) \end{bmatrix}.$$

Thus, x_n satisfies the second order difference equation

$$x_{n+2} - \operatorname{tr}(A)x_{n+1} + \det(A)x_n = 0;$$

since the trace of A is $e^{-a} - Q(a)$ and the determinant is $\frac{1-e^{-a}}{a}P(a) - e^{-a}Q(a)$, we immediately obtain (2.6). $\square$

Remark 2.3. By Lemma 2.2 the values of (1.1), (2.1) at integer points satisfy the difference equation (2.6), with x_0, x_{-1} defined in (2.2).

Let us also note that for any x_0, x_{-1} there exists φ satisfying (A2) which leads to these x_0, x_{-1} in (2.2). Really, since $R(s)$ has a nonzero variation, then there exists a continuous function $g : [-1, 0] \to \mathbb{R}$ such that $\int_0^1 g(s-1)dR(s) = c \neq 0$. Besides, $R(s)$ is left continuous, so the relevant measure has no atom at $x = 1$, thus

$\int_0^1 g_1(s-1)dR(s) = \int_0^1 g(s-1)dR(s)$, where $g_1(s)$ coincides with $g(s)$ everywhere in $[-1,0]$ but probably at $s=0$ (and the left integral always exists). Then

$$\varphi(s) = \begin{cases} x_{-1}g(s)/c, & s \in [-1,0), \\ x_0, & s = 0, \end{cases}$$

leads to any prescribed x_{-1}, x_0.

Definition 2.4. A solution of (1.1) *oscillates* if it is neither eventually positive nor eventually negative. Equation (1.1) is *oscillatory* if all its solutions oscillate.

A solution of (2.6) *oscillates* if the sequence $\{x_n\}$ is neither eventually positive nor eventually negative. Equation (2.6) is *oscillatory* if all its solutions oscillate.

Corollary 1. *Equation* (1.1) *is oscillatory if and only if* (2.6) *is oscillatory.*

Proof. Obviously if a solution of (2.6) oscillates then the relevant solution of (1.1) (with an appropriate initial function, see Remark 2.3) cannot be eventually positive or negative. Let us notice that by (2.5) a solution of (1.1) increases in $[n, n+1)$ if $ax_n + K_{n-1} < 0$ and decreases if $ax_n + K_{n-1} > 0$. Thus, if $x(n)$, $x(n+1)$ have the same sign, so are all the points between n and $n+1$, hence oscillation of (1.1) implies that (2.6) is also oscillating. □

According to (A2), the initial function is bounded, so we can define the sup-norm:

$$\|\varphi\| = \sup_{t \in [-1,0]} |\varphi(t)|.$$

Definition 2.5. Equation (1.1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any φ satisfying (A2) inequality $\|\varphi\| < \delta$ implies $|x(t)| < \varepsilon$ for $t \geq 0$. Equation (1.1) is *asymptotically stable* if it is stable and $\lim_{t\to\infty} x(t) = 0$ for any initial conditions. Equation (1.1) is *exponentially stable* if there exist positive numbers N, γ such that any solution satisfies

$$|x(t)| \leq Ne^{-\gamma t}\|\varphi\|.$$

Eq. (2.6) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\max\{|x_0|, |x_{-1}|\} < \delta$ implies $|x_n| < \varepsilon$ for any $n \geq 0$. Equation (2.6) is *asymptotically stable* if it is stable and $\lim_{n\to\infty} x_n = 0$ for any initial conditions. Equation (2.6) is *exponentially stable* if there exist positive numbers N, γ such that any solution satisfies

$$|x_n| \leq Ne^{-\gamma n} \max\{|x_0|, |x_{-1}|\}.$$

Corollary 2. *Equation* (1.1) *is stable (asymptotically stable, exponentially stable) if and only if* (2.6) *is stable (asymptotically stable, exponentially stable).*

Proof. As in the previous corollary, for any solution of (1.1), $\max_{t\in[n,n+1]} |x(t)|$ is attained at the ends and equals either $|x(n)| = |x_n|$ or $|x(n+1)| = |x_{n+1}|$. Thus any type of stability of (1.1) is equivalent to the appropriate stability kind for (2.6). □

3. Stability and Oscillation Tests

In this section we will obtain necessary and sufficient conditions for oscillation, stability and exponential stability of equation (1.1) with a distributed delay. In the following we will also apply the well known result for second order difference equations with constant coefficients (see, for example, [10, p. 53–65]).

Lemma 3.1. *A difference equation with constant coefficients*

$$x_{n+1} - p_1 x_{n+1} + p_2 x_n = 0, \quad n = -1, 0, 1, \dots, \tag{3.1}$$

is stable if both roots of the characteristic equation $\lambda^2 - p_1\lambda + p_2 = 0$ are in the unit circle and is exponentially stable if the roots are inside the unit circle. The latter condition is satisfied if $|p_1| < p_2 + 1 < 2$ and is also equivalent to the asymptotic stability of (3.1)*. The former condition is satisfied if $|p_1| \le p_2 + 1 \le 2$.*

Equation (3.1) *is oscillatory if and only if its characteristic equation has no positive roots, which is valid if either the discriminant is negative ($p_1^2 < 4p_2$) or all coefficients are nonnegative ($p_1 \le 0$, $p_2 \ge 0$).*

Lemma 3.1 together with the form of (2.6) and Corollaries 1, 2 imply the following oscillation and stability tests for equation (1.1).

Theorem 3.2. *Suppose (A1)-(A2) are satisfied. Equation* (1.1) *is oscillatory if and only if at least one of the two following conditions holds:*

$$\frac{1}{4}\left(e^{-a} + Q(a)\right)^2 < \frac{1 - e^{-a}}{a} P(a), \tag{3.2}$$

$$e^{-a} \le Q(a) \le \frac{e^a - 1}{a} P(a). \tag{3.3}$$

Proof. By Lemma 3.1 equation (2.6) is oscillatory if and only if either

$$\frac{1}{4}\left(e^{-a} - Q(a)\right)^2 < \frac{1 - e^{-a}}{a} P(a) - e^{-a} Q(a)$$

or

$$e^{-a} \le Q(a) \le \frac{1 - e^{-a}}{a} e^a P(a),$$

where the former inequality is equivalent to (3.2) and the latter to (3.3). □

Theorem 3.3. *Suppose (A1)-(A2) are satisfied. Equation* (1.1) *is stable if and only if*

$$\left|Q(a) - e^{-a}\right| \le \frac{1 - e^{-a}}{a} P(a) - e^{-a} Q(a) + 1 \le 2 \tag{3.4}$$

and is exponentially stable if and only if

$$\left|Q(a) - e^{-a}\right| < \frac{1 - e^{-a}}{a} P(a) - e^{-a} Q(a) + 1 < 2. \tag{3.5}$$

To illustrate Theorems 3.2, 3.3, let us consider two particular cases of (1.1). First, let $a, b \in \mathbb{R}$. Consider the integro-differential equation

$$x'(t) + a x(t) + b \int_{[t-1]}^{[t]} x(s) ds = 0, \tag{3.6}$$

which is a special case of (1.1), with $R(s) = bs$. Then $P(a), Q(a)$ in (2.4) have the form

$$P(a) = \int_0^1 e^{-as} b ds = b\frac{1 - e^{-a}}{a}, \quad Q(a) = \int_0^1 \frac{1 - e^{-as}}{a} b ds = b\frac{a - 1 + e^{-a}}{a^2}. \tag{3.7}$$

The following results are corollaries of Theorems 3.2, 3.3, where $P(a)$ and $Q(a)$ are substituted from (3.7). However, the straightforward computation is long and thus is presented in the Appendix.

Theorem 3.4. *The following two statements are equivalent.*

(1) *Equation (3.6) is oscillatory.*
(2) *If $a \neq 0$ then*

$$b > a^2 \left(\frac{|1 - e^{-a}| - \sqrt{1 - e^{-a} - ae^{-a}}}{a - 1 + e^{-a}} \right)^2 ; \tag{3.8}$$

if $a = 0$ then

$$b > 6 - 4\sqrt{2}. \tag{3.9}$$

The domain of parameters a, b where (3.6) oscillates is illustrated in Fig 1.

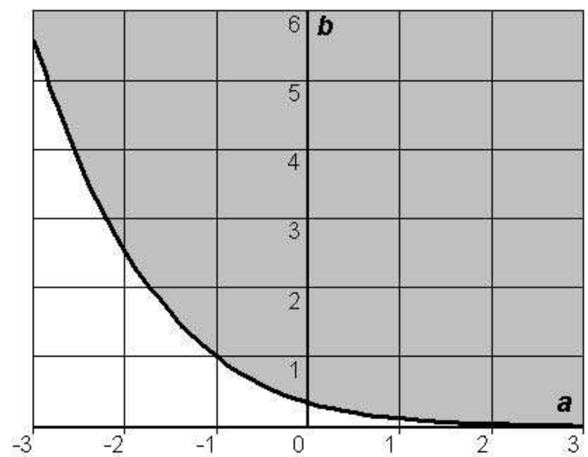

FIGURE 1. Illustration for the Theorem 3.4. The shaded area designates the set of parameters (a, b) where any solution of the initial value problem (3.6), (2.1) oscillates.

Theorem 3.5. *The following two statements are equivalent.*

(1) *Equation (3.6) is exponentially stable.*
(2) *If $a < 0$ then*

$$-a < b < \frac{a^2}{1 - e^{-a} - ae^{-a}}; \tag{3.10}$$

if $a = 0$ then

$$0 < b < 2; \tag{3.11}$$

if $a > 0$ then

$$-a < b < \min \left\{ -\frac{a^2(1 + e^{-a})}{2 - 2e^{-a} - ae^{-a} - a}, \frac{a^2}{1 - e^{-a} - ae^{-a}} \right\}. \tag{3.12}$$

Figure 2 illustrates Theorem 3.5. For any value of the parameters a, b such that point (a, b) is in the grey area, equation (3.6) is exponentially stable. For any value of the parameters a, b such that point (a, b) is in the white area the equation is unstable and it is stable (but not exponentially) on the boundary.

Next, let $R(s)$ be a step function $R(s) = b\chi_{(r,1]}(t)$, $0 \le r < 1$. Then (1.1) has the form

$$x'(t) + ax(t) + bx(r + [t - 1]) = 0, \tag{3.13}$$

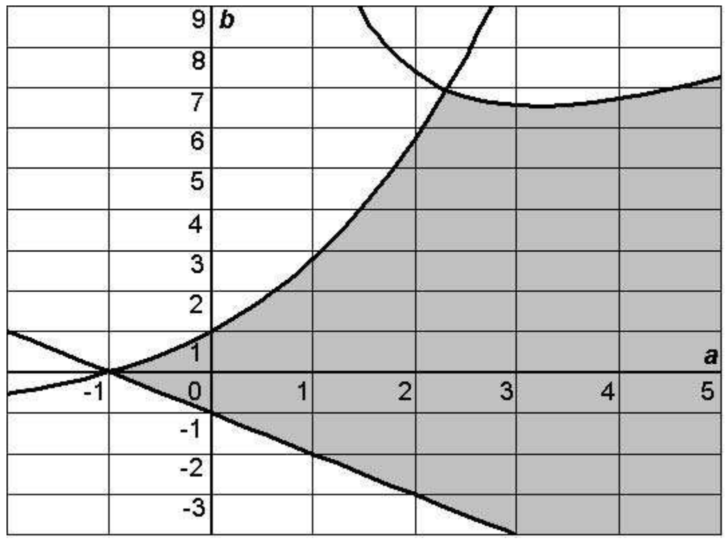

FIGURE 2. The shaded area designates the set of parameters (a, b) where (3.6) is exponentially stable, the equation is stable (but not asymptotically) for values (a, b) on the boundary.

equations for $r = 0$ or $r = 1$ were considered in [1, 2, 8]. Then

$$P(a) = be^{-ar}, \quad Q(a) = b\frac{1-e^{-ar}}{a}.$$

Thus, Theorems 3.2 and 3.3 immediately imply the following oscillation and stability criteria for (3.13).

Equation (3.13) is oscillatory if at least one of the following two inequalities holds

$$\frac{1}{4}\Big(e^{-a} + b\frac{1-e^{-ar}}{a}\Big)^2 < b\frac{1-e^{-a}}{a}e^{-ar}, \tag{3.14}$$

$$e^{-a} \le b\frac{1-e^{-ar}}{a} \le b\frac{e^{a}-1}{a}e^{-ar}. \tag{3.15}$$

Equation (3.13) is stable if and only if

$$\big|b\frac{1-e^{-ar}}{a} - e^{-a}\big| \le b\frac{e^{-ar}-e^{-a}}{a} + 1 \le 2 \tag{3.16}$$

and is exponentially stable if and only if

$$\big|b\frac{1-e^{-ar}}{a} - e^{-a}\big| < b\frac{e^{-ar}-e^{-a}}{a} + 1 < 2. \tag{3.17}$$

Theorem 3.6. *Let* $0 < r < 1$. *Equation* (3.13) *is oscillatory if and only if*

$$b > \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big[\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} - \sqrt{\frac{e^{-ar}-e^{-a}}{a}}\Big]^2. \tag{3.18}$$

Proof. Using (3.14), (3.15), we will obtain explicit conditions for b, if a is given. Since $\frac{1-e^{-ar}}{a} \ge 0$, then the left inequality in (3.15) becomes $b \ge \frac{ae^{-a}}{1-e^{-ar}}$, while the right inequality $b\frac{1-e^{a(1-r)}}{a} \le 0$ is $b \ge 0$. Thus, (3.15) has the form

$$b \ge \max\left\{0, \frac{ae^{-a}}{1-e^{-ar}}\right\} = \frac{ae^{-a}}{1-e^{-ar}}.$$

Inequality (3.14) can be rewritten as a quadratic inequality in b:

$$\Big(\frac{1-e^{-ar}}{a}\Big)^2 b^2 + 2\frac{e^{-a}-2e^{-ar}+e^{-a(r+1)}}{a}b + e^{-2a} < 0. \tag{3.19}$$

The discriminant of the above quadratic inequality in b is

$$\begin{aligned} D &= \frac{(e^{-a}-2e^{-ar}+e^{-a(r+1)})^2 - (e^{-a}-e^{-a(r+1)})^2}{a^2} \\ &= 4\frac{e^{-ar}-e^{-a(r+1)}}{a}\frac{e^{-ar}-e^{-a}}{a} \end{aligned}$$

which is positive as a product of two positive factors. A solution of inequality (3.19) is between the two roots $b_1 < b_2$ of the relevant quadratic equation, the largest of them is

$$\begin{aligned} b_2 &= \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big[\frac{2e^{-ar}-e^{-a}-e^{-a(r+1)}}{a} + 2\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}}\sqrt{\frac{e^{-ar}-e^{-a}}{a}}\Big] \\ &= \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big[\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} + \sqrt{\frac{e^{-ar}-e^{-a}}{a}}\Big]^2, \end{aligned}$$

similarly,

$$b_1 = \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big[\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} - \sqrt{\frac{e^{-ar}-e^{-a}}{a}}\Big]^2 \tag{3.20}$$

is obviously nonnegative.

If $b_1 < b < b_2$, then (3.14) is satisfied. Let us demonstrate that $b_2 \ge \frac{ae^{-a}}{1-e^{-ar}}$, then for $b \ge b_2$ inequality (3.15) is satisfied, thus for $b > b_1$ all solutions are oscillatory. Hence $b > b_1$, where b_1 is defined in (3.20), immediately implies oscillation condition (3.18). Consider

$$\begin{aligned} & b_2 - \frac{ae^{-a}}{1-e^{-ar}} \\ &= \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big[\Big(\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} + \sqrt{\frac{e^{-ar}-e^{-a}}{a}}\Big)^2 - \frac{e^{-a}-e^{-a(r+1)}}{a}\Big] \\ &= \Big(\frac{a}{1-e^{-ar}}\Big)^2\Big(\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} + \sqrt{\frac{e^{-ar}-e^{-a}}{a}} + \sqrt{\frac{e^{-a}-e^{-a(r+1)}}{a}}\Big) \\ &\times \Big(\sqrt{\frac{e^{-ar}-e^{-a(r+1)}}{a}} + \sqrt{\frac{e^{-ar}-e^{-a}}{a}} - \sqrt{\frac{e^{-a}-e^{-a(r+1)}}{a}}\Big) \ge 0 \end{aligned}$$

as a product of two nonnegative terms, the latter term is nonnegative since $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$ for any nonnegative x, y. Consequently, (3.18) is necessary and sufficient for oscillation, which completes the proof. $\square$

Remark. However, Theorem 3.6 does not consider the cases $r = 0$, $r \to 1$, which correspond to equations (1.4) and (1.5), respectively. First, let $r = 0$. Then (3.15) is never valid (it involves $e^{-a} \le 0b = 0$), we stay with (3.14) which has the form $\frac{1}{4}e^{-2a} < b\frac{1-e^{-a}}{a}$. Hence

$$b > \frac{ae^{-2a}}{4(1-e^{-a})} = \frac{ae^{-a}}{4(e^{a}-1)},$$

which was obtained in [2] as a necessary and sufficient oscillation condition for (1.4).

If $r \to 1$, then (3.15) tends to $e^{-a} \le b\frac{1-e^{-a}}{a}$, while (3.14) has the form

$$\Big(e^{-a} + b\frac{1-e^{-a}}{a}\Big)^2 < 4b\Big(\frac{1-e^{-a}}{a}\Big)e^{-a}, \quad \text{or} \quad \Big(e^{-a} - b\frac{1-e^{-a}}{a}\Big)^2 < 0,$$

which is impossible. The inequality $b > \frac{a}{e^{a}-1}$ is sufficient for oscillation of (1.5), see [1].

Now let us proceed to stability of (3.13).

Theorem 3.7. *Let $0 < r < 1$. Equation (3.13) is stable if and only if*

$$-a \le b \le C, \tag{3.21}$$

where

$$C = \begin{cases} \min\big\{\frac{a(1+e^{-a})}{1+e^{-a}-2e^{-ar}}, \frac{a}{e^{-ar}-e^{-a}}\big\}, & \text{if } \frac{1+e^{-a}-2e^{-ar}}{a} > 0, \\ \frac{a}{e^{-ar}-e^{-a}}, & \text{if } \frac{1+e^{-a}-2e^{-ar}}{a} \le 0, \end{cases} \tag{3.22}$$

and is exponentially stable if and only if

$$-a < b < C. \tag{3.23}$$

Proof. By (3.17) exponential stability is equivalent to the following inequalities

$$-b\frac{e^{-ar}-e^{-a}}{a} - 1 < b\frac{1-e^{-ar}}{a} - e^{-a} < b\frac{e^{-ar}-e^{-a}}{a} + 1, \tag{3.24}$$

$$b\frac{e^{-ar}-e^{-a}}{a} < 1. \tag{3.25}$$

The latter inequality can be rewritten as $b < \frac{a}{e^{-ar}-e^{-a}}$, while the left inequality of (3.24) is

$$b\frac{1-e^{-a}}{a} > e^{-a} - 1, \quad \text{or} \quad b > \frac{e^{-a}-1}{1-e^{-a}}a = -a.$$

Further, consider the right inequality in (3.24) which is equivalent to

$$\frac{1+e^{-a}-2e^{-ar}}{a}b < 1 + e^{-a}. \tag{3.26}$$

The right hand side is positive, so if the left hand side is nonpositive then (3.26) holds. Thus, to prove that (3.23) is sufficient for exponential stability, it is enough to consider the case $\frac{1+e^{-a}-2e^{-ar}}{a} < 0$, $b < 0$. Since $b > -a$, then we deduce $a < 0$. We have $|b|/|a| < 1$ and

$$\Big|\frac{1+e^{-a}-2e^{-ar}}{a}b\Big| = \Big|\frac{b}{a}\Big|\big|1+e^{-a}-2e^{-ar}\big| < \big|1+e^{-a}-2e^{-ar}\big| \le 1 + e^{-a},$$

since $2e^{-ar} \le 2$ for $a > 0$, $r \ge 0$, which completes the proof for the exponential stability. Stability is considered similarly. □

Remark. In the case $r = 0$ we have $C = \frac{a}{1-e^{-a}}$; the exponential stability condition $-a < b < \frac{a}{1-e^{-a}}$ is well known for (1.4), see [8]. If $r \to 1$, then $C = \frac{a(1+e^{-a})}{1-e^{-a}}$ and the exponential stability condition $-a < b < \frac{a(1+e^{-a})}{1-e^{-a}}$ for (1.5) is also known [8].

4. Discussion and Open Problems

We have obtained sharp oscillation and stability conditions for equation (1.1) and some of its particular cases. After some straightforward computations, we have the following relation between the properties of equations (3.6), (1.4) and (1.5).

(1) Exponential stability of (1.4) implies exponential stability of (1.5) and (3.6). However, we cannot compare conditions of exponential stability of (1.5) and (3.6). For example, if $a = -1$, $b = 2$, then equation (1.5) is exponentially stable and (3.6) is not stable, while for $a = 2$, $b = 6$ equation (3.6) is exponentially stable, unlike (1.5).

(2) Oscillation domains of (1.4), (1.5) and (3.6) in (a, b)-plane also cannot be compared: for each pair of equations there are two examples when one oscillates while the other does not for the same values of parameters a, b.

Let us discuss some possible applications and generalizations of our results, as well as relevant open problems.

(1) Apply the results of the present paper to obtain sharp stability and oscillation conditions for the equation
$$x'(t) + ax(t) + \sum_{j=1}^{k} b_j x(\alpha_j + [t-1]) = 0, \quad 0 \le \alpha_j < 1, \quad j = 1, \dots, k,$$
which is a partial case of (1.1) with $R(s) = \sum_{j=1}^{k} b_j \chi_{(\alpha_j,1]}(s)$.

(2) The present paper contains a comprehensive analysis of (1.1) which can be reduced to an autonomous second order difference equation. Using the same method, reduce
$$x'(t) + ax(t) + \int_0^1 x(s + [t-1]) d_s R(t, s) = 0, \tag{4.1}$$
to the second order nonautonomous difference equation: in (2.4) we will have $P_n(a)$, $Q_n(a)$ rather than $P(a)$ and $Q(a)$. Deduce sufficient stability, oscillation and nonoscillation conditions for (4.1).

(3) Consider the equation, where the derivative depends on the solution in some previous intervals
$$x'(t) + ax(t) + \sum_{j=1}^{k} \int_0^1 x(s + [t-j]) d_s R_j(s) = 0 \tag{4.2}$$
and its nonautonomous version. Reduce (4.2) to a high order difference equation, establish oscillation and stability conditions.

(4) Equations with piecewise constant delays are sometimes considered as a semidiscretization of delay equations [12]. If (1.1) is a semidiscretization of the integro-differential equation
$$x'(t) + ax(t) + \int_0^1 x(s + t - 1) d_s R(s) = 0,$$
study the relation between oscillation and stability conditions of two equations. Consider a more accurate semidiscretization of type (4.2)
$$x'(t) + ax(t) + \sum_{j=1}^{k} \int_0^r x\Big(s + r\big[\frac{t-j}{r}\big]\Big) d_s R_j(s) = 0 \tag{4.3}$$

and study the convergence of solutions. If $a = 0$ and $R_j(s)$ are step functions then we obtain the well known convergence problem for a finite difference approximation.

5. Appendix

In the proof of Theorems 3.4 and 3.5 we will apply the following obvious result.

Lemma 5.1. *For any real number x the following inequalities hold:*
1. *If $x \neq 0$ then $(1-e^{-x})/x > 0$;*
2. *If $x \neq 0$ then $xe^{-x} + e^{-x} - 1 < 0$;*
3. *If $x < 0$ then $2 - xe^{-x} - 2e^{-x} - x > 0$, if $x > 0$ then $2 - xe^{-x} - 2e^{-x} - x < 0$;*
4. *$xe^{-x} - 1 < 0$;*
5. *If $x < 0$ then $1 - xe^{-x} - e^{-x} - x > 0$, if $x > 0$ then $1 - xe^{-x} - e^{-x} - x < 0$;*
6. *If $x \neq 0$ then $e^{-x} + x - 1 > 0$.*

Proof of Theorem 3.4. We recall that we have to prove that a solution of (3.6), (2.1) oscillates for any initial function satisfying (A2) if and only if

$$b > a^2\left(\frac{|1-e^{-a}| - \sqrt{1-e^{-a}-ae^{-a}}}{a-1+e^{-a}}\right)^2, \quad \text{if } a \neq 0,$$
$$\text{and} \quad b > 6 - 4\sqrt{2}, \quad \text{if } a = 0.$$

First, consider $a \neq 0$. By Theorem 3.2, the solution of the initial value problem (3.6), (2.1) oscillates for any φ if and only if either (3.2) or (3.3) holds. Substitute $P(a)$, $Q(a)$ from (3.7) into (3.2), (3.3) and obtain that at least one of the following two inequalities should hold:

$$b\frac{1-e^{-a}}{a} > \frac{a}{4}\Big(e^{-a} + b\frac{a-1+e^{-a}}{a^2}\Big)^2/(1-e^{-a}),$$
$$e^{-a} \le b\frac{a-1+e^{-a}}{a^2} \le b\frac{1-e^{-a}}{a}\frac{e^a-1}{a}.$$

The first inequality above is equivalent to

$$4b\Big(\frac{1-e^{-a}}{a}\Big)^2 > \Big(e^{-a} + b\frac{a-1+e^{-a}}{a^2}\Big)^2$$

which is a quadratic inequality in b

$$(a-1+e^{-a})^2b^2 + 2a^2(e^{-a}(a-1+e^{-a}) - 2(1-e^{-a})^2)b + (a^2e^{-a})^2 < 0.$$

The corresponding quadratic equation has the discriminant

$$\begin{aligned}
&4a^4(e^{-a}(a-1+e^{-a}) - 2(1-e^{-a})^2)^2 - 4(a-1+e^{-a})^2(a^2e^{-a})^2 \\
&= 4a^4\Big[e^{-2a}(a-1+e^{-a})^2 - 4(1-e^{-a})^2(a-1+e^{-a})e^{-a} + 4(1-e^{-a})^4 \\
&\quad - e^{-2a}(a-1+e^{-a})^2\Big] \\
&= 16a^4(1-e^{-a})^2((1-e^{-a})^2 - (a-1+e^{-a})e^{-a}) \\
&= 16a^4(1-e^{-a})^2\,(1-e^{-a}-ae^{-a}).
\end{aligned}$$

Note that by Lemma 5.1, Part 2, we have $1 - e^{-a} - ae^{-a} > 0$ for any $a \neq 0$. Therefore, the quadratic equation has two real solutions, $b_1 < b_2$, given by the quadratic formula

$$\begin{aligned} b &= \frac{2a^2\left(2(1-e^{-a})^2 - e^{-a}(a-1+e^{-a})\right) \pm 4a^2|1-e^{-a}|\sqrt{1-e^{-a}-ae^{-a}}}{2(a-1+e^{-a})^2} \\ &= a^2\frac{(1-e^{-a})^2 + (1-e^{-a}-ae^{-a}) \pm 2|1-e^{-a}|\sqrt{1-e^{-a}-ae^{-a}}}{(a-1+e^{-a})^2} \\ &= a^2\Big(\frac{|1-e^{-a}| \pm \sqrt{1-e^{-a}-ae^{-a}}}{a-1+e^{-a}}\Big)^2. \end{aligned} \tag{5.1}$$

Then the solution b of the quadratic inequality satisfies $b_1 < b < b_2$.

Consider the second inequality of the system:

$$e^{-a} \le b\frac{a-1+e^{-a}}{a^2} \le b\frac{1-e^{-a}}{a}\frac{e^a-1}{a}$$

if and only if

$$a^2e^{-a} \le b(a-1+e^{-a}) \le b(e^a+e^{-a}-2).$$

Since $a-1+e^{-a} > 0$ by Lemma 5.1, Part 6, then the latter inequality is equivalent to $b \ge \frac{a^2e^{-a}}{a-1+e^{-a}}$. Moreover, $b(a-1+e^{-a}) \le b\,(e^a+e^{-a}-2)$ can be rewritten as $b(e^a-a-1) \ge 0$, which is equivalent to $b \ge 0$, since $e^a-a-1 > 0$ by Lemma 5.1, Part 6.

Lemma 5.1, Part 6, implies $(a^2e^{-a})/(a-1+e^{-a}) > 0$ for $a \ne 0$, so

$$e^{-a} \le b\frac{a-1+e^{-a}}{a^2} \le b\frac{1-e^{-a}}{a}\frac{e^a-1}{a} \quad\Leftrightarrow\quad b \ge \frac{a^2e^{-a}}{a-1+e^{-a}}.$$

Thus, the solution of the initial value problem (3.6), (2.1) oscillates for any φ if and only if

$$\text{either}\quad b_1 < b < b_2 \quad\text{or}\quad b \ge (a^2e^{-a})/(a-1+e^{-a}),$$

where b_1, b_2 are defined in (5.1). To simplify this system let us prove that $b_2 > a^2e^{-a}/(a-1+e^{-a})$. In fact,

$$\begin{aligned} b_2 - \frac{a^2e^{-a}}{a-1+e^{-a}} &= a^2\Big(\frac{|1-e^{-a}| + \sqrt{1-e^{-a}-ae^{-a}}}{a-1+e^{-a}}\Big)^2 - \frac{a^2e^{-a}}{a-1+e^{-a}} \\ &= a^2\Big(|1-e^{-a}|^2 + 2|1-e^{-a}|\sqrt{1-e^{-a}-ae^{-a}} + 1 - e^{-a} \\ &\quad - ae^{-a} - e^{-a}(1-e^{-a}-ae^{-a})\Big)\big/\big(1-e^{-a}-ae^{-a}\big)^2 \\ &\le a^2\frac{|1-e^{-a}|^2 + 1 - e^{-a} - ae^{-a} - e^{-a}(1-e^{-a}-ae^{-a})}{(1-e^{-a}-ae^{-a})^2} \\ &= 2a^2\frac{1-e^{-a}-ae^{-a}}{(a-1+e^{-a})^2}. \end{aligned}$$

By Lemma 5.1, Part 6, we have $e^a - a - 1 > 0$, so $2a^2\frac{1-e^{-a}-ae^{-a}}{(a-1+e^{-a})^2} > 0$ and thus $b_2 > \frac{a^2e^{-a}}{a-1+e^{-a}}$. Therefore, the oscillation condition becomes

$$b > a^2\Big(\frac{|1-e^{-a}| - \sqrt{1-e^{-a}-ae^{-a}}}{a-1+e^{-a}}\Big)^2.$$

Next let $a = 0$. By Theorem 3.2, the solution of the initial value problem (3.6), (2.1) oscillates for any φ if and only if either (3.2) or (3.3) holds. Substituting

$P(0) = b$, $Q(0) = b/2$ from (3.7) into (3.2), (3.3), we obtain

$$\begin{aligned} & b > \frac{1}{4}\big(\frac{b}{2}+1\big)^2 \text{ or } 1 \le \frac{b}{2} \le b \\ \Leftrightarrow \quad & b^2 - 12b + 4 < 0 \text{ or } b \ge 2 \\ \Leftrightarrow \quad & 6 - 4\sqrt{2} < b < 6 + 4\sqrt{2} \text{ or } b \ge 2 \\ \Leftrightarrow \quad & b > 6 - 4\sqrt{2}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.5. We remark that we have to prove that the exponential stability of (3.6) is equivalent to the following systems (in each of the three cases $a < 0$, $a = 0$, $a > 0$):

$$b > -a \text{ and } b < \frac{a^2}{1 - e^{-a} - ae^{-a}} \quad \text{if } a < 0, \tag{5.2}$$

$$0 < b < 2, \quad \text{if } a = 0, \tag{5.3}$$

$$b > -a, \quad b < -\frac{a^2(1+e^{-a})}{2 - 2e^{-a} - ae^{-a} - a}, \quad b < \frac{a^2}{1 - e^{-a} - ae^{-a}} \quad \text{if } a > 0. \tag{5.4}$$

First, consider $a \ne 0$. By Theorem 3.3, equation (3.6) is exponentially stable if and only if inequalities (3.5) hold. After substituting $P(a)$, $Q(a)$ from (3.7) into (3.5) we obtain

$$\begin{gathered} b\frac{1-e^{-a}}{a} > -a\Big(b\frac{a-1+e^{-a}}{a^2} + 1\Big), \\ b\frac{1-e^{-a}}{a} > a\Big(b\frac{a-1+e^{-a}}{a^2} - 1\Big)\frac{1+e^{-a}}{1-e^{-a}}, \\ b\frac{1-e^{-a}}{a} < \frac{a}{1-e^{-a}}\Big(1 + e^{-a}b\frac{a-1+e^{-a}}{a^2}\Big), \end{gathered}$$

which can be rewritten as

$$\begin{gathered} b\Big(\frac{1-e^{-a}}{a} + \frac{a-1+e^{-a}}{a}\Big) > -a, \\ b\Big(\frac{1-e^{-a}}{a} - \frac{a-1+e^{-a}}{a}\cdot\frac{1+e^{-a}}{1-e^{-a}}\Big) > -a\frac{1+e^{-a}}{1-e^{-a}}, \\ b\Big(\frac{1-e^{-a}}{a} - \frac{e^{-a}(a-1+e^{-a})}{a(1-e^{-a})}\Big) < \frac{a}{1-e^{-a}}. \end{gathered}$$

These inequalities can be simplified to the form

$$\begin{gathered} b > -a, \\ b\frac{2 - 2e^{-a} - a - ae^{-a}}{a(1-e^{-a})} > -a\frac{1+e^{-a}}{1-e^{-a}}, \\ b\frac{1 - e^{-a} - ae^{-a}}{a(1-e^{-a})} < \frac{a}{1-e^{-a}}. \end{gathered} \tag{5.5}$$

Consider $a < 0$. For negative a we have $1 - e^{-a} < 0$ and $a(1 - e^{-a}) > 0$. Moreover, $2 - 2e^{-a} - a - ae^{-a} > 0$ and $1 - e^{-a} - ae^{-a} > 0$ by Lemma 5.1, Parts 3 and 2. Inequalities (5.5) are equivalent to

$$
\begin{gathered}
b > -a, \\
b > -\frac{a^2(1+e^{-a})}{2-2e^{-a}-ae^{-a}-a}, \\
b < \frac{a^2}{1-e^{-a}-ae^{-a}}.
\end{gathered}
$$

To simplify this system let us compare $-a$ and $-(a^2(1+e^{-a}))/(2-2e^{-a}-ae^{-a}-a)$. Since

$$
-\frac{a^2(1+e^{-a})}{2-2e^{-a}-ae^{-a}-a} - (-a) = 2a\frac{1-e^{-a}-a-ae^{-a}}{2-2e^{-a}-ae^{-a}-a},
$$

then by Lemma 5.1, Parts 3 and 5, $a<0$ implies $1-e^{-a}-a-ae^{-a}>0$ and $2-2e^{-a}-ae^{-a}-a>0$. Hence $a\frac{1-e^{-a}-a-ae^{-a}}{2-2e^{-a}-ae^{-a}-a}<0$ and $-\frac{a^2(1+e^{-a})}{2-2e^{-a}-ae^{-a}-a}<-a$; therefore,

$$
\begin{gathered}
b > -a, \\
b > -\frac{a^2(1+e^{-a})}{2-2e^{-a}-ae^{-a}-a}, \\
b < \frac{a^2}{1-e^{-a}-ae^{-a}}
\end{gathered}
$$

is equivalent to

$$
\begin{gathered}
b > -a, \\
b < \frac{a^2}{1-e^{-a}-ae^{-a}}.
\end{gathered}
$$

Thus, if $a<0$ then (3.6) is exponentially stable if and only if system (5.2) holds.

Consider $a>0$. If $a>0$ then $1-e^{-a}>0$ and $a(1-e^{-a})>0$. Moreover, by Lemma 5.1, Parts 2 and 3, we have $2-ae^{-a}-2e^{-a}-a<0$ and $1-e^{-a}-ae^{-a}>0$. Applying these inequalities, we obtain that (5.5) is equivalent to

$$
\begin{gathered}
b > -a, \\
b < -\frac{a^2(1+e^{-a})}{2-2e^{-a}-ae^{-a}-a}, \\
b < \frac{a^2}{1-e^{-a}-ae^{-a}}.
\end{gathered}
$$

So, if $a>0$ then (3.6) is exponentially stable if and only if inequalities (5.4) hold.

Finally, let $a=0$. By Theorem 3.3 equation (3.6) is exponentially stable if and only if inequalities (3.5) hold. Substitute $P(0)=b$, $Q(0)=b/2$ from (3.7) into (3.5):

$$
b>0, \quad b>b-2, \quad b<\frac{b}{2}+1.
$$

which can be rewritten as

$$
b>0, \quad b<2.
$$

So, if $a=0$ then (3.6) is exponentially stable if and only if inequalities (5.3) hold. ∎

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Elena Braverman
Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, AB, Canada T2N 1N4
E-mail address: maelena@math.ucalgary.ca Fax (403)-282-5150 Phone (403)-220-3956

Sergey Zhukovskiy
Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, AB, Canada T2N 1N4
E-mail address: sergey@math.ucalgary.ca; s-e-zhuk@yandex.ru