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A GLOBAL APPROACH TO GROUND STATE SOLUTIONS

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Abstract. We study radial solutions of semilinear Laplace equations. We try to understand all solutions of the problem, regardless of the boundary behavior. It turns out that one can study uniqueness or multiplicity properties of ground state solutions by considering curves of solutions of the corresponding Dirichlet and Neumann problems. We show that uniqueness of ground state solutions can sometimes be approached by a numerical computation.

1. Introduction

The question of uniqueness of ground state solutions of semilinear elliptic problems has been of great interest in recent years, given its importance for applications to physics and other sciences, see e.g. Serrin and Tang [18] and Peletier and Serrin [15], which contain references to many other papers. Namely, one is looking for the positive solutions of

$$\Delta u(x) + f(u(x)) = 0, \quad x \in \mathbb{R}^n, \quad u(x) \to 0, \quad \text{as } |x| \to \infty. \tag{1.1}$$

It follows from Gidas, Ni and Nirenberg [6] that (under a mild smoothness assumption, and for functions $f(u)$ consistent with the ones we consider below) such solutions are symmetric with respect to some point, which we take to be the origin (see also [5] and [19] for more recent results on symmetry). Then $u = u(r)$ ($r = |x|$) satisfies the ODE

$$\begin{gathered} u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0, \quad 0 < r < \infty \\ u'(0) = u(\infty) = 0 \\ u(r) > 0, \quad u'(r) < 0, \quad 0 < r < \infty. \end{gathered} \tag{1.2}$$

As was pointed out by Peletier and Serrin [16], it is reasonable to study radial ground states; i.e., solutions of (1.2), even for $f(u)$ for which no symmetry result is available for (1.1). We will approach the ground state solutions by trying to understand *all* solutions of the equation in (1.2).

Namely, consider shooting for

$$u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0 \quad u(0) = \alpha, \quad u'(0) = 0. \tag{1.3}$$

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We shall assume that $f(u)$ satisfies

$$\begin{gathered} f(u) \in C^1[0,\infty), \quad f(0)=0, \text{ and for some } b>0 \text{ we have:} \\ f(u)<0 \text{ on } (0,b), \text{ and } f(u)>0 \text{ for } b<u<c, \text{ with } c \le \infty \;\; f'(b)>0; \\ f(c)=0, \text{ in case } c<\infty. \end{gathered} \tag{1.4}$$

For most of our results we shall also assume that

$$\text{there is } \theta > b, \text{ such that } \quad \int_0^\theta f(u)\,du = 0. \tag{1.5}$$

If $u(0)=\alpha>b$, then $u''(0)=-\frac{1}{n}f(\alpha)<0$, and the solution is decreasing for small r. If also $f(u)$ is bounded or subcritical, then, for α large enough, $u(r)$ will become zero at some point, see e.g. Ouyang and Shi [14] for a collection of existence results for the Dirichlet problem. After rescaling, we can identify the corresponding solution with a solution of the Dirichlet problem

$$u''(r)+\frac{n-1}{r}\,u'(r)+\lambda f(u)=0, \quad 0<r<1, \quad u'(0)=0, u(1)=0, \tag{1.6}$$

for some value of $\lambda>0$. We then say that the initial value α belongs to the *Dirichlet range*. The key is to prove uniqueness of the Dirichlet curve (i.e. the solution curve of (1.6)), and that this curve extends to infinity. It follows that the Dirichlet range is (d,∞) for some $d>\theta$ ($\int_0^\theta f(u)\,du=0$), which means that the range of α's giving rise to ground states is compact.

Below the Dirichlet range there is *Neumann range*; i.e., when solution of (1.3) develops a zero slope. After stretching, we identify the corresponding solution with a solution of the Neumann problem

$$\begin{gathered} u''(r)+\frac{n-1}{r}u'(r)+\lambda f(u)=0, \quad 0<r<1 \\ u'(0)=u'(1)=0 \\ u(r)>0, \quad u'(r)<0, \quad 0<r<1, \end{gathered} \tag{1.7}$$

where λ is a positive parameter. There is a curve of solutions bifurcating from b at $\lambda_1>0$, the principal Neumann eigenvalue. Assume that all solutions of (1.7) lie on a unique solution curve (i.e. continuation in λ produces all solutions of (1.7)). Then the Neumann range is (b,n) for some $n<d$. It follows that (n,d) is the ground state range. So, the number of ground states is either one or infinite. (The number of the ground states can be also zero, in case $f(u)$ behaves like u^s, with $0<s<1$ for small u. Then the Neumann range and Dirichlet range come together to produce a dead core, i.e. solutions that are positive on $[0,r_1)$ and identically zero on $[r_1,1)$, for some $r_1\in(0,1)$. Our condition (1.4) prevents that from happening, in view of the Hopf's boundary lemma.) By studying the variational equation, one finally shows that it is impossible to have a continuum of ground states.

We were able to prove uniqueness of the Neumann curve under the condition

$$\frac{f(u)}{u-b} > f'(u) \quad \text{ for all } u>0. \tag{1.8}$$

This condition (going back to Opial [13] in the ODE case, see also Schaaf [17], and Korman [11]) is rather restrictive (it holds if e.g. $f''(u)(u-b)<0$ for $u>0$). We found Neumann curves (i.e. solutions of (1.7)) to be hard to handle. If the Neumann curve bifurcating from $u\equiv b$ is not unique, we show that any other Neumann curve has at least one turn. Computing numerically solutions of variational equation, we are able to rule out the possibility of turns on Neumann curves, implying that the Neumann curve bifurcating from $u \equiv b$ is unique. We believe that the problem (1.7) deserves further study, even for its own sake.

By contrast, for the Dirichlet problem (1.6)) we were able to give a general condition for the uniqueness of the solution curve. This is a crucial result, since it allows us to "compactify" the set of $\alpha = u(0)$ for which ground states may occur, since this Dirichlet curve typically extends to infinity. On a compact set it is feasible to do a computer assisted proof, by studying the corresponding variational equation. (Essentially the same numerical computation rules out both the possibility of Neumann curves with a turn, and intervals of ground states.) We show that uniqueness of the Dirichlet curve (i.e. there is only one solution curve of (1.6)) follows from positivity of any non-trivial solution of the corresponding linearized problem, exactly the same condition that was prominent in the study of exact multiplicity of solutions of (1.6), see Korman, Li and Ouyang [10], Ouyang and Shi [14], and Korman and Ouyang [11]. The same condition occurs in the study of symmetry breaking on a ball, see Korman [11]. A number of conditions for positivity was given in the above papers, and we add yet another one here.

We now summarize our results. We show that a positivity property of the linearized equation implies that ground states are possible only if $u(0)$ belongs to a finite interval. Then we show that the assumption of non-uniqueness of ground states implies that either there is an interval of $u(0)$ giving ground states, or there is a Neumann branch with a turn (or both). Finally, we give conditions ruling out these two possibilities, opening a way to do a computer assisted proof. We conclude with a description of a numerical experiment.

We wish to mention a recent paper of Arioli et al [1], which also contains a computer assisted study of a related problem.

Throughout the paper we consider mostly the classical solutions. We shall use the following bifurcation theorem of Crandall and Rabinowitz [3].

Theorem 1.1 ([3]). *Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and let $\operatorname{codim} R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2. Uniqueness for a class of Neumann problems

We shall study solution curves for the Neumann problem (1.7), with $f(u)$ satisfying (1.4). Since solutions of (1.7) are decreasing, and $u''(0) = -\frac{\lambda}{n} f(u(0))$, it follows that $u(0) > b$. Similarly, $u''(1) = -\lambda f(u(1))$, which implies that $u(1) < b$ (the possibilities of $u(0) = b$ or $u(1) = b$ are ruled out by the uniqueness theorem, which would imply $u \equiv b$).

The linearized problem corresponding to (1.7) is

$$\begin{gathered} w''(r) + \frac{n-1}{r} w'(r) + \lambda f'(u) w = 0, \quad 0 < r < 1 \\ w'(0) = w'(1) = 0. \end{gathered} \tag{2.1}$$

For any non-trivial solution of (2.1) we may assume that $w(0) > 0$. We have the following rather general lemma.

Lemma 2.1. *Let $u(r)$ be a solution of (1.7), and assume that $f(u)$ satisfies (1.4). Then any non-trivial solution of (2.1) must be sign changing.*

Proof. Assume on the contrary that $w(r) > 0$. Differentiating (1.7), we have

$$u''' + \frac{n-1}{r} u''(r) + \lambda f'(u) u' - \frac{n-1}{r^2} u' = 0.$$

Combining this with (2.1)

$$[r^{n-1}(w'u' - wu'')]' + (n-1) r^{n-3} u' w = 0.$$

It follows that the function $q(r) \equiv r^{n-1}(w'u' - wu'')$ is increasing on $(0,1)$. But $q(0) = 0$, while

$$q(1) = -w(1) u''(1) = \lambda w(1) f(u(1)) < 0,$$

which is a contradiction. □

Lemma 2.2. *Assume $f(u)$ satisfies the condition (1.4) and (1.8). Then the problem (2.1) has no non-trivial solutions.*

Proof. Let $\xi \in (0,1)$ be the point where $u(\xi) = b$. According to the previous lemma, $w(r)$ has to change sign on $(0,1)$. However, we shall show that $w(r)$ cannot vanish on $(0,\xi]$, and on $[\xi,1)$.

We rewrite the equation in (1.7) in the form

$$(u-b)''(r) + \frac{n-1}{r} (u-b)'(r) + \lambda \frac{f(u)}{u-b} (u-b) = 0. \tag{2.2}$$

Multiplying (2.2) by w, equation (2.1) by $u-b$, and subtracting

$$[r^{n-1}\big(wu' - w'(u-b)\big)]' + \lambda r^{n-1} \big[\frac{f(u)}{u-b} - f'(u)\big](u-b) w = 0. \tag{2.3}$$

If we now assume that $w(r)$ vanishes on $(0,\xi]$, we can find $\eta \in (0,\xi]$ (the first zero), so that $w(r) > 0$ on $(0,\eta)$, and $w(\eta) = 0$. Integrating (2.3) over $(0,\eta)$, we obtain

$$-\eta^{n-1} w'(\eta)(u(\eta) - b) + \lambda \int_0^\eta r^{n-1} [\frac{f(u)}{u-b} - f'(u)](u-b) w \, dr = 0.$$

Since both quantities on the left are positive, we have a contradiction. In case $w(r)$ vanishes on $[\xi,1)$, we can find $\eta \in [\xi,1)$ (the last zero), so that $w(r) > 0$ on $(\eta,1]$, and $w(\eta) = 0$. Integrating (2.3) over $(\eta,1)$, we have

$$\eta^{n-1} w'(\eta)(u(\eta) - b) + \lambda \int_\eta^1 r^{n-1} \big[\frac{f(u)}{u-b} - f'(u)\big](u-b) w \, dr = 0.$$

Since $u(r) < b$ over $(\eta,1]$, both terms on the left are negative, which is a contradiction. □

The following lemma will be used for continuation of Neumann branches.

Lemma 2.3. *Assume that $u(r)$ is a singular solution of the Neumann problem* (1.7)*; i.e., the linearized problem* (2.1) *has a non-trivial solution $w(r)$. Then*

$$\int_0^1 f(u)wr^{n-1}\,dr = -\frac{1}{2\lambda}w(1)u''(1). \tag{2.4}$$

Proof. The function $z(r) = ru_r$ satisfies the equation

$$z''(r) + \frac{n-1}{r}z'(r) + \lambda f'(u)z = -2\lambda f(u). \tag{2.5}$$

Combining this with (2.1)

$$[r^{n-1}(z'w - zw')]' = -2\lambda f(u)wr^{n-1}.$$

Integrating over $(0,1)$, we obtain (2.4). □

We have the following existence and uniqueness result for the Neumann problem (1.7), whose proof we postpone for the next section. We denote by $\lambda_1 > 0$ the first positive eigenvalue of the Neumann problem for $-\Delta$, acting on radial functions on a unit ball.

Theorem 2.1. *The problem* (1.7)*, with $f(u)$ satisfying* (1.4)*,* (1.5)*, and* (1.8) *has a unique solution for every $\lambda > \lambda_1/f'(b)$. Moreover, all solutions of* (1.7) *lie on a unique curve of solutions. This curve bifurcates from the trivial solution $u = b$, and continues without any turns for all $\lambda > \lambda_1/f'(b)$. Moreover, the maximum value of solution, $u(0,\lambda)$, is monotone increasing in λ, with a finite limit $\lim_{\lambda\to\infty} u(0,\lambda) > \theta$.*

3. General properties of Dirichlet and Neumann branches

We shall discuss some properties of positive solutions of the Dirichlet problem

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0, \quad 0 < r < 1, \quad u'(0) = 0, \quad u(1) = 0, \tag{3.1}$$

depending on a positive parameter λ.

The following result from Korman [7] gives a detailed description of the shape of solutions for large λ. It holds for general $f(u)$, in particular we do not assume the condition (1.4) here. Recall a customary convention that root α of $f(u)$ is called *stable* if $f(\alpha) = 0$ and $f'(\alpha) < 0$. It turns out that for large λ solutions of (3.1) concentrate at stable roots of $f(u)$, as the following result shows.

Theorem 3.1. *Let $u(r,\lambda)$ be a branch of positive solutions of* (3.1)*, that exists for all $\lambda > \bar{\lambda}$, with some $\bar{\lambda} > 0$. Assume that $u(r,\lambda)$ does not tend to infinity over any subinterval of $(0,1)$ (which may happen if e.g. $\lim_{u\to\infty}\frac{f(u)}{u} = \infty$, or if there is $u_0 > 0$, so that $f(u) \le 0$ for $u \ge u_0$). Then the interval $(0,1)$ can be decomposed into a union of open intervals, whose total length $= 1$, so that on each such subinterval $u(r,\lambda)$ tends to a stable root of $f(u)$, as $\lambda \to \infty$.*

If for some reason solutions cannot be of that shape, it follows that there are no positive solutions of (1.6) for large λ, as happens in the following example.

Example. Assume that $f(u) < 0$ for $0 \le u < b$, with some $b > 0$, $f(u) > 0$ for $u > b$, and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, e.g. $f(u) = u^p - 1$, with $p > 1$. Then the problem (3.1) has no positive solution for λ large enough. Indeed, since $f(u)$ has no stable roots, solutions cannot exhibit the behavior described in the above theorem, and hence solutions cannot continue to exist for all large λ.

In this example $f(0) < 0$. If, instead, we have $f(0) = 0$ (as is the case when $f(u)$ satisfies our condition (1.4)), then it is possible for solution to exist for large λ, concentrating at zero. If condition (1.4) holds, but $u(r, \lambda)$ does not concentrate at zero, then it must tend to infinity over a subinterval of $[0, 1)$, in view of Theorem 3.1, in particular its maximum value $u(0, \lambda)$ goes to infinity.

The following lemma is essentially known.

Lemma 3.1. *The value of $u(0) = \alpha$ uniquely identifies the solution pair $(\lambda, u(r))$ for both the Dirichlet problem* (3.1) *and the Neumann problem* (1.7) *(i.e. there is at most one λ, with at most one solution $u(r)$, so that $u(0) = \alpha$).*

Proof. This result is well known for the Dirichlet case, see e.g. [14] or [8], so let us prove it for the Neumann problem (1.7). Assume on the contrary that we have two solution pairs $(\lambda, u(r))$ and $(\mu, v(r))$, with $u(0) = v(0) = \alpha$. Clearly, $\lambda \ne \mu$, since otherwise we have a contradiction with uniqueness of initial value problems, which was established for this class of problems by Peletier and Serrin [15]. Then $u(\frac{1}{\sqrt{\lambda}} r)$ and $v(\frac{1}{\sqrt{\mu}} r)$ are both solutions of the same initial value problem

$$u''(r) + \frac{n-1}{r} u'(r) + f(u) = 0 \quad u(0) = \alpha, \quad u'(0) = 0,$$

and hence $u(\frac{1}{\sqrt{\lambda}} r) = v(\frac{1}{\sqrt{\mu}} r)$, but that is impossible, since the derivative of the first function has its first root at $r = \sqrt{\lambda}$, while the derivative of the second one has its first root at $r = \sqrt{\mu}$. $\square$

The linearized problem corresponding to (3.1) is

$$w''(r) + \frac{n-1}{r} w'(r) + \lambda f'(u) w = 0, \quad 0 < r < 1, \quad w'(0) = w(1) = 0. \tag{3.2}$$

Theorem 3.2. *Assume that $f(u)$ satisfies the conditions* (1.4) *and* (1.5)*, and assume that any non-trivial solution of* (3.2) *is of one sign. Then all positive solutions of the Dirichlet problem* (3.1) *lie on a unique solution curve.*

Proof. It is known that all solutions of the Dirichlet problem (3.1) lie on smooth solution curves, see [10], [14]. What this means is that at any regular solution $(\lambda, u(r))$, i.e. when the problem (3.2) has only the trivial solution, one can apply the implicit function theorem, and continue the solution in λ, while at any singular solution $(\lambda_0, u_0(r))$, i.e. when the problem (3.2) admits a non-trivial solution, the Crandall and Rabinowitz Theorem 1.1 applies. This theorem implies that only simple turns are possible at singular solutions. Moreover, this theorem shows that the solution curve near a turning point has the form

$$u(r, s) = u_0(r) \pm s w(r) + o(s) \quad \text{for } s \text{ close to } 0, \tag{3.3}$$

where s is some parameter (e.g. $s = u(0) - u_0(0)$). It follows that near a turning point $u_\lambda(r, \lambda)$ is proportional to $w(r)$, and hence $u_\lambda(r, \lambda)$ is positive (negative) on the upper branch, and negative (positive) on the lower branch if a turn to the right (left) occurs (see [10] for more details). It was proved in [10] that positivity of $u_\lambda(r,\lambda)$ persists on a branch (until a possible next turning point).

Let us now start with an arbitrary solution of the Dirichlet problem (3.1) and continue it for decreasing λ. Let us assume first that $c = \infty$ in (1.4). We will show that the maximum value $u(0,\lambda)$ has to tend to infinity along this curve. This will imply uniqueness of the solution curve, because by Lemma 3.1 solutions are identified by the values of $u(0)$, and our solution curve has "taken up" all possible large values of $u(0)$. It might happen that $u(0,\lambda)$ goes to infinity as $\lambda \downarrow \lambda_1$ for some $\lambda_1 \geq 0$, then we are done. Otherwise, the curve will have to make a turn to the right at some (λ_0, u_0). (It cannot go to zero by maximum principle, and it cannot cross the line $\lambda = 0$.)

At the turning point (λ_0, u_0), we have an upper branch which is increasing in λ for all r. This upper branch, let us refer to it as $u_0(r,\lambda)$, might continue without any more turns for all $\lambda > \lambda_0$. But then $u(0,\lambda)$ on this branch tends to infinity in view of Theorem 3.1 (see remarks after it), since solutions on this branch cannot tend to zero, the only stable root of $f(u)$. Another possibility is for $u_0(r,\lambda)$ to reach another turning point (λ_1, u_1), at some $\lambda_1 > \lambda_0$. At the second turning point we have another upper branch, denoted $u_1(r,\lambda)$, which we continue for decreasing λ. By the Crandall-Rabinowitz theorem 1.1, $u_1(r,\lambda) > u_0(r,\lambda)$ for λ close to λ_1 (here we use the positivity of $w(r)$ again). The inequality $u_1(r,\lambda) > u_0(r,\lambda)$ persists for all λ, since otherwise we would get a contradiction at a point where these solution "touch" (by maximum principle). Hence either $u_1(r,\lambda)$ goes to infinity for decreasing λ, or it will make a turn to the right at some (λ_2, u_2), with the upper branch (denote it by $u_2(r,\lambda)$) increasing in λ, resuming its march to infinity. (We have $u_2(r,\lambda) > u_1(r,\lambda) > u_0(r,\lambda)$, for all λ where all three branches are defined.) In either case, $u(0,\lambda)$ on this branch tends to infinity. (This branch may go to infinity as λ tends to either infinity or to some finite number. The second case is quite possible, since we do not restrict the behavior of $f(u)$ at infinity.)

In case $c < \infty$ in (1.4), similar arguments show that one end of every solution curve tends to c, implying uniqueness of the solution curve, in view of Lemma 3.1. ☐

Remark. If solution curve turns at some (λ_0, u_0), then (λ_0, u_0) is a singular solution, i.e. the linearized problem (3.2) has a nontrivial solution $w(r)$. Clearly, not every singular solution is a turning point. Examining the proof, we see that it suffices to have $w(r) > 0$ at turning points only.

A number of conditions are known for positivity of $w(r)$, see [14], [10], [11]. We will formulate another such condition, which is based on Peletier and Serrin [15], [16].

Theorem 3.3. *Assume that $f(u)$ satisfies the conditions* (1.4) *and* (1.5)*, and moreover*

$$\frac{f(u)}{u-\theta} \quad \text{is non-increasing for } u > \theta. \tag{3.4}$$

Then at any turning point of (3.1) *any non-trivial solution of* (3.2) *is of one sign.*

Proof. It was shown in [10] that Crandall and Rabinowitz bifurcation Theorem 1.1 applies at any turning point of (3.1). Namely, near a turning point (λ_0, u_0) the solution set of (3.1) consists of a curve described in (3.3). It follows from (3.3) that two branches near a turning point (λ_0, u_0) do not intersect if and only if $w(r)$ is of one sign. Hence, it suffices to show that under the condition (3.4) any two positive solutions of (3.1) cannot intersect. This result is essentially due to Peletier and Serrin [15], and we review its proof next.

One begins by showing that any two solutions cannot intersect in the region, where $u(r) < \theta$. This was essentially done by Peletier and Serrin [15], who proved the same result for ground state solutions. Later Kwong and Zhang [12] observed that the same proof works also for the Dirichlet problem, and the details can be found in [9]. Then one shows that two solutions cannot intersect in the region, where $u(r) \geq \theta$ too. This fact was also proved by Peletier and Serrin [15]. We present next a considerably simpler proof.

So assume for contradiction that two solutions $u(r)$ and $v(r)$ intersect at a point R, with $u(R) = v(R) \geq \theta$. We may assume ξ to be the first point of intersection, and $u(r) > v(r) \geq \theta$ on $[0, R)$. Set $U = u - \theta$, and $V = v - \theta$. Then $U > V > 0$ on $[0, R)$. We rewrite the equation (3.1) as

$$U'' + \frac{n-1}{r}U' + \lambda\frac{f(u)}{u-\theta}U = 0.$$

Doing the same for v, and combining both equations, we have

$$[r^{n-1}(U'V - UV')]' + \lambda r^{n-1}\big[\frac{f(u)}{u-\theta} - \frac{f(v)}{v-\theta}\big]UV = 0.$$

By our condition (3.4) it follows that the function $p(r) \equiv r^{n-1}(U'V - UV')$ is non-decreasing on $[0, R)$. But $p(0) = 0$, while

$$p(R) = \mathbb{R}^{n-1}U(R)(u'(R) - v'(R)) < 0,$$

a contradiction. □

Lemma 3.2. *Let $u(r)$ be a positive solution of either ground state problem* (1.2) *or the Dirichlet problem* (3.1). *Then $u(0) > \theta$.*

Proof. Let us consider the problem (1.2), since in the Dirichlet case the proof is similar, and the result is better known, see e.g. [10]. The "energy" $E(r) = \frac{1}{2}u'^2(r) + F(u(r))$ is decreasing, since $E'(r) = -\frac{n-1}{r}u'^2(r)$. If we assume that $u(0) \leq \theta$, then $E(0) \leq 0$, and it follows that $E(r)$ is bounded above by a negative constant for large r. This is inconsistent with $F(u(r)) \to 0$, as $r \to \infty$. □

The following theorem describes the Neumann curve bifurcating from $u \equiv b$ at $\lambda = \lambda_1/f'(b)$, where $\lambda_1 > 0$ is the first positive eigenvalue of the Neumann problem for $-\Delta$ on a unit ball.

Theorem 3.4. *Consider the Neumann problem* (1.7)*, with $f(u)$ satisfying* (1.4) *and* (1.5)*. Assume that all solutions of the corresponding Dirichlet problem lie on a unique solution curve, and $u(0, \lambda)$ on this curve extends to c. Then there is a curve of solutions of* (1.7)*, bifurcating from the trivial solution $u = b$ at $\lambda = \lambda_1/f'(b)$. This curve continues globally, with the maximum value $u(0)$ strictly increasing, and with λ eventually tending to infinity. Moreover, $\lim_{\lambda\to\infty} u(0, \lambda) > \theta$. Any other solution curve has at least one turn to the right, with both branches extending to infinity along the λ axis (i.e. λ tends to infinity on both branches).*

If, in addition, the linearized problem (2.1) *admits only the trivial solution (this happens if e.g. the condition* (1.8) *holds), then the solution curve bifurcating from $u = b$ does not turn, and there are no other solution curves of* (1.7) *(this, in particular, justifies the Theorem 2.1 above).*

Proof. According to the Crandall - Rabinowitz theorem on bifurcation from simple eigenvalue [4], there is a curve of solutions of (1.7), bifurcating from the trivial solution $u = b$ at $\lambda = \lambda_1/f'(b)$ (on a ball all eigenvalues are simple, since the value of $w(1)$ parameterizes the eigenfunctions). Solutions on the bifurcating curve are positive (since they are close to b), and decreasing (since $u - b$ is asymptotically proportional to the first eigenfunction, as $\lambda \to \lambda_1$). For the same reason, the maximum value $u(0, \lambda)$ is increasing along the solution curve. We claim that all three of these properties are preserved along the solution curve. Indeed, the maximum value remains increasing, in view of Lemma 3.1. Writing the equation as $(r^{n-1}u')' = -\lambda r^{n-1} f(u)$, we see that $r^{n-1}u' < 0$ on $(0, 1)$ (since $f(u(r))$ changes sign exactly once), and so $u(r)$ is decreasing. Finally, if $u(r, \lambda)$ were to become zero, this would have to happen at $r = 1$, where we also have $u'(1) = 0$. Hence $u \equiv 0$, contradicting $u(0, \lambda) > b$.

We now continue this curve. If the linearized problem (2.1) admits only the trivial solution, we may use the implicit function theorem to continue this solution curve in λ. We show next that at singular solutions of (1.7), i.e. when the linearized problem (2.1) admits non-trivial solutions, the Crandall - Rabinowitz Theorem 1.1 applies. If $w(r)$ is a non-trivial solution of (2.1), then $w(1) \neq 0$ (since otherwise $w(r) \equiv 0$). As we mentioned above, for decreasing solutions of (1.7) $u(0) > b$, while $u(1) < b$. This implies that $u''(1) = -\lambda f(u(1)) > 0$, and hence by (2.4)

$$\int_0^1 f(u(r))w(r)r^{n-1}\, dr \neq 0. \tag{3.5}$$

We now recast the problem (1.7) in the operator form $F(\lambda, u) : R \times \{H^2(0, 1)|$
$u'(0 = u'(1) = 0\} \to L^2(0, 1)$

$$F(\lambda, u) = u''(r) + \frac{n-1}{r}\, u'(r) + \lambda f(u).$$

The operator $F_u(\lambda, u)$ is then given by the left hand side of the equation (2.1), and hence its null space is one diminsional, spanned by $w(r)$. Since $F(\lambda, u)$ is a Fredholm operator, its range has codimension one. And the crucial condition $F_\lambda \notin R(F_u)$ holds in view of (3.5). Hence, the Crandall - Rabinowitz Theorem 1.1 applies, and we have a curve of H^2 solutions through a critical point, which we then boot-strap to classical solutions.

It follows that any Neumann curve extends globally. Next we observe that there is a "free" a priori estimate for Neumann curves: the maximum value of solution lies below that of any solution of the corresponding Dirichlet problem. Hence solution curves cannot go to infinity at a finite λ. It follows that the curve bifurcating from $u \equiv b$ has one end where $\lambda \to \infty$, while for any other curve both ends extend to infinity in λ, which implies existence of at least one turn to the right on such curves. □

Remarks.

(1) We believe that the curve of solutions bifurcating from $u = b$ exhausts the set of solutions of the Neumann problem (1.7), but we have no proof.

(2) Assume that $c = \infty$, and the condition (1.5) fails, i.e. $\int_0^\infty f(u)\, du \leq 0$. Then the Neumann curve, bifurcating from b, extends to infinity. In this case there are no solutions of both the Dirichlet and ground state problems.

(3) We had assumed existence of solutions for the corresponding Dirichlet problem. For a collection of relevant existence results see T. Ouyang and J. Shi [14].

Theorem 3.5. *With $f(u)$ satisfying* (1.4) *and* (1.5)*, assume that the solution sets for both the Dirichlet problem* (1.6) *and the Neumann problem* (1.7) *consist of a single curve each. Then the set of $\alpha = u(0)$ giving ground state solutions (i.e. solutions of* (1.2)*) is either one point or an interval.*

Proof. Since the Dirichlet curve continues globally for both decreasing and increasing values of $u(0)$, it follows that the set of $\alpha = u(0)$ for which solutions of (1.2) become eventually zero, is an interval (d, c), with some $d < c \leq \infty$. Similarly, the set of $\alpha = u(0)$ giving rise to the Neumann range is (b, n), with some $n \leq d$. If $n = d$, we conclude that $u(0) = n$ gives rise to a unique ground state, otherwise $[d, n]$ produces an interval of ground states. □

4. Numerical computations

If there is an interval of α's leading to ground states $u(r, \alpha)$ (i.e. solutions of (1.2)), then $w \equiv u_\alpha(r, \alpha)$ satisfies

$$\begin{gathered} w''(r) + \frac{n-1}{r} w'(r) + f'(u)w(r) = 0, \quad 0 < r < \infty \\ w'(0) = 0, \quad w(0) = 1, \quad w(\infty) = 0. \end{gathered} \tag{4.1}$$

Similarly, if there is a turn on a curve of solutions of Neumann problem (1.7), then the corresponding linearized problem (2.1) has a non-trivial solution, i.e. after rescaling

$$\begin{gathered} w''(r) + \frac{n-1}{r} w'(r) + f'(u)w(r) = 0, \quad 0 < r < \lambda^2 \\ w'(0) = 0, \quad w(0) = 1, \quad w'(\lambda^2) = 0, \end{gathered} \tag{4.2}$$

for some $\lambda > 0$. Both possibilities can be usually ruled out by the same set of computations, justifying uniqueness of ground state solutions, provided that uniqueness of solution curve for the Dirichlet problem is known. (Ruling out turns on Neumann curves implies that the Neumann curve bifurcating from $u = b$ is unique by the Theorem 3.4, and hence by Theorem 3.5 the set of α's giving rise to ground states is either a point or an interval. And it is a point, once we rule out the case of an interval.)

Example. $f(u) = -u + u^3$, $n = 3$. I.e. we consider "shooting" for the problem

$$u'' + \frac{2}{r}u' - u + u^3 = 0, \quad u(0) = \alpha, \; u'(0) = 0. \tag{4.3}$$

For this $f(u)$ it is known that any non-trivial solution of the linearized problem (3.2) is of one sign, see [14], and hence by Theorem 3.2 the Dirichlet problem (3.1) has a unique solution curve. It follows that the set of α's giving rise to the Dirichlet solutions (i.e. decreasing solutions of (4.3) which vanish eventually) is an interval (d, ∞). Our numerical computations show that $d < 4.34$. We cannot prove uniqueness of the Neumann curve for this $f(u)$, however our Theorem 3.4 shows that the Neumann curve bifurcating from $u \equiv 1$ extends above $u(0) = \theta = \sqrt{2}$. Hence the interval $(1, \sqrt{2})$ is definitely in the Neumann range (i.e. decreasing positive solutions of (4.3) develop zero slope). Hence intervals of ground states, or Neumann branches with a turn, are only possible for $\alpha \in (\sqrt{2}, 4.34)$. This is a finite interval, for which one can give computations (of the type we give below) to show that the Neumann range is actually $(1, d)$, and then $\alpha = d$ gives rise to the unique ground state solution. Things turn out to be even easier. Our computations leave no doubt that the maximum value on the Neumann curve goes much higher than $\theta = \sqrt{2}$. For example, in Figure 1 we take $\alpha = 4$, and see that this α is clearly in the Neumann range.

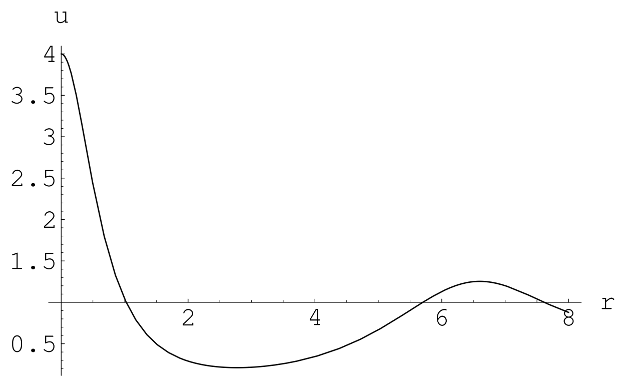

FIGURE 1. Solution of (4.3) with $\alpha = 4$

Similar computations show that the Neumann range extends above $\alpha = 4.3$. This leaves us with the interval $(4.3, 4.34)$, where intervals of ground states, or Neumann branches with a turn, might happen. (Computations show that the Neumann range goes above 4.3 and the Dirichlet range goes below 4.34, but we leave some “safety”, so that the numbers 4.3 and 4.34 could be validated.)

We took $\alpha = 4.32$ in the middle of the range, and computed the solution $u(r)$ of (4.3), and then solved the corresponding variational equation

$$w''(r) + \frac{2}{r} w'(r) + (-1 + 3u^2(r))w(r) = 0, \quad w(0) = 1, \quad w'(0) = 1 \tag{4.4}$$

on the interval $r \in [0, 1.4]$. The result is plotted in Figure 2.

We see that $w(r)$ is strictly decreasing on this interval, with $w(1.4) \simeq -0.29$, and $w'(1.4) \simeq -0.017$. We claim that $w(r)$ continues to decrease for all $r > 1.4$, which means that $w(r)$ cannot be solution of either the problem (4.1) or (4.2). One computes $u(1.4) \simeq 0.48$. Since $u(r)$ is decreasing, it follows that $-1 + 3u^2(r) < 0$ for all $r > 1.4$. Now, $w(r)$ is negative and decreasing at $r = 1.4$, and it cannot turn around to go to zero as $r \to \infty$, since at the turning point, where $w'(r) = 0$ the left hand side of the equation in (4.4) would be positive.

Our computations showed similar results for other α's in the critical range $(4.3, 4.34)$, and that was to be expected, since small changes in α produce small changes in both $u(r)$ and $w(r)$. We believe that these computations can be validated, to produce a computer assisted proof of uniqueness of the ground state, although we did not carry this out. For this equation the uniqueness result of Serrin and Tang [18] also applies.

Finally, we mention that we used *Mathematica*'s **NDSolve** command to solve the problems (4.3), and (4.4). To avoid a singularity at $r = 0$, we took a small

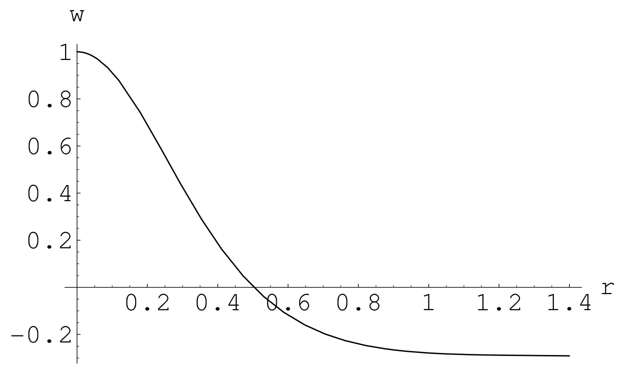

FIGURE 2. Solution of variational equation when $\alpha = 4.32$

$h = 0.0001$, and approximated $u(h) \simeq \alpha + \frac{1}{2}u''(0)h^2 = \alpha - \frac{f(\alpha)}{2n}h^2$, and $u'(h) \simeq u''(0)h = -\frac{f(\alpha)}{n}h$, and then used *Mathematica* to compute for $r > h$ (and we solved (4.4) similarly).

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