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WELL-POSED INITIAL-BOUNDARY VALUE PROBLEMS FOR THE ZAKHAROV-KUZNETSOV EQUATION

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ABSTRACT. This paper deals with non-homogeneous initial-boundary value problems for the Zakharov-Kuznetsov equation, which is one of the variants of multidimensional generalizations of the Korteweg de Vries equation. Results on local and global well-posedness are established in a scale of Sobolev-type spaces under natural assumptions on initial and boundary data.

1. INTRODUCTION

The goal of the present paper is to study initial-boundary value problems for the Zakharov–Kuznetsov (ZK) equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = f(t,x,y) \tag{1.1}$$

($u = u(t,x,y)$) in three domains:

$$\begin{aligned} \Pi_T^+ &= \{(t,x,y) : t \in (0,T), x > 0, y \in \mathbb{R}\} \equiv (0,T) \times \mathbb{R}_+^2, \\ \Pi_T^- &= \{(t,x,y) : t \in (0,T), x < 0, y \in \mathbb{R}\} \equiv (0,T) \times \mathbb{R}_-^2, \\ Q_T &= \{(t,x,y) : t \in (0,T), x \in (0,1), y \in \mathbb{R}\} \equiv (0,T) \times \Sigma, \end{aligned}$$

where $T > 0$. In all three cases we set an initial condition

$$u(0,x,y) = u_0(x,y) \tag{1.2}$$

(where respectively $(x,y) \in \mathbb{R}_+^2$, $(x,y) \in \mathbb{R}_-^2$, $(x,y) \in \Sigma$) and the following boundary conditions for $(t,y) \in B_T = (0,T) \times \mathbb{R}$:
(1) for the problem in Π_T^+ one condition:

$$u(t,0,y) = u_1(t,y), \tag{1.3}$$

(2) for the problem in Π_T^- two conditions:

$$u(t,0,y) = u_2(t,y), \quad u_x(t,0,y) = u_3(t,y), \tag{1.4}$$

(3) for the problem in Q_T three conditions:

$$u(t,0,y) = u_1(t,y), \quad u(t,1,y) = u_2(t,y), \quad u_x(t,1,y) = u_3(t,y). \tag{1.5}$$

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The ZK equation is one of the variants of multidimensional generalizations of the famous Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} + uu_x = f(t,x). \tag{1.6}$$

It describes nonlinear wave processes in dispersive media, when waves propagate in the x-direction and are deformated in the transverse y-direction. In particular, it is a model equation for ion-acoustic waves in magnetized plasma, [22].

Initial-boundary value problems for the KdV equation in domains similar to Π_T^+, Π_T^- and Q_T (without the variable y) have been intensively studied in the recent years (see [6, 2, 3, 10, 15, 12] for the last results and references there). In the first approximation the scheme of these investigations is similar and consists of 1) a proof of local well-posedness based on the contraction principle, where solutions are constructed as fixed points of mappings $u = \Lambda v$ such that u is a solution to a corresponding initial-boundary value problem for an equation $u_t + u_{xxx} = f - vv_x$, and 2) global a priori estimates based on conservation laws for the initial value problem for KdV ($f \equiv 0$):

$$\int_{\mathbb{R}} u^2\,dx = \text{const}, \quad \int_{\mathbb{R}} (u_x^2 - \frac{1}{3}u^3)\,dx = \text{const}, \quad \int_{\mathbb{R}} (u_{xx}^2 + \frac{5}{6}u^2u_{xx} + \frac{5}{36}u^4)\,dx = \text{const}. \tag{1.7}$$

For the initial value problem for KdV this scheme was for the first time implemented in [16].

One of the common features of the aforementioned papers is an idea to establish well-posedness under natural assumptions on initial and boundary data, namely, $u_0 \in H^s$, $u_1, u_2 \in H^{(s+1)/3}$, $u_3 \in H^{s/3}$. Such assumptions originate from internal properties of the operator $\partial_t + \partial_x^3$. In fact, if $u(t,x) \in C(\mathbb{R}^t; H^s(\mathbb{R}^x))$ is a solution to the initial value problem

$$u_t + u_{xxx} = 0, \quad u\big|_{x=0} = u_0(x) \in H^s(\mathbb{R}), \quad s \in \mathbb{R}, \tag{1.8}$$

then for any $x \in \mathbb{R}$

$$\|D_t^{1/3} u(\cdot,x)\|_{H^{s/3}(\mathbb{R}^t)} = \|u_x(\cdot,x)\|_{H^{s/3}(\mathbb{R}^t)} = c(s)\|u_0\|_{H^s(\mathbb{R})} \tag{1.9}$$

(see, for example, [16]).

The pointed out approach requires the necessity of study of the corresponding initial-boundary value problems for the linearized KdV equation. In [10, 12] solutions to such problems are constructed via combination of solutions to the initial value problem and solutions to the initial-boundary value problems for the homogeneous linearized equation (1.8) with zero initial data, which can be referred as "boundary potentials". For the problem in a right half-strip $(0,T) \times \mathbb{R}_+$ such a boundary potential J was for the first time introduced in [5] with the use of the Airy function. Alternative representations for this function J were obtained in the papers [2, 10]. For example, in [10] the following formula was derived:

$$J(t,x;u_1) = \mathcal{F}_t^{-1}\big[e^{r(\lambda)x}\widehat{u}_1(\lambda)\big](t) \tag{1.10}$$

for $x \geq 0$, where $r(\lambda) = -\frac{1}{2}(\sqrt{3}|\lambda|^{1/3} + i\lambda^{1/3})$ is the unique root of the algebraic equation $r^3 + i\lambda = 0$, $\lambda \neq 0$, with the negative real part. Similar boundary potentials for the problem in a left half-strip $(0,T) \times \mathbb{R}_-$ were constructed in [12]. All these boundary potentials were also used in that paper for the problem in a bounded rectangle $(0,T) \times (0,1)$.

As a result, local well-posedness under natural assumptions on initial and boundary data was established for all three initial-boundary value problems for the KdV equation if $s > -3/4$, $s \neq 3m + 1/2$, $s \neq 3m + 3/2$ (for the last two problems), $m \geq 0$ – integer, [10, 15, 12]. Solutions to these problems were constructed, in particular, in functional spaces of Bourgain type, first introduced in [4] for the initial value problem and modified in [6] for initial-boundary value problems.

In comparison with the initial value problem presence of boundary conditions produces additional difficulties for global a priori estimates in the case of initial-boundary value problems. Consider, for example, an estimate in L_2. Let I be either $\mathbb{R}$ or $\mathbb{R}_+$ or $\mathbb{R}_-$ or $(0,1)$ and let ∂I denotes the finite part of its boundary. Let $u(t,x)$ be a solution of the equation (1.6), where $f \equiv 0$, in $(0,T) \times I$ sufficiently smooth and decaying at infinity. Multiplying (1.6) by $2u$ and integrating over I one obtains an equality

$$\frac{d}{dt}\int_I u^2\,dx + \big(2uu_{xx} - u_x^2 + \frac{2}{3}u^3\big)\big|_{\partial I} = 0. \tag{1.11}$$

For $I = \mathbb{R}$ this equality coincides with the first conservation law (1.7). For the initial-boundary value problems in the case $u|_{\partial I} = 0$ an estimate on the solution u in $L_2(I)$ uniform with respect to $t \geq 0$ also succeeds from (1.11). But in the case of non-homogeneous boundary data the presence of the term $uu_{xx}|_{\partial I}$ makes it impossible to derive such an estimate directly from (1.11). Then it is quite natural to introduce an auxiliary function $\varphi(t,x)$ such that $\varphi|_{\partial I} = u|_{\partial I}$, define a new function $U(t,x) \equiv u(t,x) - \varphi(t,x)$ and try to obtain the desired estimate first for the function U. This function satisfies a more complicated equation, so this approach implies, that the function φ can be chosen such that its properties ensure such a possibility. In the papers [10, 12] the function φ was constructed on the base of the boundary potential J and the estimates in L_2 were obtained under ε-close to natural $u_1, u_2 \in H^{1/3+\varepsilon}$, $u_3 \in L_2$ assumptions on the boundary data.

Further obstacles appear for estimates in more smooth spaces, e.g. in H^1 and H^2. The difficulties on this way can be shown even in the linear case and zero boundary data. Multiplying the equation (1.8) by $-2u_{xx}(t,x)$ and integrating over I one derives an equality

$$\frac{d}{dt}\int_I u_x^2\,dx - u_{xx}^2\big|_{\partial I} = 0, \tag{1.12}$$

so an estimate on u_x in $L_2(I)$ can be obtained only for the problem in the right half-strip. Next, multiplying this equation by $2u_{xxxx}(t,x)$ and integrating over I one derives an equality

$$\frac{d}{dt}\int_I u_{xx}^2\,dx - 2u_{tx}u_{xx}\big|_{\partial I} = 0 \tag{1.13}$$

and here the estimate on u_{xx} in $L_2(I)$ can be obtained only for the problem in the left half-strip.

Note that differentiation with respect to t leads to the initial-boundary value problem of the same type for the derivative u_t. Therefore, for example, an estimate on the solution u in $H^3(I)$ can be obtained from an estimate for u_t in $L_2(I)$ via expressing the third derivative u_{xxx} from the equation (1.6) itself.

On this way estimates on solutions in H^{3k}, H^{3k+1}, $k \geq 0$ – integer, to the problem in the right half-strip, in H^{3k}, H^{3k+2} to the problem in the left half-strip and in H^{3k} to the problem in the bounded rectangle were established in

[10, 12] and for intermediate orders of smoothness, following the approach from [2], nonlinear interpolation was used. As a result, global well-posedness of all three considered initial-boundary value problems for the KdV equation was established in these papers under natural assumptions on initial and boundary data for $s > 0$, $s \neq 3m + 1/2$, $s \neq 3m + 3/2$ (for the last two problems), $m \geq 0$ – integer, and ε-close to natural for $s = 0$.

The study of the ZK equation in comparison with KdV besides traditional difficulties originating from the transfer from the line to the plane has some additional obstacles. First of all, Bourgain-type spaces, which turned out to be very useful for KdV, are not found yet for this equation. Next, in contrast to (1.7) only two conservation laws are known for (1.1), $f \equiv 0$:

$$\iint_{\mathbb{R}^2} u^2\,dx\,dy = \text{const}, \quad \iint_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{3}u^3)\,dx\,dy = \text{const}. \tag{1.14}$$

Note that first global existence result (without uniqueness) for the initial value problem for ZK in the space $L_\infty(0,T;H^1(\mathbb{R}^2))$ in the case $u_0 \in H^1(\mathbb{R}^2)$ was, in particular, established in [20] just on the base of these conservation laws.

On the other hand, the so-called local smoothing effect is valid for this equation as for KdV. Let $u(t,x,y)$ be a smooth and decaying at infinity solution to the initial value problem (1.1), (1.2), where $f \equiv 0$. Multiplying (1.1) by $2u(t,x,y)\rho(x)$ for certain smooth, non-negative and non-decreasing function ρ one can easily derive after integration that

$$\frac{d}{dt}\iint_{\mathbb{R}^2} u^2\rho\,dx\,dy + \iint_{\mathbb{R}^2}(3u_x^2+u_y^2)\rho'\,dx\,dy - \iint_{\mathbb{R}^2}\big(u^2\rho''' + \frac{2}{3}u^3\rho'\big)\,dx\,dy = 0, \tag{1.15}$$

and after an appropriate choice of ρ, making use of the first conservation law (1.14), establish an estimate

$$\lambda(u;T) = \sup_{m\in\mathbb{Z}} \int_0^T \int_m^{m+1} \int_{\mathbb{R}} (u_x^2 + u_y^2)\,dy\,dx\,dt \leq c(T, \|u_0\|_{L_2(\mathbb{R}^2)}). \tag{1.16}$$

The estimate (1.16) gave an opportunity in the paper [7] to establish global existence result for the problem (1.1), (1.2) in the class $\{u : u \in L_\infty(0,T;L_2(\mathbb{R}^2)), \lambda(u;T) < \infty\}$ for $u_0 \in L_2(\mathbb{R}^2)$ (in fact, in [20, 7] more general quasilinear evolution equations of an arbitrary high odd order in the multidimensional case were considered). Moreover, if, in addition, $xu_0 \in L_2(\mathbb{R}^2_+)$, a class of global well-posedness was constructed for this problem in [7].

In the paper [8] results from [16] on global well-posedness of the initial value problem for KdV were transferred to ZK, namely, classes of global well-posedness for the problem (1.1), (1.2) were constructed for $u_0 \in H^k(\mathbb{R}^2)$, k – natural (see Remark 2.10 below).

In [17] gain of regularity for solutions to the initial value problem for ZK under decaying at infinity initial data was established.

The study of initial-boundary value problems for the ZK equation started only in recent years (with the only exception in [19], where one problem in a bounded domain for an equation, which can be reduced to ZK by a simple transformation, was considered). Certain results in the case $u_0 \in L_2$ on global existence and uniqueness of weak solutions to the problem (1.1)–(1.3) in Π_T^+ were obtained in [9] and similar results on global existence to the problem (1.1), (1.2), (1.4) in Π_T^- – in [13]. These results are as in [7] based on the first conservation law (1.14) and the local smoothing effect (1.15), (1.16) (in more details they are discussed further in Section 6).

The approach of the present paper repeats the one from [10, 12] (besides the use of Bourgain-type spaces, of course). Special solutions of the "boundary potential" type are constructed and studied for a linearized ZK equation and further used for linear problems in Π_T^+, Π_T^- and Q_T. Solutions to the corresponding linear initial value problem, which was previously studied in [8], are also used here. Moreover, properties of this initial value problem show, that by analogy with (1.9) smoothness assumptions $u_0 \in H^k$, $u_1, u_2 \in H_{t,y}^{(k+1)/3,k+1}$, $u_3 \in H_{t,y}^{k/3,k}$, where H^{s_1,s_2} are anysotropic Sobolev spaces, are natural for the considered initial-boundary value problems (see Remark 3.2 below).

Then local well-posedness of all three considered problems for the ZK equation under natural assumptions on boundary data is established for $u_0 \in H^k$ via the contraction principle.

Global a priori estimates in L_2 for the considered problems are obtained by methods similar to ones for KdV (see (1.11) and subsequent arguments). Obstacles similar to (1.12), (1.13) also appear for the ZK equation, so an estimate in H^1 is established (by methods similar to ones for the second conservation law (1.14)) for the problem in Π_T^+. The absence of an analogue for ZK of the third conservation law (1.7) has not allowed to establish a global estimate in H^2 for the problem in Π_T^-.

Global a priori estimate in H^3 for the problem in Q_T are obtained via differentiation of the equation with respect to t and to y. Note that in comparison with KdV an extra obstacle to establish such an estimate is that one can express from the equation (1.1) not a single derivative of the third order but the term $(u_{xxx} + u_{xyy})$.

Other global estimates in more smooth classes are obtained on the basis of the aforementioned ones.

As a result, global well-posedness is established for the problem (1.1)–(1.3) in Π_T^+ for $u_0 \in H^k(\mathbb{R}_+^2)$, k – natural, and for the problem (1.1), (1.2), (1.5) in Q_T for $u_0 \in H^k(\Sigma)$, $k \geq 3$ – natural, under natural assumptions on the boundary data (the result for the problem in Π_T^+ in the case $k = 1$ was previously published in [11]). Global well-posedness for the problem (1.1), (1.2), (1.4) in Π_T^- is an open problem.

The paper is organized as follows. Section 2 contains main notation and a statement of the main result on local and global well-posedness of the considered problems. In Section 3 potentials for the linearized ZK equation are studied. Section 4 is devoted to the corresponding initial-boundary value problems for this linear equation. The proof of the main result is accomplished in Section 5. Certain remarks on global weak solutions to the considered problems can be found in Section 6.

2. Notation and Statement of the main result

In what follows (if there are no other conditions) in introduced notation we use a symbol I for an arbitrary interval (bounded or unbounded) on the real axis, Ω – for a domain in $\mathbb{R}^n$, k, l, m, n, j – non-negative integers, $p \in [1, +\infty]$, $s \in \mathbb{R}$.

Let $[s]$ be the integer part of s ($s - [s] \in [0, 1)$).

Let $C_b^k(\overline{\Omega})$ be a space of functions with all derivatives up to the order k continuous and bounded in $\overline{\Omega}$. Define $C_b(\overline{\Omega}) = C_b^0(\overline{\Omega})$. If Ω is bounded, the index b is omitted.

Let $\widehat{f} \equiv \mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ be respectively the direct and inverse Fourier transforms of a function f, considered as operations in $L_2(\mathbb{R}^n)$. In particular, for $f \in \mathcal{S}(\mathbb{R})$

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx, \quad \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} f(\xi)\,d\xi.$$

Define the fractional order Sobolev space

$$H^s(\mathbb{R}^n) = \big\{ f : \mathcal{F}^{-1}[(1+|\xi|)^s \widehat{f}(\xi)] \in L_2(\mathbb{R}^n) \big\}$$

and let $H^s(\Omega)$ be a space of restrictions on Ω of functions from $H^s(\mathbb{R}^n)$. Note that $H^k(\Omega) = W_2^k(\Omega)$. Define

$$H_0^s(\Omega) = \big\{ f \in H^s(\mathbb{R}^n) : \operatorname{supp.} f \subset \overline{\Omega} \big\}.$$

Properties of the spaces H^s and H_0^s can be found, for example, in [18].

For domains $\Omega \subset \mathbb{R}^2$ with regular boundaries (in particular, for $\Omega = \mathbb{R}^2_+$ and $\Omega = \Sigma$) the following interpolation inequality is valid:

$$\|f\|_{L_p(\Omega)} \le c(p,\Omega)\Big[\|\nabla f\|_{L_2(\Omega)}^{(p-2)/p} \|f\|_{L_2(\Omega)}^{2/p} + \|f\|_{L_2(\Omega)} \Big], \tag{2.1}$$

where $2 \le p < +\infty$ (see, e.g. [1]).

For description of properties of boundary data we also use the anysotropic Sobolev spaces for $s_1, s_2 \ge 0$:

$$H^{s_1,s_2}(\mathbb{R}^2) = H^{s_1,s_2}_{t,y}(\mathbb{R}^2) = \big\{ \mu(t,y) : \mathcal{F}^{-1}[(1+|\lambda|^{s_1}+|\eta|^{s_2})\widehat{\mu}(\lambda,\eta)] \in L_2(\mathbb{R}^2) \big\}.$$

For $\Omega \subset \mathbb{R}^2$ a symbol $H^{s_1,s_2}(\Omega)$ is also used for a space of corresponding restrictions.

If $\mathcal{B}$ is a certain Banach space, define by $C_b(\overline{I};\mathcal{B})$ a space of continuous bounded mappings from $\overline{I}$ to $\mathcal{B}$ (for bounded I the index b is omitted). The symbol $L_p(I;\mathcal{B})$ is used in the conventional sense.

Solutions to the considered problems are constructed in special functional spaces X_k.

Definition 2.1. For any $T > 0$ let $X_k((0,T)\times I \times \mathbb{R})$ be a space of functions $u(t,x,y)$ such that

$$\partial_t^m u \in C\big([0,T]; H^{k-3m}(I\times\mathbb{R})\big), \quad m \le [k/3], \tag{2.2}$$

$$\partial_x^l u \in C_b\big(\overline{I}; H^{(k-l+1)/3,k-l+1}(B_T)\big), \quad l \le k+1, \tag{2.3}$$

$$\partial_t^m \partial_x^l \partial_y^j u \in L_2\big(0,T; C_b(\overline{I}\times\mathbb{R})\big), \quad 3m+l+j \le k, \tag{2.4}$$

$$\partial_t^m \partial_x^l \partial_y^j u \in L_2\big(I; C_b(\overline{B}_T)\big), \quad k \ge 1, \quad 3m+l+j \le k-1. \tag{2.5}$$

Remark 2.2. For small k such solutions are interpreted in a weak (distributional) sense (see, e.g. [10, 12] for corresponding definitions in similar situations for KdV).

For description of properties of the right part of the equation introduce the following spaces M_k.

Definition 2.3. For any $T > 0$ let $M_k((0,T)\times I\times\mathbb{R})$ be a space of functions $f(t,x,y)$ such that

$$\partial_t^m f \in L_2\big(0,T; H^{k-3m}(I\times\mathbb{R})\big), \quad m \le m_0 = [(k+1)/3].$$

For simplicity we often use shortened symbols X_k and M_k. Let $\partial_{x,y}$ denotes either ∂_x or ∂_y.

Lemma 2.4. *For any $T > 0$ and $I \subset \mathbb{R}$*

$$\|u\partial_{x,y}v\|_{M_0} + \|v\partial_{x,y}u\|_{M_0} \le c\|u\|_{X_1}\|v\|_{X_0}, \tag{2.6}$$
$$\|u\partial_{x,y}v\|_{M_k} \le c(k)\|u\|_{X_k}\|v\|_{X_k}, \quad k \ge 1, \tag{2.7}$$
$$\|u\partial_{x,y}u\|_{M_k} \le c(k)\|u\|_{X_{k-1}}\|u\|_{X_k}, \quad k \ge 2. \tag{2.8}$$

Proof. Note that $M_0 = L_2$, so (2.6) follows from obvious inequalities

$$\|u\partial_{x,y}v\|_{L_2((0,T)\times I\times\mathbb{R})} \le \|u\|_{L_2(I;C_b(\overline{B}_T))}\|\partial_{x,y}v\|_{C_b(\overline{I};L_2(B_T))}, \tag{2.9}$$
$$\|v\partial_{x,y}u\|_{L_2((0,T)\times I\times\mathbb{R})} \le \|u\|_{C([0,T];H^1(I\times\mathbb{R}))}\|v\|_{L_2(0,T;C_b(\overline{I}\times\mathbb{R}))}. \tag{2.10}$$

Let in (2.7) and (2.8) $k = 3n + j$, $0 \le j \le 2$. If $j \le 1$, then $m_0 = n$ and these inequalities can be derived similarly to (2.9), (2.10). Let $j = 2$, then $m_0 = n+1$ and in addition to the previous cases we must evaluate $\partial_t^{m_0}(u\partial_{x,y}v)$ in $L_2(0,T;H^{-1})$. Here

$$u\partial_t^{m_0}\partial_{x,y}v = \partial_{x,y}(u\partial_t^{m_0}v) - \partial_{x,y}u\partial_t^{m_0}v$$

and similarly to (2.9)

$$\|\partial_{x,y}(u\partial_t^{m_0}v)\|_{L_2(0,T;H^{-1}(I\times\mathbb{R}))} \le \|u\partial_t^{m_0}v\|_{L_2((0,T)\times I\times\mathbb{R})} \le \|u\|_{X_1}\|v\|_{X_k}, \tag{2.11}$$
$$\|\partial_{x,y}u\partial_t^{m_0}v\|_{L_2(0,T;H^{-1}(I\times\mathbb{R}))} \le \|\partial_{x,y}u\partial_t^{m_0}v\|_{L_2((0,T)\times I\times\mathbb{R})} \le \|u\|_{X_2}\|v\|_{X_k}.$$

Thus (2.8) for $k \ge 3$ and (2.7) are established. Finally, note that if $k = 2$, then $\partial_t(u\partial_{x,y}u) = \partial_{x,y}(uu_t)$, and the inequality (2.8) in this case follows from (2.11). □

In order to describe properties of boundary data we introduce some special notation common for all three considered problems.

Definition 2.5. Let $n = 1$, $I = \mathbb{R}_+ = (0, +\infty)$ for the problem in Π_T^+; $n = 2$, $I = \mathbb{R}_- = (-\infty, 0)$ for the problem in Π_T^-; $n = 3$, $I = (0, 1)$ for the problem in Q_T. Let $\mathcal{B}_n^k(T)$ be a space of ordered assemblies $\mathcal{U}^n$, where

$$\mathcal{U}^1 = (u_1), \quad \mathcal{U}^2 = (u_2, u_3), \quad \mathcal{U}^3 = (u_1, u_2, u_3),$$

such that

$$u_1, u_2 \in H^{(k+1)/3,k+1}(B_T), \quad u_3 \in H^{k/3,k}(B_T),$$

with the natural norm.

We also need to formulate compatibility conditions for the considered problems.

Definition 2.6. Let $\Phi_0(x, y) \equiv u_0(x, y)$ and for $m \ge 1$

$$\begin{aligned} \Phi_m(x, y) \equiv {} & \partial_t^{m-1} f(0, x, y) - (\partial_x^3 + \partial_x\partial_y^2)\Phi_{m-1}(x, y) \\ & - \sum_{l=0}^{m-1} \binom{m-1}{l} \Phi_l(x, y)\partial_x\Phi_{m-l-1}(x, y). \end{aligned} \tag{2.12}$$

We say that the compatibility conditions of the order k are satisfied if

(1) $\partial_t^m u_1(0, y) \equiv \Phi_m(0, y)$ for $m < k/3$ in the case of the problem in Π_T^+;
(2) $\partial_t^m u_2(0, y) \equiv \Phi_m(0, y)$ for $m < k/3$, $\partial_t^m u_3(0, y) \equiv \partial_x\Phi_m(0, y)$ for $m < (k-1)/3$ in the case of the problem in Π_T^-;
(3) $\partial_t^m u_1(0, y) \equiv \Phi_m(0, y)$, $\partial_t^m u_2(0, y) \equiv \Phi_m(1, y)$ for $m < k/3$, $\partial_t^m u_3(0, y) \equiv \partial_x\Phi_m(1, y)$ for $m < (k-1)/3$ in the case of the problem in Q_T.

Now we can present the main result of the paper.

Theorem 2.7. *Let either $n = 1$, $I = \mathbb{R}_+$ for the problem in Π_T^+ or $n = 2$, $I = \mathbb{R}_-$ for the problem in Π_T^- or $n = 3$, $I = (0,1)$ for the problem in Q_T. Let $u_0 \in H^k(I \times \mathbb{R})$, $\mathcal{U}^n \in \mathcal{B}_n^k(T)$, $f \in M_k((0,T) \times I \times \mathbb{R})$ for certain $T > 0$, $k \geq 1$. Assume also that the compatibility conditions of the order k are satisfied for the considered problem. Then respectively*

1. *the problem (1.1)–(1.3) is well-posed in $X_k(\Pi_T^+)$;*
2. *there exists $t_0 \in (0,T]$ such that the problem (1.1), (1.2), (1.4) is well-posed in $X_k(\Pi_{t_0}^-)$;*
3. *the problem (1.1), (1.2), (1.5) is well-posed in $X_k(Q_T)$ if $k \geq 3$ and there exists $t_0 \in (0,T]$ such that this problem is well-posed in $X_k(Q_{t_0})$ if $k = 1$ or $k = 2$.*

Remark 2.8. We mean that the problem is well-posed in the space X_k, if there exists a unique solution $u(t,x,y)$ in this space and the map $(u_0, \mathcal{U}^n, f) \mapsto u$ is Lipschitz continuous on any ball in the norm of the map $H^k(I \times \mathbb{R}) \times \mathcal{B}_n^k(T) \times M_k((0,T) \times I \times \mathbb{R})$ into X_k.

Remark 2.9. All these well-posedness results can be easily generalized for an equation of the (1.1) type with a nonlinear term $g(u)u_x$, where the sufficiently smooth function g has not more than linear rate of growth (more precisely, g' is bounded on $\mathbb{R}$) and $g(0) = 0$.

Remark 2.10. In the paper [8] global well-posedness of the initial value problem (1.1), (1.2) was established under assumptions $u_0 \in H^k(\mathbb{R}^2)$, $f \in L_1(0,T; H^k(\mathbb{R}^2))$, $k \geq 1$, in the classes similar to X_k but without smoothness properties with respect to t.

3. Potentials

Consider a linear equation

$$u_t + u_{xxx} + u_{xyy} = f(t,x,y). \tag{3.1}$$

Solution to the initial value problem in a domain $\Pi_T = (0,T) \times \mathbb{R}^2$ with the initial profile (1.2) can be constructed in a form (see [8])

$$u(t,x,y) = S(t,x,y;u_0) + K(t,x,y;f), \tag{3.2}$$

where potentials S and K are given by formulae

$$\begin{aligned} S(t,x,y;u_0) &\equiv \mathcal{F}_{x,y}^{-1}\big[e^{it(\xi^3+\xi\eta^2)}\widehat{u}_0(\xi,\eta)\big](x,y), \\ K(t,x,y;f) &\equiv \int_0^t S(t-\tau,x,y;f(\tau,\cdot,\cdot))\,d\tau. \end{aligned} \tag{3.3}$$

By analogy with (2.12) let $\widetilde{\Phi}_0(x,y) \equiv u_0(x,y)$ and for $m \geq 1$

$$\widetilde{\Phi}_m(x,y) \equiv \partial_t^{m-1} f(0,x,y) - (\partial_x^3 + \partial_x\partial_y^2)\widetilde{\Phi}_{m-1}(x,y). \tag{3.4}$$

Lemma 3.1. *If $u_0 \in H^k(\mathbb{R}^2)$, $f \in M_k(\Pi_T)$ for some $T > 0$ and $k \geq 0$, then a unique solution $u(t,x,y)$ to the problem (3.1), (1.2) exists and for any $t_0 \in (0,T]$*

$$\begin{aligned} &\|u\|_{X_k(\Pi_{t_0})} \\ &\leq c(T,k)\Big(\|u_0\|_{H^k(\mathbb{R}^2)} + t_0^{1/6}\|f\|_{M_k(\Pi_{t_0})} + \sum_{m=0}^{m_0-1} \|\partial_t^m f\big|_{t=0}\|_{H^{k-3(m+1)}(\mathbb{R}^2)}\Big). \end{aligned} \tag{3.5}$$

Proof. First of all note that

$$\partial_t^m S(t,x,y;u_0) + \partial_t^m K(t,x,y;f) = S(t,x,y;\widetilde{\Phi}_m) + K(t,x,y;\partial_t^m f). \tag{3.6}$$

For $m=0$ corresponding estimates on the solution u in the norms (2.2), (2.4), (2.5) (where $I=\mathbb{R}$) by $\|u_0\|_{H^k(\mathbb{R}^2)}$ and $\|f\|_{L_1(0,t_0;H^k(\mathbb{R}^2))}$ are established in [8] (moreover, in (2.4) L_2 with respect to t can be enlarged to L_3).

In [8] it was also proved that

$$\|\nabla_{x,y} u\|_{C_b(\mathbb{R};L_2(B_{t_0}))} \le c(T,k)\left(\|u_0\|_{H^1(\mathbb{R}^2)} + \|f\|_{L_1(0,t_0;H^1(\mathbb{R}^2))}\right). \tag{3.7}$$

Similarly to (3.7) a corresponding estimate on $\partial_x^l S$ in $C_b(\mathbb{R};H^{(k-l+1)/3,k-l+1}(B_{t_0}))$ by $\|u_0\|_{H^k(\mathbb{R}^2)}$ for $l \le k+1$ can be also derived.

For the potential K first of all we show that for $s\in[0,1]$

$$\|K(\cdot,\cdot,\cdot;f)\|_{C_b(\mathbb{R};H^{s,3s}(B_{t_0}))} \le c(T) t_0^{(1-s)/2} \|f\|_{L_2(0,t_0;H^{3s-1}(\mathbb{R}^2))}. \tag{3.8}$$

In fact, if $s=0$ then this inequality is similar to (3.7), if $s=1$ it succeeds from an equality

$$K_t(t,x,y;f) = f(t,x,y) - \int_0^t (\partial_x^3 + \partial_x\partial_y^2) S(t-\tau,x,y;f(\tau,\cdot,\cdot))\,d\tau$$

and the already established estimates on the potential S, for intermediate values of s (3.8) is obtained via interpolation.

Finally, it is suffice to note that if one applies (3.8) to $K(t,x;\partial_t^m f)$, where $m=[(k-l+1)/3]$, $s=(k-l+1)/3-m$, the minimal value $1/6$ for the degree $(1-s)/2$ is achieved if $k-l+1=3m+2$. □

Remark 3.2. By the methods from [8] it is easy to show that for the function $S=S(t,x,y;u_0)$, where $u_0\in H^s(\mathbb{R}^2)$, uniformly with respect to $x\in\mathbb{R}$

$$\big\|D_t^{1/3} S\big\|^2_{H^{s/3,s}_{t,y}(\mathbb{R}^2)} + \big\|\partial_x S\big\|^2_{H^{s/3,s}_{t,y}(\mathbb{R}^2)} + \big\|\partial_y S\big\|^2_{H^{s/3,s}_{t,y}(\mathbb{R}^2)} \sim \|u_0\|^2_{H^s(\mathbb{R}^2)}$$

(here D^α denotes the Riesz potential of the order $-\alpha$).

In what follows we need simple properties of solutions to an algebraic equation

$$r^3 - r\eta^2 + i\lambda = 0, \quad (\lambda,\eta)\ne(0,0). \tag{3.9}$$

This equation has one root $r_0(\lambda,\eta)$ with the negative real part, one root $r_1(\lambda,\eta)$ with the positive real part and one pure imaginary root $r_2(\lambda,\eta)$. These roots can be written in a form

$$r_0 = -p(\lambda,\eta) + iq(\lambda,\eta), \quad r_1 = p(\lambda,\eta) + iq(\lambda,\eta), \quad r_2 = i\kappa(\lambda,\eta), \tag{3.10}$$

where $p>0$, $q\in\mathbb{R}$ and the function κ for a fixed η is the inverse function to $\varphi(\xi)\equiv \xi^3+\xi\eta^2$. Moreover, for certain positive constants c, c_1 and any (λ,η)

$$p(\lambda,\eta) \ge c(|\lambda|^{1/3}+|\eta|), \tag{3.11}$$

$$|r_j(\lambda,\eta)| \le c_1(|\lambda|^{1/3}+|\eta|) \quad \forall j, \tag{3.12}$$

$$|r_j(\lambda,\eta) - r_k(\lambda,\eta)| \ge c(|\lambda|^{1/3}+|\eta|), \quad j\ne k. \tag{3.13}$$

Now we can introduce boundary potentials for the homogeneous equation (3.1).

Definition 3.3. Let $\mu, \nu \in L_2(\mathbb{R}^2)$. Define for $x \ge 0$

$$J_+(t,x,y;\mu) \equiv \mathcal{F}^{-1}_{t,y}\left[e^{r_0 x}\widehat{\mu}(\lambda,\eta)\right](t,y) \tag{3.14}$$

and for $x \le 0$,

$$J_-(t,x,y;\mu,\nu) \equiv \mathcal{F}^{-1}_{t,y}\Big[\frac{r_1 e^{r_2 x} - r_2 e^{r_1 x}}{r_1 - r_2}\widehat{\mu}(\lambda,\eta) + \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}\widehat{\nu}(\lambda,\eta)\Big](t,y), \tag{3.15}$$

where $r_j = r_j(\lambda,\eta)$ are the aforementioned roots of the equation (3.9).

Lemma 3.4. *Let $\mu \in H^{(k+1)/3,k+1}(\mathbb{R}^2)$ for some $k \ge 0$, then for any $T > 0$*

$$\|J_+(\cdot,\cdot,\cdot;\mu)\|_{X_k(\Pi_T^+)} \le c(T,k)\|\mu\|_{H^{(k+1)/3,k+1}(\mathbb{R}^2)}. \tag{3.16}$$

Proof. In order to obtain an estimate in the norm (2.2) we use the following fundamental inequality from [2]: if certain continuous function $\gamma(\theta)$ satisfies an inequality $\operatorname{Re}\gamma(\theta) \le -\varepsilon|\theta|$ for some $\varepsilon > 0$ and all $\theta \in \mathbb{R}$, then

$$\Big\|\int_{\mathbb{R}} e^{\gamma(\theta)x} f(\theta)\,d\theta\Big\|_{L_2(\mathbb{R}^x_+)} \le c(\varepsilon)\|f\|_{L_2(\mathbb{R})}. \tag{3.17}$$

Therefore, changing variables $\lambda = \theta^3$ we derive from (3.14) with the use of (3.11) and (3.12) that for $3m + l + j \le k$ uniformly with respect to $t \in \mathbb{R}$,

$$\begin{aligned} \|\partial_t^m \partial_x^l \partial_y^j J_+(t,\cdot,\cdot;\mu)\|_{L_2(\mathbb{R}^2_+)} &\le c\,\big\|\theta^{3m+2}\eta^j(|\theta|^l + |\eta|^l)\widehat{\mu}(\theta^3,\eta)\big\|_{L_2(\mathbb{R}^2)} \\ &\le c_1\big\|\lambda^{m+1/3}\eta^j(|\lambda|^{l/3} + |\eta|^l)\widehat{\mu}(\lambda,\eta)\big\|_{L_2(\mathbb{R}^2)} \\ &\le c_2\|\mu\|_{H^{(k+1)/3,k+1}(\mathbb{R}^2)}. \end{aligned} \tag{3.18}$$

Similarly to (3.18), for $3m + l + j \le k - 1$,

$$\begin{aligned} &\big\|\sup_{(t,y)\in\mathbb{R}^2}|\partial_t^m \partial_x^l \partial_y^j J_+(\cdot,t,y;\mu)|\big\|_{L_2(\mathbb{R}^x_+)} \\ &\le c\int_{\mathbb{R}}\Big\|\int_{\mathbb{R}} e^{-p(\lambda,\eta)x}|\lambda^m\eta^j|(|\lambda|^{l/3} + |\eta|^l)|\widehat{\mu}(\lambda,\eta)|\,d\lambda\Big\|_{L_2(\mathbb{R}^x_+)}d\eta \\ &\le c_1\int_{\mathbb{R}}|\eta^j|\big\|\theta^{3m+2}(|\theta|^l + |\eta|^l)\widehat{\mu}(\theta^3,\eta)\big\|_{L_2(\mathbb{R}^\theta)}d\eta \\ &\le c_2\|\lambda^{m+1/3}\eta^{j+1}(|\lambda|^{l/3} + |\eta|^l)\widehat{\mu}(\lambda,\eta)\|_{L_2(\mathbb{R}^2)} \\ &\le c_3\|\mu\|_{H^{(k+1)/3,k+1}(\mathbb{R}^2)} \end{aligned} \tag{3.19}$$

and we obtain the desired estimate in the norm (2.5).

The estimate in the norm (2.3) simply follows from the equality (3.14) since $\operatorname{Re} r_0 \le 0$.

Finally, the estimate in the norm (2.4) succeeds by virtue of the well-known embedding $H^{1+\varepsilon}(\Omega) \subset C_b(\overline{\Omega})$ for domains $\Omega \subset \mathbb{R}^2$ from the following inequality: for $s \ge 0$

$$\|\partial_t^m J_+(\cdot,\cdot,\cdot;\mu)\|_{L_2(0,T;H^{s-3m}(\mathbb{R}^2_+))} \le c(T,s)\|\mu\|_{H^{(2s-1)/6,s-1/2}(\mathbb{R}^2)}. \tag{3.20}$$

It is suffice to prove (3.20) for $m = 0$. Let

$$\mu_0(t,y) \equiv \mathcal{F}^{-1}_{t,y}\left[\chi(\lambda,\eta)\widehat{\mu}(\lambda,\eta)\right](t,y), \quad \mu_1(t,y) \equiv \mu(t,y) - \mu_0(t,y), \tag{3.21}$$

where χ denotes the characteristic function of the unit circle $\{(\lambda,\eta) : \lambda^2 + \eta^2 < 1\}$. Then it follows from the already established estimate (3.18) that for any $s \ge 0$

$$\|J_+(\cdot,\cdot,\cdot;\mu_0)\|_{L_2(0,T;H^s(\mathbb{R}^2_+))} \le T^{1/2}\sup_{t\in[0,T]}\|J_+(t,\cdot,\cdot;\mu_0)\|_{H^{[s]+1}(\mathbb{R}^2_+)}$$

$$\le c(T,s)\|\mu\|_{H^{-1}(\mathbb{R}^2)}.$$

Next,

$$\begin{aligned}
&\|\partial_x^l \partial_y^j J_+(\cdot,\cdot,\cdot;\mu_1)\|_{L_2(\mathbb{R}^t\times\mathbb{R}_+^2)}\\
&= \Big\| r_0^l \eta^j \widehat{\mu}_1(\lambda,\eta)\Big(\int_{\mathbb{R}_+} e^{-2p(\lambda,\eta)x}\,dx\Big)^{1/2}\Big\|_{L_2(\mathbb{R}^2)}\\
&\le c\big\|(|\lambda|^{l/3}+|\eta|^l)\eta^j p^{-1/2}(\lambda,\eta)\widehat{\mu}(\lambda,\eta)\big(1-\chi(\lambda,\eta)\big)\big\|_{L_2(\mathbb{R}^2)}\\
&\le c_1\|\mu\|_{H^{(2(l+j)-1)/6,\,l+j-1/2}(\mathbb{R}^2)}
\end{aligned}$$

and using interpolation we complete the proof of (3.20). □

Remark 3.5. It follows from the proof of Lemma 3.4 that the estimates on J_+ are valid in norms of the (2.2), (2.3), (2.5) type, where the domain of the variable t is the whole real axis.

The potential J_+ possesses also certain additional properties.

Lemma 3.6. *Let $\mu \in L_2(\mathbb{R}^2)$ and $\mu(t,y)=0$ for $t<0$. Then the function $J_+(t,x,y;\mu)$ is infinitely differentiable for $x>0$, $J_+(t,x,y;\mu)=0$ for $t\le 0$ and for any $T>0$, $x_0>0$, $\beta\ge 0$ and m,l,j*

$$\sup_{t\in[0,T],x\ge x_0}(1+x)^\beta\|\partial_t^m\partial_x^l J_+(t,x,\cdot;\mu)\|_{H^j(\mathbb{R})}\le c(T,x_0,\beta,m,l,j)\|\mu\|_{L_2(\mathbb{R}^2)}. \quad (3.22)$$

Proof. These properties succeed from the following representation of the function J_+ for $x>0$:

$$J_+(t,x,y;\mu)=\int_{-\infty}^t\int_{\mathbb{R}}(3\partial_x^2+\partial_y^2)G(t-\tau,x,y-z)\mu(\tau,z)\,dzd\tau, \quad (3.23)$$

$$G(t,x,y)\equiv\frac{1}{t^{2/3}}A\left(\frac{x}{t^{1/3}},\frac{y}{t^{1/3}}\right),\quad A(x,y)\equiv\mathcal{F}_{x,y}^{-1}\big[e^{i(\xi^3+\xi\eta^2)}\big](x,y). \quad (3.24)$$

This formula was proved in [11]. We reproduce here the scheme of the proof. Changing variables $\xi=\kappa(\lambda,\eta)$, where κ is the function from (3.10), we can write an equality

$$G(t,x,y)=\mathcal{F}_{t,y}^{-1}\big[\partial_\lambda\kappa(\lambda,\eta)e^{i\kappa(\lambda,\eta)x}\big](t,y).$$

So if we denote by J the right part of (3.23) then

$$\begin{aligned}
\mathcal{F}_{t,y}[J](\lambda,\eta)&=\mathcal{F}_{t,y}\big[(3\partial_x^2+\partial_y^2)G(t,x,y)\vartheta(t)\big](\lambda,\eta)\widehat{\mu}(\lambda,\eta)\\
&=-\frac{1}{4\pi^2}\Big(e^{i\kappa(\lambda,\eta)x}*\big(\widehat{\vartheta}(\lambda)\times\delta(\eta)\big)\Big)\widehat{\mu}(\lambda,\eta)\\
&=-\Big(\frac12 e^{i\kappa(\lambda,\eta)x}+\frac{i}{2\pi}\,\mathrm{v.p.}\int_{\mathbb{R}}\frac{e^{i\kappa(\zeta,\eta)x}}{\zeta-\lambda}d\zeta\Big)\widehat{\mu}(\lambda,\eta),
\end{aligned}$$

where ϑ is the Heaviside function. The last integral can be easily calculated:

$$\mathrm{v.p.}\int_{\mathbb{R}}\frac{e^{i\kappa(\zeta,\eta)x}}{\zeta-\lambda}d\zeta=\mathrm{v.p.}\int_{\mathbb{R}}\frac{e^{izx}(3z^2+\eta^2)}{z^3+z\eta^2-\lambda}dz=2\pi i e^{r_0(\lambda,\eta)x}+\pi i e^{i\kappa(\lambda,\eta)x}$$

and, consequently, $J=J_+$.

The function A was studied in [9] (in fact, more general one). In particular, it was proved that $A\in\mathcal{S}(\overline{\mathbb{R}}_+^2)$ – the space of restrictions on $\overline{\mathbb{R}}_+^2$ of functions from $\mathcal{S}(\mathbb{R}^2)$. This property applied to (3.23) provides the assertion of the lemma (see [9] for more details). □

Now we consider properties of the potential J_-.

Lemma 3.7. *Let $\mu \in H^{(k+1)/3,k+1}(\mathbb{R}^2)$, $\nu \in H^{k/3,k}(\mathbb{R}^2)$ for some $k \geq 0$, then for any $T > 0$*

$$\|J_-(\cdot,\cdot,\cdot;\mu,\nu)\|_{X_k(\Pi_T^-)} \leq c(T,k)\big(\|\mu\|_{H^{(k+1)/3,k+1}(\mathbb{R}^2)} + \|\nu\|_{H^{k/3,k}(\mathbb{R}^2)}\big). \tag{3.25}$$

Proof. First consider the part of J_- containing the term $e^{r_1 x}$. Since the root r_1 has the properties similar to r_0, taking into account the inequality (3.13) one can derive corresponding analogues of (3.18)–(3.20) for this part by the same methods as for J_+ (the analogues of (3.18), (3.19) are supplemented with $\|\nu\|_{H^{k/3,k}(\mathbb{R}^2)}$, of (3.20) – with $\|\nu\|_{H^{(2s-3)/6,s-3/2}(\mathbb{R}^2)}$).

The estimate in the norm (2.3) is obvious just as for J_+ except the case $l = 0$ because of the denominator near the point $(0,0)$ in the part containing $\widehat{\nu}$, but here one can use the partition of ν similar to (3.21) and for ν_0 apply the already established estimate in the (2.2) norm.

In order to evaluate the part of J_- containing the term $e^{r_2 x}$ consider an expression

$$\mathcal{I}(t,x,y) \equiv \mathcal{F}^{-1}_{t,y}\big[e^{r_2(\lambda,\eta)x} f(\lambda,\eta)\big](t,y).$$

By virtue of (3.3), (3.10) and the change of variables $\lambda = \xi^3 + \xi\eta^2$ it can be written in a form

$$\mathcal{I} = S\big(t,x,y;\mathcal{F}^{-1}\big[(3\xi^2+\eta^2) f(\xi^3+\xi\eta^2,\eta)\big]\big).$$

It is easy to see that

$$\big\|\mathcal{F}^{-1}\big[(3\xi^2+\eta^2) f(\xi^3+\xi\eta^2,\eta)\big]\big\|_{H^k(\mathbb{R}^2)} \leq c\|(|\lambda|^{1/3}+|\eta|) f(\lambda,\eta)\|_{H^{k/3,k}(\mathbb{R}^2)}$$

and so the desired estimates on the rest part of J_- succeed from Lemma 3.1. $\square$

4. Linear problems

Consider for the equation (3.1) initial-boundary value problems in the domains Π_T^+, Π_T^-, Q_T with the initial data (1.2) and the boundary data (1.3), (1.4) or (1.5) respectively. First we establish one auxiliary lemma for the first two problems. Note that solutions to these problems are unique in the spaces $L_2(\Pi_T^+)$ and $L_2(\Pi_T^-)$ respectively because of the already proved solvability in smooth classes (see [21], these problems are, in fact, adjoint to each other).

Lemma 4.1. *Let $u_0 \equiv 0$, $f \equiv 0$, $u_1, u_2 \in H^{1/3,1}(\mathbb{R}^2)$, $u_3 \in L_2(\mathbb{R}^2)$ and $u_1(t,y) = u_2(t,y) = u_3(t,y) = 0$ for $t < 0$. Then $J_+(t,x,y;u_1)$ and $J_-(t,x,y;u_2,u_3)$ are respectively (unique) solutions to the problems (3.1), (1.2), (1.3) in Π_T^+ or (3.1), (1.2), (1.4) in Π_T^- for any $T > 0$ in the classes $X_0(\Pi_T^+)$ or $X_0(\Pi_T^-)$.*

Proof. By virtue of Lemmas 3.4 and 3.7 without loss of generality one can assume that $u_j \in C_0^\infty(\mathbb{R}^t_+ \times \mathbb{R}^y)$.

It is obvious that the functions J_+ and J_- satisfy the homogeneous equation (3.1) if $x \geq 0$ or $x \leq 0$ respectively and satisfy the corresponding boundary conditions (1.3) or (1.4). Lemma 3.6 provides also that the function J_+ satisfies the zero initial condition (1.2), so for the problem in Π_T^+ the proof is complete.

Since for the function J_- we don't have an equality of the (3.23) type, we choose in this case an indirect way and prove that a solution to the problem in Π_T^- coincides with J_-. According to [21] (see also [9]) there exists a solution $u(t,x,y)$ to the problem (3.1), (1.2), (1.4) and $\partial_t^m u \in C([0,T];H^l(\mathbb{R}^2_-))$ for any $T > 0$, $m,l \geq 0$.

Moreover, if $u_2(t,y) = u_3(t,y) = 0$ for $t \geq T_0 > 0$ then multiplying (3.1) by $2u(t,x,y)$ and integrating over $\mathbb{R}^2_-$ one can easily derive that for $t \geq T_0$

$$\frac{d}{dt}\|u(t,\cdot,\cdot)\|_{L_2(\mathbb{R}^2_-)} = 0. \tag{4.1}$$

Obviously similar equality can be obtained for any derivative $\partial_t^m \partial_y^j u$. Derivatives with respect to x can be expressed from the equation (3.1) itself (see for more details [9] or the following arguments in the proof of Lemma 5.4). Finaly, we obtain that $\partial_t^m u \in C_b(\overline{\mathbb{R}}^t_+; H^l(\mathbb{R}^2_-))$ for any $m, l \geq 0$.

Therefore for any $p = \varepsilon + i\lambda$, where $\varepsilon > 0$, and $\eta \in \mathbb{R}$ we can define the Laplace transform with respect to t and the Fourier transform with respect to y:

$$\widetilde{u}(p,x,\eta) \equiv \iint_{\mathbb{R}^2_+} e^{-pt-i\eta y} u(t,x,y)\, dtdy.$$

The function $\widetilde{u}$ solves a problem

$$\begin{gathered} p\widetilde{u}(p,x,\eta) + \widetilde{u}_{xxx}(p,x,\eta) - \eta^2 \widetilde{u}_x(p,x,\eta) = 0, \quad x \leq 0, \\ \widetilde{u}(p,0,\eta) = \widetilde{u}_2(p,\eta), \quad \widetilde{u}_x(p,0,\eta) = \widetilde{u}_3(p,\eta), \end{gathered} \tag{4.2}$$

where $\widetilde{u}_2$, $\widetilde{u}_3$ are the similar Laplace–Fourier transforms of u_2, u_3. The corresponding characteristic equation for (4.2) $r^3 - \eta^2 r + i\lambda + \varepsilon = 0$ has exactly two roots $r_1(\lambda,\eta,\varepsilon)$ and $r_2(\lambda,\eta,\varepsilon)$ with the positive real parts (and one root with the negative one), so since $\widetilde{u}(p,x,\eta) \to 0$ as $x \to -\infty$ it follows that

$$\widetilde{u}(p,x,\eta) = \frac{r_1 e^{r_2 x} - r_2 e^{r_1 x}}{r_1 - r_2}\widetilde{u}_2(p,\eta) + \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}\widetilde{u}_3(p,\eta).$$

Applying the formulae of invertion of the Laplace and the Fourier transforms and passing to the limit as $\varepsilon \to +0$ we derive that $u \equiv J_-$. $\square$

For the considered initial-boundary value problems for the equation (3.1) introduce the notion of compatibility conditions of the order k similar to Definition 2.6, where only Φ_m must be substituted by $\widetilde{\Phi}_m$ (see (3.4)).

Now we can establish the main lemma for the linear problems.

Lemma 4.2. *Let $n = 1$, $I = \mathbb{R}_+$ for the problem in Π_T^+; $n = 2$, $I = \mathbb{R}_-$ for the problem in Π_T^-; $n = 3$, $I = (0,1)$ for the problem in Q_T. Let $u_0 \in H^k(I \times \mathbb{R})$, $\mathcal{U}^n \in \mathcal{B}^k_n(T)$, $f \in M_k((0,T) \times I \times \mathbb{R})$ for certain $T > 0$, $k \geq 0$. Assume also that the compatibility conditions of the order k are satisfied for each of the considered problems. Then there exists a unique solution $u(t,x,y)$ to each problem in the space $X_k((0,T) \times I \times \mathbb{R})$ and for any $t_0 \in (0,T]$*

$$\begin{aligned} \|u\|_{X_k((0,t_0)\times I\times\mathbb{R})} \leq c(T,k)\Big(&\|u_0\|_{H^k(I\times\mathbb{R})} + \|\mathcal{U}^n\|_{\mathcal{B}^k_n(T)} + t_0^{1/6}\|f\|_{M_k((0,t_0)\times I\times\mathbb{R})} \\ &+ \sum_{m=0}^{m_0-1} \|\partial_t^m f\big|_{t=0}\|_{H^{k-3(m+1)}(I\times\mathbb{R})}\Big). \end{aligned} \tag{4.3}$$

Proof. Consider first the problems in Π_T^+ and Π_T^-. Extend u_0 and f to the whole real axis with respect to x in the classes $H^k(\mathbb{R}^2)$ and $M_k(\Pi_T)$ respectively and consider a solution $U(t,x,y)$ to the initial value problem (3.1), (1.2) in the class $X_k(\Pi_T)$ given by Lemma 3.1. Note that by virtue of the compatibility conditions

$$u_1 - U|_{x=0},\ u_2 - U|_{x=0} \in H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)\big|_{B_T}, \quad u_3 - U_x|_{x=0} \in H_0^{k/3,k}(\mathbb{R}^2_+)\big|_{B_T},$$

so these functions can be extended by zero to the whole plane $\mathbb{R}^2$ in the same classes. Then the desired result succeeds from Lemmas 4.1, 3.4 and 3.7.

Solutions to the last problem (similarly to the corresponding problem for KdV in [12]) are constructed with the help of solutions to the first two in the form
$$u(t,x,y) = w(t,x,y) + v(t,x,y), \tag{4.4}$$
where $w(t,x,y)$ is a solution to an initial-boundary value problem in $\Pi^-_{T,1} = (0,T)\times(-\infty,1)$ for the equation (3.1) with the initial and boundary conditions (1.2) and (1.4) (for $x=1$) in the class $X_k(\Pi^-_{T,1})$. Then
$$\begin{aligned}\|w\|_{X_k(\Pi^-_{T,1})} &\le c(T,k)\big(\|u_0\|_{H^k(\Sigma)} + \|\mathcal{U}^2\|_{\mathcal{B}^k_2(T)} \\ &\quad + t_0^{1/6}\|f\|_{M_k(Q_{t_0})} + \sum_{m=0}^{m_0-1}\|\partial_t^m f\big|_{t=0}\|_{H^{k-3(m+1)}(\Sigma)}\big).\end{aligned} \tag{4.5}$$
Moreover, by virtue of the compatibility conditions on the line $(0,0,y)$
$$v_1(t,y) \equiv u_1(t,y) - w(t,0,y) \in H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)\big|_{B_T}$$
and
$$\begin{aligned}\|v_1\|_{H^{(k+1)/3,k+1}(B_T)} &\le c(T,k)\big(\|u_0\|_{H^k(\Sigma)} + \|\mathcal{U}^3\|_{\mathcal{B}^k_3(T)} \\ &\quad + t_0^{1/6}\|f\|_{M_k(Q_{t_0})} + \sum_{m=0}^{m_0-1}\|\partial_t^m f\big|_{t=0}\|_{H^{k-3(m+1)}(\Sigma)}\big).\end{aligned} \tag{4.6}$$

Consider in Q_T a problem for the function v:
$$v_t + v_{xxx} + v_{xyy} = 0, \tag{4.7}$$
$$v\big|_{t=0} = 0, \quad v\big|_{x=0} = v_1, \quad v\big|_{x=1} = v_x\big|_{x=1} = 0. \tag{4.8}$$
In order to construct a solution to this problem we consider the boundary potential $J_+(t,x,y;\mu)$ for an arbitrary function $\mu \in H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)\big|_{B_T}$. According to Lemma 3.6 for any $\delta\in(0,T]$
$$\|J_+(\cdot,1,\cdot;\mu)\|_{H^{(k+1)/3,k+1}(B_\delta)} + \|\partial_x J_+(\cdot,1,\cdot;\mu)\|_{H^{k/3,k}(B_\delta)} \le c(T,k)\delta^{1/2}\|\mu\|_{L_2(B_\delta)}. \tag{4.9}$$
Moreover, $J_+(\cdot,1,\cdot;\mu)\in H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)|_{B_T}$, $\partial_x J_+(\cdot,1,\cdot;\mu)\in H_0^{k/3,k}(\mathbb{R}^2_+)|_{B_T}$.

Consider in the domain $\Pi^-_{\delta,1}$ the problem of the (3.1), (1.2), (1.4) (for $x=1$) type, where $u_0\equiv 0$, $f\equiv 0$, $u_2\equiv -J_+(\cdot,1,\cdot;\mu)$, $u_3\equiv -\partial_x J_+(\cdot,1,\cdot;\mu)$. A solution to this problem $V\in X_k(\Pi^-_{\delta,1})$ exists and, in particular,
$$\begin{aligned}&\|V(\cdot,0,\cdot)\|_{H^{(k+1)/3,k+1}(B_\delta)} \\ &\le c(T,k)\left(\|J_+(\cdot,1,\cdot;\mu)\|_{H^{(k+1)/3,k+1}(B_\delta)} + \|\partial_x J_+(\cdot,1,\cdot;\mu)\|_{H^{k/3,k}(B_\delta)}\right).\end{aligned} \tag{4.10}$$
Moreover, it is obvious that $V(\cdot,0,\cdot)\in H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)|_{B_\delta}$.

Consider a linear operator $\Gamma:\mu\mapsto V(\cdot,0,\cdot)$ in the space $H_0^{(k+1)/3,k+1}(\mathbb{R}^2_+)|_{B_\delta}$. For small $\delta=\delta(T,k)$ the estimates (4.9) and (4.10) provide that the operator $(E+\Gamma)$ is invertible (E is the identity operator) and setting $\mu\equiv(E+\Gamma)^{-1}v_1$ we obtain the desired solution to the problem (4.7), (4.8)
$$v(t,x,y)\equiv J_+(t,x,y;\mu) + V(t,x,y),$$
where
$$\|v\|_{X_k(Q_\delta)} \le c(T,k)\|v_1\|_{H^{(k+1)/3,k+1}(B_T)}. \tag{4.11}$$

Thus the solution $u(t,x,y)$ to the problem (3.1), (1.2), (1.5) in the domain Q_δ is constructed and according to (4.4)–(4.6) and (4.11) is evaluated in the space $X_k(Q_\delta)$ by the right part of (4.3). Moving step by step (δ is constant) we obtain the desired solution in the whole domain Q_T.

Uniqueness of weak solutions to the problem (3.1), (1.2), (1.5) in $L_2(Q_T)$ succeeds from existence of smooth solutions to the adjoint problem

$$\phi_t + \phi_{xxx} + \phi_{xyy} = f \in C_0^\infty(Q_T),$$
$$\phi\big|_{t=T} = 0, \quad \phi\big|_{x=0} = \phi_x\big|_{x=0} = \phi\big|_{x=1} = 0,$$

which after simple change of variables transforms to the original one. □

For global a priori estimates for solutions to nonlinear problems we also need certain integral inequalities.

Lemma 4.3. *Let the hypothesis of Lemma 4.2 be satisfied for* $n=1$, $I=\mathbb{R}_+$, $k=0$ *and, in addition,* $u_1 \equiv 0$. *Consider a solution to the problem* (3.1), (1.2), (1.3) *in the class* $X_0(\Pi_T^+)$. *Then for any* $t \in (0,T]$

$$\begin{aligned} &\iint_{\mathbb{R}_+^2} u^2(t,x,y)\,dxdy + \iint_{B_t} u_x^2(\tau,0,y)\,dyd\tau \\ &= \iint_{\mathbb{R}_+^2} u_0^2\,dxdy + 2\iiint_{\Pi_t^+} fu\,dxdy\,d\tau. \end{aligned} \tag{4.12}$$

Proof. For smooth solutions (4.12) is obtained obviously by multiplication of the equation (3.1) by $2u(t,x,y)$ and consequent integration (compare with (4.1)) and then for weak ones via closure. □

Lemma 4.4. *Let the hypothesis of Lemma 4.2 be satisfied for* $n=1$, $I=\mathbb{R}_+$, $k=1$ *and, in addition,* $u_1 \equiv 0$. *Consider a solution to the problem* (3.1), (1.2), (1.3) *in the class* $X_1(\Pi_T^+)$. *Then for any* $t \in (0,T]$

$$\begin{aligned} &\iint_{\mathbb{R}_+^2} \Big(u_x^2 + u_y^2 - \frac{1}{3}u^3\Big)\rho(x)\,dxdy + \frac{1}{2}\iiint_{\Pi_t^+} (u_{xx}^2 + u_{xy}^2 + u_{yy}^2)\rho'(x)\,dx\,dy\,d\tau \\ &+ 2\iiint_{\Pi_t^+} uu_x(u_{xx}+u_{yy})\rho\,dx\,dy\,d\tau \\ &\le \iint_{\mathbb{R}_+^2} \Big(u_{0x}^2 + u_{0y}^2 - \frac{1}{3}u_0^3\Big)\rho\,dx\,dy + 2\iiint_{\Pi_t^+} (f_x u_x + f_y u_y)\rho\,dx\,dy\,d\tau \\ &\quad - \iiint_{\Pi_t^+} fu^2\rho\,dx\,dy\,d\tau + c\iint_{B_t} (f^2+u_x^2)\Big|_{x=0} dyd\tau \\ &\quad + c\Big(1 + \|u\|^2_{C([0,t];L_2(\mathbb{R}_+^2))}\Big)\iiint_{\Pi_t^+} (u_x^2 + u_y^2 + u^2)\rho\,dx\,dy\,d\tau, \end{aligned} \tag{4.13}$$

where $\rho(x) \equiv 2 - (1+x)^{-1/2}$.

Proof. This lemma was proved in [11]. We represent here the slightly modified version of the proof.

As in the previous lemma it it sufficient to consider smooth solutions. Multiplying (3.1) by $-\big(2(u_x(t,x,y)\rho(x))_x + 2u_{yy}(t,x,y)\rho(x) + u^2(t,x,y)\rho(x)\big)$ and integrating over $\mathbb{R}^2_+$ we derive an equality

$$
\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^2_+}(u_x^2+u_y^2-\frac{1}{3}u^3)\rho\,dxdy + \iint_{\mathbb{R}^2_+}(3u_{xx}^2+4u_{xy}^2+u_{yy}^2)\rho'\,dxdy\\
&+\int_{\mathbb{R}}\big(u_{xx}^2\rho+2u_{xx}u_x\rho'-u_x^2\rho''\big)\big|_{x=0}dy - \iint_{\mathbb{R}^2_+}(u_x^2+u_y^2)\rho'''\,dxdy\\
&+2\iint_{\mathbb{R}^2_+}uu_x(u_{xx}+u_{yy})\rho\,dxdy + \iint_{\mathbb{R}^2_+}u^2(u_{xx}+u_{yy})\rho'\,dxdy\\
&=2\iint_{\mathbb{R}^2_+}(f_xu_x+f_yu_y)\rho\,dxdy + 2\int_{\mathbb{R}}(fu_x\rho)\big|_{x=0}dy - \iint_{\mathbb{R}^2_+}fu^2\rho\,dxdy.
\end{aligned}
\tag{4.14}
$$

Applying the interpolational inequality (2.1) in the case $p=4$ we find that

$$
\begin{aligned}
&\Big|\iint_{\mathbb{R}^2_+}u^2(u_{xx}+u_{yy})\rho'\,dxdy\Big|\\
&\le \frac{1}{2}\iint_{\mathbb{R}^2_+}(u_{xx}^2+u_{yy}^2)\rho'\,dxdy + c\iint_{\mathbb{R}^2_+}(u_x^2+u_y^2+u^2)\rho\,dxdy\iint_{\mathbb{R}^2_+}u^2\,dxdy
\end{aligned}
$$

and derive (4.13) from (4.14). □

Lemma 4.5. *Let the hypothesis of Lemma 4.2 be satisfied for $n=3$, $I=(0,1)$, $k=0$ and, in addition, $u_1=u_2\equiv 0$. Consider a solution to the problem* (3.1), (1.2), (1.5) *in the class $X_0(Q_T)$. Then for any $t\in(0,T]$*

$$
\begin{aligned}
&\iint_{\mathbb{R}^2_+}u^2(t,x,y)\rho(x)\,dxdy + \iiint_{Q_t}(3u_x^2+u_y^2)\rho'(x)\,dx\,dy\,d\tau + \iint_{B_t}u_x^2\big|_{x=0}\,dyd\tau\\
&=\iint_{\mathbb{R}^2_+}u_0^2\rho\,dxdy + \rho(1)\iint_{B_t}u_3^2\,dyd\tau + 2\iiint_{Q_t}fu\rho\,dxdy\,d\tau,
\end{aligned}
\tag{4.15}
$$

where either $\rho\equiv 1$ or $\rho\equiv 1+x$.

Proof. Similarly to Lemma 4.3 for smooth solutions (4.15) is obtained via multiplication of (3.1) by $2u(t,x,y)\rho(x)$ and consequent integration and then for weak ones via closure. □

5. Proof of the main result

This section contains the proof of Theorem 2.7 consisting of several lemmas. The first one is devoted to local well-posedness.

Lemma 5.1. *Let the hypothesis of Theorem 2.7 be satisfied. Then there exists $t_0\in(0,T]$ such that any of the considered initial-boundary value problems for the equation* (1.1) *is well posed in $X_k((0,t_0)\times I\times\mathbb{R})$.*

Proof. For $t_0\in(0,T]$ introduce a set of functions

$$
Y_k((0,t_0)\times I\times\mathbb{R}) = \big\{v\in X_k((0,t_0)\times I\times\mathbb{R}) : \partial_t^m v\big|_{t=0}=\Phi_m \text{ for } m\le m_0-1\big\}
$$

and define on this set a map Λ in such a way: $u=\Lambda v$ is a solution in $Y_k((0,t_0)\times I\times\mathbb{R})$ to a corresponding initial-boundary value linear problem for an equation

$$
u_t+u_{xxx}+u_{xyy}=f-vv_x \tag{5.1}
$$

with the initial profile (1.2) and one of the three boundary conditions (1.3), (1.4) or (1.5). Note that the functions $\widetilde{\Phi}_m$, written for these problems, coincide for $m < k/3$ with the functions Φ_m written for the original problems. Therefore the compatibility conditions of the order k are satisfied. Moreover, by virtue of Lemma 2.4 $vv_x \in M_k((0,t_0)\times I\times\mathbb{R})$, so Lemma 4.2 provides existence of the map Λ and according to (4.3) and (2.7)

$$\|u\|_{X_k((0,t_0)\times I\times\mathbb{R})} \le c(T,k)\Big(\widetilde{c} + t_0^{1/6}\|v\|^2_{X_k((0,t_0)\times I\times\mathbb{R})}\Big), \tag{5.2}$$

where the constant $\widetilde{c}$ depends on the norms of u_0, $\mathcal{U}^n$ and f in the corresponding spaces. It follows from (5.2) that for considerably large $R > 0$ and considerably small $t_0^* \in (0,T]$ the map Λ transforms for any $t_0 \in (0,t_0^*]$ a ball $Y_{k,R}((0,t_0)\times I\times\mathbb{R}) = \{v \in Y_k((0,t_0)\times I\times\mathbb{R}) : \|v\|_{X_k((0,t_0)\times I\times\mathbb{R})} \le R\}$ into itself.

Next, consider two functions v and $\widetilde{v}$ from the set $Y_{k,R}((0,t_0)\times I\times\mathbb{R})$. Similarly to (5.2)

$$\|\Lambda v - \Lambda\widetilde{v}\|_{X_k((0,t_0)\times I\times\mathbb{R})} \le c(T,k)t_0^{1/6}R\|v-\widetilde{v}\|_{X_k((0,t_0)\times I\times\mathbb{R})}$$

and therefore Λ is a contraction in $Y_{k,R}((0,t_0)\times I\times\mathbb{R})$ for considerably small t_0.

Continuous dependence is established in a similar way. □

The next lemma is devoted to one global conditional a priori estimate valid for all three considered problems.

Lemma 5.2. *Let the hypothesis of Theorem 2.7 be satisfied for $k \ge 2$. Let $u(t,x,y)$ be a solution to any of the three initial-boundary value problems for the equation* (1.1) *in the class $X_k((0,T')\times I\times\mathbb{R})$ for some $T' \in (0,T]$. Then uniformly with respect to T',*

$$\begin{aligned}&\|u\|_{X_k((0,T')\times I\times\mathbb{R})}\\ &\le c\big(T,k,\|u_0\|_{H^k(I\times\mathbb{R})},\|\mathcal{U}^n\|_{\mathcal{B}_n^k(T)},\|f\|_{M_k((0,T)\times I\times\mathbb{R})},\|u\|_{X_{k-1}((0,T')\times I\times\mathbb{R})}\big).\end{aligned} \tag{5.3}$$

Proof. Consider u as a solution to the corresponding problem for the equation (5.1), where $v \equiv u$. Then the inequalities (2.8) and (4.3) yield that similarly to (5.2)

$$\|u\|_{X_k((0,t_0)\times I\times\mathbb{R})} \le c(T,k)\Big(\widetilde{c} + t_0^{1/6}\|u\|_{X_{k-1}((0,T')\times I\times\mathbb{R})}\|u\|_{X_k((0,t_0)\times I\times\mathbb{R})}\Big),$$

whence (5.3) follows by the standard argument. □

The next lemma provides a global a priori estimate for the problem in Π_T^+ in the class $X_1(\Pi_T^+)$ and thus completes the proof of Theorem 2.7 in this case.

Lemma 5.3. *Let the hypothesis of Theorem 2.7 be satisfied for $n=1$, $I=\mathbb{R}_+$, $k=1$. Let $u(t,x,y)$ be a solution to the problem* (1.1)–(1.3) *in the class $X_1(\Pi_{T'}^+)$ for some $T' \in (0,T]$. Then uniformly with respect to T'*

$$\|u\|_{C([0,T'];H^1(\mathbb{R}_+^2))} \le c\big(T,\|u_0\|_{H^1(\mathbb{R}_+^2)},\|u_1\|_{H^{2/3,2}(B_T)},\|f\|_{L_2(0,T;H^1(\mathbb{R}_+^2))}\big). \tag{5.4}$$

Proof. We reproduce here in brief the proof from [11]. Extend the function u_1 in the class $H^{2/3,2}$ to the whole plane $\mathbb{R}^2$ such that $u_1(t,y)=0$ for $t \le -1$. Let

$$U(t,x,y) \equiv u(t,x,y) - J_+(t,x,y;u_1). \tag{5.5}$$

Write down for the function U the equality (4.12), then for $t \in [0,T']$

$$\iint_{\mathbb{R}_+^2} U^2(t,x,y)\,dx\,dy + \iint_{B_t} U_x^2\Big|_{x=0}dy\,d\tau \le c + 2\iiint_{\Pi_t^+}(f-uu_x)U\,dx\,dy\,d\tau. \tag{5.6}$$

Since

$$uu_xU = \Big(\frac{U^3}{3} + J_+\frac{U^2}{2}\Big)_x + \partial_x J_+\Big(\frac{U^2}{2} + J_+U\Big) \tag{5.7}$$

and $U\big|_{x=0} = 0$, it follows from (5.6) and (3.16) that

$$\|u\|_{C([0,T'];L_2(\mathbb{R}^2_+))} + \|u_x\big|_{x=0}\|_{L_2(B_{T'})} \le c. \tag{5.8}$$

Next, write down for the function U the inequality (4.13), then by virtue of the already established estimate (5.8) for any $t \in [0,T']$

$$\begin{aligned}
&\iint_{\mathbb{R}^2_+} (U_x^2 + U_y^2 - \frac{1}{3}U^3)\rho\,dx\,dy + \frac{1}{2}\iiint_{\Pi_t^+} (U_{xx}^2 + U_{xy}^2 + U_{yy}^2)\rho'\,dx\,dy\,d\tau \\
&\le c + c\iiint_{\Pi_t^+} (U_x^2 + U_y^2)\rho\,dx\,dy\,d\tau + \iiint_{\Pi_t^+} uu_xU^2\rho\,dx\,dy\,d\tau \\
&\quad + c\iint_{B_t} u_1^2u_x^2\big|_{x=0} dy\,d\tau + 2\iiint_{\Pi_t^+} uu_xU_x\rho'\,dx\,dy\,d\tau \\
&\quad + 2\iiint_{\Pi_t^+} (J_+U_x + u\partial_x J_+)(U_{xx} + U_{yy})\rho\,dx\,dy\,d\tau.
\end{aligned} \tag{5.9}$$

The inequality (2.1) and the estimates (3.16), (5.8) yield that

$$\begin{aligned}
\iiint_{\Pi_t^+} uu_xU^2\rho\,dx\,dy\,d\tau &= \iiint_{\Pi_t^+}\Big(\partial_x J_+ uU^2\rho - \frac{1}{3}(J_+\rho)_xU^3 - \frac{1}{4}U^4\rho'\Big)\,dx\,dy\,d\tau \\
&\le c\iiint_{\Pi_t^+} (U_x^2 + U_y^2)\rho\,dx\,dy\,d\tau + c,
\end{aligned}$$

$$\begin{aligned}
&2\iiint_{\Pi_t^+} uu_xU_x\rho'\,dx\,dy\,d\tau \\
&= -\iint_{B_t} u_1^2(U_x\rho')\big|_{x=0} dy\,d\tau - \iiint_{\Pi_t^+} u^2(U_{xx}\rho' + U_x\rho'')\,dx\,dy\,d\tau \\
&\le \frac{1}{6}\iiint_{\Pi_t^+} U_{xx}^2\rho'\,dx\,dy\,d\tau + c\iiint_{\Pi_t^+} (U_x^2 + U_y^2)\rho\,dx\,dy\,d\tau + c,
\end{aligned}$$

$$\begin{aligned}
\iint_{B_t} u_1^2u_x^2\big|_{x=0} dy\,d\tau &\le \varepsilon\iiint_{\Pi_t^+} U_{xx}^2\rho'\,dx\,dy\,d\tau \\
&\quad + c(\varepsilon)\int_0^t \Big(1 + \sup_{y\in\mathbb{R}} u_1^4\Big)\iint_{\mathbb{R}^2_+} U_x^2\rho\,dx\,dy\,d\tau + c,
\end{aligned}$$

where $\|u_1\|_{L_4(0,T;C_b(\mathbb{R}))} \le c\|u_1\|_{H^{2/3,2}(\mathbb{R}^2)}$ (see, e.g. [1]) and $\varepsilon > 0$ can be chosen arbitrarily small. In the above arguments we also use the obvious interpolational inequality

$$|\varphi(0)| \le c\Big(\int_{\mathbb{R}_+} (\varphi')^2\rho'\,dx\Big)^{1/4}\Big(\int_{\mathbb{R}_+} \varphi^2\rho\,dx\Big)^{1/4} + c\Big(\int_{\mathbb{R}_+} \varphi^2\rho\,dx\Big)^{1/2}. \tag{5.10}$$

Finally,

$$\begin{aligned}
&2\iiint_{\Pi_t^+} (J_+U_x + u\partial_x J_+)(U_{xx} + U_{yy})\rho\,dx\,dy\,d\tau \\
&\le \frac{1}{6}\iiint_{\Pi_t^+} (U_{xx}^2 + U_{yy}^2)\rho'\,dx\,dy\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ c\int_0^t \sup_{(x,y)\in\mathbb{R}^2_+}\Big[\big((\partial_x J_+)^2 + J_+^2\big)\frac{\rho^2}{\rho'}\Big]\iint_{\mathbb{R}^2_+}(U_x^2\rho + u^2)\,dx\,dy\,d\tau \\
&\le \frac{1}{6}\iiint_{\Pi_t^+}(U_{xx}^2 + U_{yy}^2)\rho'\,dx\,dy\,d\tau + \int_0^t \gamma(\tau)\iint_{\mathbb{R}^2_+} U_x^2\rho\,dx\,dy\,d\tau + c,
\end{aligned}
$$

where $\|\gamma\|_{L_1(0,T)} \le c$ since $\rho^2(\rho')^{-1} \le c(1+x)^{3/2}$ and we can apply the inequalities (3.16) and (3.22). Therefore the inequality (5.9) provides (5.4). $\square$

The last lemma establishes a global a priori estimate for the problem in Q_T in the class $X_3(Q_T)$ and thus completes the proof of Theorem 2.7.

Lemma 5.4. *Let the hypothesis of Theorem 2.7 be satisfied for $n = 3$, $I = (0,1)$, $k = 3$. Let $u(t,x,y)$ be a solution to the problem* (1.1), (1.2), (1.5) *in the class $X_3(Q_{T'})$ for some $T' \in (0,T]$. Then uniformly with respect to T'*

$$
\|u\|_{C([0,T'];H^3(\Sigma))} \le c\big(T, \|u_0\|_{H^3(\Sigma)}, \|\mathcal{U}^3\|_{\mathcal{B}^3_3(T)}, \|f\|_{M_3(Q_T)}\big). \tag{5.11}
$$

Proof. Let $v(t,x,y) \in X_3(Q_T)$ be a solution to the linear problem (3.1), (1.2), (1.5). Let

$$
U(t,x,y) \equiv u(t,x,y) - v(t,x,y), \tag{5.12}
$$

then the function $U \in X_3(Q_{T'})$ is a solution to a problem

$$
\begin{gathered}
U_t + U_{xxx} + U_{xyy} = -uu_x, \\
U\big|_{t=0} = 0, \quad U\big|_{x=0} = U\big|_{x=1} = U_x\big|_{x=1} = 0.
\end{gathered}
$$

First write down the equality (4.15) for the function U in the case $\rho \equiv 1$. Then using the equality (5.7), where J_+ is substituted by v, similarly to (5.8) we derive an estimate

$$
\|u\|_{C([0,T'];L_2(\Sigma))} + \|u_x\big|_{x=0}\|_{L_2(B_{T'})} \le c. \tag{5.13}
$$

Next, again use the equality (4.15) for the function U but now in the case $\rho \equiv 1+x$. Then by virtue of the already proved estimate (5.13)

$$
\iiint_{Q_{T'}}(U_x^2 + U_y^2)\,dx\,dydt \le c - 2\iiint_{Q_{T'}} uu_xU\rho\,dx\,dydt. \tag{5.14}
$$

Again applying the corresponding analogue of (5.7) we find that

$$
2\iint_\Sigma uu_xU\rho\,dx\,dy = \iint_\Sigma\Big(-\frac{2}{3}U^3 + (v_x\rho - v)U^2 + 2vv_xU\rho\Big)dx\,dy
$$

and by virtue of the interpolation inequality (2.1) derive from (5.14) an estimate

$$
\|u\|_{L_2(0,T';H^1(\Sigma))} \le c. \tag{5.15}
$$

Next, by induction with respect to j we prove that for $j \le 3$

$$
\|\partial_y^j u\|_{C([0,T'];L_2(\Sigma))} + \|\partial_y^j u\|_{L_2(0,T';H^1(\Sigma))} + \|\partial_y^j u_x\big|_{x=0}\|_{L_2(B_{T'})} \le c. \tag{5.16}
$$

For $j=0$ this estimate coincides with (5.13), (5.15). For $j\ge 1$ write down for the function $\partial_y^j U$ the equality (4.15) in the case $\rho\equiv 1+x$:

$$\begin{aligned} &\iint_{\mathbb{R}^2_+} (\partial_y^j U)^2\rho\,dxdy + \iiint_{Q_t} \big(3(\partial_y^j U_x)^2 + (\partial_y^{j+1}U)^2\big)\,dx\,dy\,d\tau \\ &+ \iint_{B_t} (\partial_y^j U_x)^2\big|_{x=0}\,dyd\tau \\ &= -2\iiint_{Q_t} \partial_y^j(uu_x)\partial_y^j U\rho\,dx\,dy\,d\tau. \end{aligned} \tag{5.17}$$

Here

$$\begin{aligned} &\Big|2\iiint_{Q_t}(u\partial_y^j U_x + u_x\partial_y^j U)\partial_y^j U\rho\,dx\,dy\,d\tau\Big| \\ &= \Big|\iiint_{Q_t}(u_x\rho - u)(\partial_y^j U)^2\,dx\,dy\,d\tau\Big| \\ &\le c\int_0^t \Big(\iint_\Sigma (u_x^2+u^2)\,dx\,dy\Big)^{1/2}\Big(\iint_\Sigma (\partial_y^j U)^4\,dx\,dy\Big)^{1/2}\,d\tau \\ &\le \varepsilon\iiint_{Q_t}\big((\partial_y^j U_x)^2 + (\partial_y^{j+1}U)^2\big)\,dx\,dy\,d\tau + c(\varepsilon)\int_0^t \gamma(\tau)\iint_\Sigma (\partial_y^j U)^2\,dx\,dy\,d\tau, \end{aligned} \tag{5.18}$$

where $\|\gamma\|_{L_1(0,T')}\le c$ and $\varepsilon>0$ can be chosen arbitrarily small. All other terms in the right part of (5.17) are also similarly estimated by the right part of (5.18) (plus certain appropriate constant) and so (5.17) yields the estimate (5.16).

Now let $v_1(t,x,y)\in X_0(Q_T)$ be a solution to a problem

$$\begin{gathered} v_{1t}+v_{1xxx}+v_{1xyy}=f_t,\\ v_1\big|_{t=0}=\Phi_1,\quad v_1\big|_{x=0}=u_{1t},\quad v_1\big|_{x=1}=u_{2t},\quad v_{1x}\big|_{x=1}=u_{3t}. \end{gathered}$$

Let

$$U_1(t,x,y)\equiv u_t(t,x,y)-v_1(t,x,y),$$

then the function $U_1\in X_0(Q_{T'})$ is a solution to a problem

$$\begin{gathered} U_{1t}+U_{1xxx}+U_{1xyy}=-(uu_x)_t,\\ U_1\big|_{t=0}=0,\quad U_1\big|_{x=0}=U_1\big|_{x=1}=U_{1x}\big|_{x=1}=0. \end{gathered}$$

Writing down for the function U_1 the equality (4.15) in the case $\rho\equiv 1+x$ we obtain the corresponding analogue of (5.17) and estimating nonlinear terms similarly to (5.18) derive an estimate

$$\|u_t\|_{C([0,T'];L_2(\Sigma))} + \|u_t\|_{L_2(0,T';H^1(\Sigma))} + \|u_{tx}\big|_{x=0}\|_{L_2(B_{T'})}\le c. \tag{5.19}$$

Write down the equation (1.1) in a form

$$u_{xxx}=f-u_{xyy}-uu_x-u_t. \tag{5.20}$$

By virtue of (5.16) and (5.19) the right part of this equality can be represented as a sum g_0+g_{1x}, where $g_j\in C([0,T'];L_2(\Sigma))$ with appropriate estimates on the corresponding norms. Therefore it follows from (5.20) (together with (5.16) for $j=2$) that

$$\|u\|_{C([0,T'];H^2(\Sigma))}\le c. \tag{5.21}$$

Finally, we apply the inequality (see, e.g. [18])

$$\|g\|_{H^2(\Sigma)} \le c\big(\|\Delta g\|_{L_2(\Sigma)} + \|g\big|_{\partial\Sigma}\|_{H^{3/2}(\mathbb{R})} + \|g\|_{H^1(\Sigma)}\big) \tag{5.22}$$

for the function $g \equiv u_x$. It follows from (1.1) that

$$\Delta u_x = f - uu_x - u_t \in C([0,T'];L_2(\Sigma)).$$

Moreover, by virtue of (5.16), (5.19) and embedding $H^2(\Sigma) \subset H^{3/2}(\partial\Sigma)$ (see [18])

$$\begin{aligned}
&\|u_x\big|_{x=0}\|_{C([0,T'];H^{3/2}(\mathbb{R}))}\\
&\le \|u_{0x}\big|_{x=0}\|_{H^{3/2}(\mathbb{R})} + 2\|u_{tx}\big|_{x=0}\|^{1/2}_{L_2(B_{T'})}\|u_x\big|_{x=0}\|^{1/2}_{L_2(0,T';H^3(\mathbb{R}))} \le c,
\end{aligned}$$

$$\begin{aligned}
\|u_x\big|_{x=1}\|_{C([0,T'];H^{3/2}(\mathbb{R}))} &= \|u_3\|_{C([0,T'];H^{3/2}(\mathbb{R}))}\\
&\le \|u_3(0,\cdot)\|_{H^{3/2}(\mathbb{R})} + c\|u_3\|_{H^{1,3}(B_T)} \le c_1.
\end{aligned}$$

Therefore, (5.22) together with (5.21) and (5.16) for $j=3$ provide (5.11). □

6. Weak solutions

Solutions considered in Theorem 2.7 are at least in the space X_1 because of the use of the inequality (2.7), which is valid only if $k \ge 1$. Of course, the contraction principle is not the unique method to construct solutions. For example, they can be obtained as limits of certain sequences of solutions to regularized problems, if appropriate uniform estimates on these regularized solutions are established (uniqueness in this situation requires its own special approaches). In order to describe classes of weak solutions, which are obtained on this way, we need some additional notation.

By $C_w(\overline{I};\mathcal{B})$, where $\mathcal{B}$ is a certain Banach space, we denote a space of weakly continuous mappings from $\overline{I}$ to $\mathcal{B}$. Note that $C_w(\overline{I};\mathcal{B}) \subset L_\infty(I;\mathcal{B})$ for bounded intervals I (see, e.g. [14]) and this space becomes the Banach space supplied with a norm

$$\|f\|_{C_w(\overline{I};\mathcal{B})} = \sup_{t\in\overline{I}} \|f(t)\|_{\mathcal{B}}.$$

By analogy with (1.16) let

$$\lambda^+(u;T) = \sup_{m\ge 0}\int_0^T\int_m^{m+1}\int_{\mathbb{R}} (u_x^2 + u_y^2)\,dy\,dx\,dt.$$

Theorem 6.1. *Let $(1+x)^\beta u_0 \in L_2(\mathbb{R}^2_+)$ for certain $\beta \ge 0$, $u_1 \in H^{s/3,s}(B_T)$ for certain $s > 3/2$ and $T > 0$, $(1+x)^\beta f \in L_1(0,T;L_2(\mathbb{R}^2_+))$. Then there exists a solution $u(t,x,y)$ to the problem (1.1)–(1.3) such that*

$$(1+x)^\beta u \in C_w([0,T];L_2(\mathbb{R}^2_+)), \quad \lambda^+(u;T) < \infty.$$

If, in addition, $\beta > 0$ then

$$(1+x)^{\beta-1/2}u \in L_2(0,T;H^1(\mathbb{R}^2_+)).$$

The problem (1.1)–(1.3) is well-posed in this class if $\beta \ge 1$.

Proof. This result was established in the paper [9] under other (more complicated) hypothesis on the boundary data u_1. This hypothesis in [9] ensured that the boundary potential $J_+(t,x,y;u_1)$, which was constructed in the form (3.23), possessed the following properties:

$$J_+ \in C([0,T];L_2(\mathbb{R}^2_+)) \cap L_2(0,T;H^1(\mathbb{R}^2_+)), \quad \|\partial_{x,y}J_+(t,\cdot,\cdot;u_1)\|_{C_b(\overline{\mathbb{R}}^2_+)} \in L_1(0,T), \tag{6.1}$$

which were crucial for global a priori estimates in the space $C([0,T];L_2(\mathbb{R}^2_+))$ (see (5.8)) and on the value $\lambda^+(u,T)$. But these properties are also provided by the inequalities (3.18) and (3.20) under the hypothesis of the present theorem. □

Theorem 6.2. *Let $u_0 \in L_2(\Sigma)$, $u_1, u_2 \in H^{s/3,s}(B_T)$ for certain $s > 3/2$ and $T > 0$, $u_3 \in L_2(B_T)$, $f \in L_1(0,T;L_2(\Sigma))$. Then the problem* (1.1), (1.2), (1.5) *is well-posed in a class $C_w([0,T];L_2(\Sigma)) \cap L_2(0,T;H^1(\Sigma))$.*

Proof. The main global a priori estimate in this class is obtained similarly to (5.13), (5.15), where in the formula (5.12) the function v is taken in a form

$$v(t,x,y) \equiv J_+(t,x,y;u_1)\sigma(1-x) + J_+(-t,1-x,y;\widetilde{u}_2)\sigma(x), \tag{6.2}$$

where $\widetilde{u}_2(t,y) \equiv u_2(-t,y)$, u_1 and u_2 are extended in the class $H^{s/3,s}$ to the whole plane $\mathbb{R}^2$ such that $u_1(t,y) \equiv 0$ for $t \le -1$, $u_2(t,y) \equiv 0$ for $t \ge T+1$, and $\sigma(x)$ is a certain smooth "cut-off" function, namely, $\sigma(x) = 0$ for $x \le 1/4$, $\sigma(x) = 1$ for $x \ge 3/4$, $\sigma'(x) > 0$ for $x \in (1/4, 3/4)$. Then the estimates (3.18), (3.20) and (3.22) provide the required properties of the function v similar to (6.1), where $\mathbb{R}^2_+$ is substituted by Σ, and $v_x|_{x=1}$ is evaluated in $L_2(B_T)$ by virtue of the estimate on $\partial_x J_+$ in $C_b(\overline{\mathbb{R}}_+;L_2(B_T))$.

Uniqueness and continuous dependence are established on the base of the equality (4.15) in the case $\rho \equiv 1 + x$ similarly to Theorem 6.1 (here, of course, additional decay of solutions at infinity is not required), where the boundary data are made zero similarly to (5.12), (6.2). □

In the paper [13] existence of global solutions to the problem (1.1), (1.2), (1.4) was established in a class of functions $u \in C_w([0,T];L_2(\mathbb{R}^2_-))$ such that

$$\lambda^-(u;T) = \sup_{m\ge 0} \int_0^T \int_{-m-1}^{-m} \int_{\mathbb{R}} (u_x^2 + u_y^2)\,dy\,dx\,dt < \infty$$

under the hypothesis $u_0 \in L_2(\mathbb{R}^2_-)$, $u_2 \in H^{s/3,s}(B_T)$ for certain $s > 3/2$, $u_3 \in L_2(B_T)$, $f \in L_1(0,T;L_2(\mathbb{R}^2_-))$.

References

[1] O.V. Besov, V.P. Il'in, S.M. Nikolskii; *Integral Representation of Functions and Embedding Theorems*, J. Wiley, 1978.

[2] J.L. Bona, S.M. Sun, B.-Y. Zhang; *A non-homogeneous boundary-value problem for the Korteweg–de Vries equation in a quarter plane*, Trans. Amer. Math. Soc., **354** (2002), 427–490.

[3] J.L. Bona, S.M. Sun, B.-Y. Zhang; *A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain*, Comm. Partial Differential Equations, **28** (2003), 1391–1436.

[4] J. Bourgain; *Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations. Part II: the KdV equation*, Geom. Funct. Anal., **3** (1993), 209–262.

[5] L. Cattabriga; *Una problema al contorno per una equazione parabolica di ordine dispari*, Ann. Scuola Norm. Sup Pisa Sci. Fiz. e Mat., **13** (1959), 163–203.

[6] J.E. Colliander, C.E. Kenig; *The generalized Korteweg – de Vries equation on the half line*, Comm. Partial Differential Equations, **27** (2002), 2187–2266.

[7] A.V. Faminskii; *The Cauchy problem for quasilinear equations of odd order*, Mat. Sb. **180** (1989), 1183–1210; Engl. transl. in Math. USSR-Sb., **68** (1991), 31–59.

[8] A.V. Faminskii; *The Cauchy problem for the Zakharov–Kuznetsov equation*, Differentsial'nye Uravneniya, **31** (1995), 1070–1081; Engl. transl. in Differential Equations, **31** (1995), 1002–1012.

[9] A.V. Faminskii; *On the mixed problem for quasilinear equations of the third order*, Journ. Math. Sci., **110** (2002), 2476–2507.

[10] A.V. Faminskii; *An initial boundary-value problem in a half-strip for the Korteweg–de Vries equation in fractional-order Sobolev spaces*, Comm. Partial Differential Equations, **29** (2004), 1653–1695.

[11] A.V. Faminskii; *On global well-posedness of a mixed problem for the Zakharov–Kuznetsov equation*, Sovremennaya Matematika i ee Prilozheniya, **38** (2006), 135–148, in Russian.

[12] A.V. Faminskii; *Global well-posedness of two initial boundary-value problems for the Korteweg–de Vries equation*, Differential Integral Equations, **20** (2007), 601–642.

[13] A.V. Faminskii, E.S. Baykova; *Weak solutions to a mixed problem with two boundary conditions for the generalized Zakharov–Kuznetsov equation*, in: Neklassicheskie Uravneniya Mat. Fiz., A.I. Kozhanov (ed.), Novosibirsk, 2007, 298–306, in Russian.

[14] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.

[15] J. Holmer; *The initial-boundary value problems for the Korteweg–de Vries equation*, Comm. Partial Differential Equations, **31** (2006), 1151–1190.

[16] C.E. Kenig, G. Ponce, L. Vega; *Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle*, Comm. Pure Appl. Math., **46** (1993), 527–620.

[17] Levandosky J.L.; *Smoothing properties of nonlinear dispersive equations in two spatial dimensions*, J. Differential Equations, **175** (2001), 275–352.

[18] J.-L. Lions, E. Magenes; *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968.

[19] S.G. Pyatkov; *An equation of composite type*, Differentsial'nye Uravneniya, **16** (1980), 117–123, in Russian.

[20] J.C. Saut; *Sur quelques généralizations de l'equation de Korteweg–de Vries*, J. Math. Pures Appl., **58** (1979), 21–61.

[21] S. Gindikin, L. Volevich; *Mixed problem for partial differential equations with quasihomogeneous principal part*, Translations of Mathematical Monographs, **147**, Amer. Math. Soc., Providence, RI, 1996.

[22] V.E. Zakharov, E.A. Kuznetsov; *On threedimensional solitons*, Zhurnal Eksp. Teoret. Fiz., **66** (1974), 594–597; Engl. transl. in Soviet Phys JETP, **39** (1974), 285–288.

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