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A NOTE ON THE EXISTENCE OF Ψ-BOUNDED SOLUTIONS FOR A SYSTEM OF DIFFERENTIAL EQUATIONS ON $\mathbb{R}$

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Abstract. We prove a necessary and sufficient condition for the existence of Ψ-bounded solutions of a linear nonhomogeneous system of ordinary differential equations on $\mathbb{R}$.

1. Introduction

The aim of this paper is to give a necessary and sufficient condition so that the nonhomogeneous system of ordinary differential equations

$$x' = A(t)x + f(t) \tag{1.1}$$

has at least one Ψ-bounded solution on $\mathbb{R}$ for every continuous and Ψ-bounded function f on $\mathbb{R}$.

Here, Ψ is a continuous matrix function on $\mathbb{R}$. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

The problem of boundedness of the solutions for the system (1.1) was studied in [4] The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, as e.q. [1, 3, 9, 11]. The fact that in [1] the function Ψ is a scalar continuous function and increasing, differentiable and such that $\Psi(t) \geq 1$ on $\mathbb{R}_+$ and $\lim_{t\to\infty} \Psi(t) = b \in \mathbb{R}_+$ does not enable a deeper analysis of the asymptotic properties of the solutions of a differential equation than the notions of stability or boundedness. In [3], the function Ψ is a scalar continuous function, nondecreasing and such that $\Psi(\mathrm{t}) \geq 1$ on $\mathbb{R}_+$. In [9, 11], Ψ is a scalar continuous function.

In [5, 6, 7], the author proposes a novel concept, Ψ-boundedness of solutions, Ψ being a continuous matrix function, which is interesting and useful in some practical cases and presents the existence conditions for such solutions on $\mathbb{R}_+$. In [2], the author associates this problem with the concept of Ψ-dichotomy on $\mathbb{R}$ of the system $x' = A(t)x$. Also, in [10], the authors define Ψ-boundedness of solutions for difference equations via Ψ-bounded sequences and establish a necessary and sufficient condition for existence of Ψ-bounded solutions for a nonhomogeneous linear difference equation.

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Let $\mathbb{R}^d$ be the Euclidean d-space. For $x = (x_1, x_2, x_3, \dots, x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = (a_{ij})$, we define the norm $|A|$ by $|A| = \sup_{\|x\|\le 1} \|Ax\|$. It is well-known that $|A| = \max_{1\le i\le d}\{\sum_{j=1}^{d} |a_{ij}|\}$.

Let $\Psi_i : \mathbb{R} \to (0, \infty)$, $i = 1, 2, \dots, d$, be continuous functions and

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_d].$$

Definition. A function $\varphi : \mathbb{R} \to \mathbb{R}^d$ is said to be Ψ-bounded on $\mathbb{R}$ if $\Psi\varphi$ is bounded on $\mathbb{R}$.

By a solution of (1.1), we mean a continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^d$ satisfying the system for all $t \in \mathbb{R}$.

Let A be a continuous $d \times d$ real matrix and the associated linear differential system

$$y' = A(t)y. \tag{1.2}$$

Let Y be the fundamental matrix of (1.2) for which $Y(0) = I_d$ (identity $d \times d$ matrix).

Let the vector space $\mathbb{R}^d$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(t)$ of (1.2) is Ψ-bounded on $\mathbb{R}$ if and only if $y(0) \in X_0$ and Ψ-bounded on $\mathbb{R}_+ = [0, \infty)$ if and only if $y(0) \in X_- \oplus X_0$. Also, let P_-, P_0, P_+ denote the corresponding projection of $\mathbb{R}^d$ onto X_-, X_0, X_+ respectively.

Main result

We are now in position to prove our main result.

Theorem 1.1. *If Ais a continuous $d\times d$ real matrix on $\mathbb{R}$, then, the system* (1.1) *has at least one Ψ-bounded solution on $\mathbb{R}$ for every continuous and Ψ-bounded function $f : \mathbb{R} \to \mathbb{R}^d$ if and only if there exists a positive constant K such that*

$$\begin{aligned}
&\int_{-\infty}^{t} |\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)|ds\\
&+ \int_{t}^{0} |\Psi(t)Y(t)(P_0 + P_+)Y^{-1}(s)\Psi^{-1}(s)|ds\\
&+ \int_{0}^{\infty} |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|ds \le K, \quad fort \ge 0,
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
&\int_{-\infty}^{0} |\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)|ds\\
&+ \int_{0}^{t} |\Psi(t)Y(t)(P_0 + P_-)Y^{-1}(s)\Psi^{-1}(s)|ds\\
&\int_{t}^{\infty} |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|ds \le K, \quad fort \ge 0.
\end{aligned}$$

Proof. First, we prove the "only if" part. Suppose that the system (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$ for every continuous and Ψ-bounded function $f : \mathbb{R} \to \mathbb{R}^d$ on $\mathbb{R}$.

We shall denote by B the Banach space of all Ψ-bounded and continuous functions $x : \mathbb{R} \to \mathbb{R}^d$ with the norm $\|\mathrm{x}\|_B = \sup_{t\in\mathbb{R}} \|\Psi(t)x(t)\|$.

Let D denote the set of all Ψ-bounded and continuously differentiable functions $x:\mathbb{R}\to\mathbb{R}^d$ such that $x(0)\in X_-\oplus X_+$ and $x'-Ax\in B$. Evidently, D is a vector space. We define a norm in D by setting $\|x\|_D=\|x\|_B+\|x'-Ax\|_B$.

Step 1. $(D,\|\cdot\|_D)$ is a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a fundamental sequence of elements of D. Then, $(x_n)_{n\in\mathbb{N}}$ is a fundamental sequence in B. Therefore, there exists a continuous and Ψ-bounded function $x:\mathbb{R}\to\mathbb{R}^d$ such that $\lim_{n\to\infty}\Psi(t)x_n(t)=\Psi(t)x(t)$, uniformly on $\mathbb{R}$. From the inequality

$$\|x_n(t)-x(t)\|\le|\Psi^{-1}(t)|\|\Psi(t)x_n(t)-\Psi(t)x(t)\|,$$

it follows that $\lim_{n\to\infty}x_n(t)=x(t)$, uniformly on every compact of $\mathbb{R}$. Thus, $x(0)\in X_-\oplus X_+$.

Similarly, $(x'_n-Ax_n)_{n\in\mathbb{N}}$ is a fundamental sequence in B. Therefore, there exists a continuous and Ψ-bounded function $f:\mathbb{R}\to\mathbb{R}^d$ such that

$$\lim_{n\to\infty}\Psi(t)(x'_n(t)-A(t)x_n(t))=\Psi(t)f(t),\quad\text{uniformly on }\mathbb{R}.$$

Similarly,

$$\lim_{n\to\infty}(x'_n(t)-A(t)x_n(t))=f(t),\quad\text{uniformly on every compact subset of }\mathbb{R}.$$

For any fixed t $\in\mathbb{R}$, we have

$$\begin{aligned}x(t)-x(0)&=\lim_{n\to\infty}(x_n(t)-x_n(0))\\&=\lim_{n\to\infty}\int_0^t x'_n(s)ds\\&=\lim_{n\to\infty}\int_0^t[(x'_n(s)-A(s)x_n(s))+A(s)x_n(s)]ds\\&=\int_0^t(f(s)+A(s)x(s))ds.\end{aligned}$$

Hence, the function x is continuously differentiable on $\mathbb{R}$ and

$$x'(t)=A(t)x(t)+f(t),\quad t\in\mathbb{R}.$$

Thus, $x\in D$. On the other hand, from

$$\begin{gathered}\lim_{n\to\infty}\Psi(t)x_n(t)=\Psi(t)x(t),\quad\text{uniformly on }\mathbb{R},\\ \lim_{n\to\infty}\Psi(t)(x'_n(t)-A(t)x_n(t))=\Psi(t)(x'(t)-A(t)x(t)),\quad\text{uniformly on }\mathbb{R},\end{gathered}$$

it follows that $\lim_{n\to\infty}\|x_n-x\|_D=0$. This proves that $(D,\|\cdot\|_D)$ is a Banach space.

Step 2. There exists a positive constant K_0 such that, for every $f\in B$ and for corresponding solution $x\in D$ of (1.1), we have

$$\sup_{t\in\mathbb{R}}\|\Psi(t)x(t)\|\le K_0\sup_{t\in\mathbb{R}}\|\Psi(t)f(t)\|,$$

or

$$\sup_{t\in\mathbb{R}}\max_{1\le i\le d}|\Psi_i(t)x_i(t)|\le K_0\sup_{t\in\mathbb{R}}\max_{1\le i\le d}|\Psi_i(t)f_i(t)|.\tag{1.4}$$

For this, define the mapping $T:D\to B$, $Tx=x'-Ax$. This mapping is obviously linear and bounded, with $\|T\|\le 1$.

Let $Tx=0$. Then, $x'=Ax$, $x\in D$. This shows that x is a Ψ-bounded solution on $\mathbb{R}$ of (1.2). Then, $x(0)\in X_0\cap(X_-\oplus X_+)=\{0\}$. Thus, $x=0$, such that the mapping T is "one-to-one".

Finally, the mapping T is “onto”. In fact, for any $f \in B$, let x be the Ψ-bounded solution on $\mathbb{R}$ of the system (1.1) which exists by assumption. Let z be the solution of the Cauchy problem

$$x' = A(t)x + f(t), \quad z(0) = (P_- + P_+)x(0).$$

Then, $u = x - z$ is a solution of (1.2) with $u(0) = x(0) - (P_- + P_+)x(0) = P_0x(0)$. From the Definition of X_0, it follows that u is Ψ-bounded on $\mathbb{R}$. Thus, z belongs to D and Tz = f. Consequently, the mapping T is “onto”. From a fundamental result of *S.Banach*: “If T is a bounded one-to-one linear operator of one Banach space onto another, then the inverse operator T^{-1} is also bounded”. We have $\|\mathrm{T}^{-1}f\|_D \le \|T^{-1}\|\|f\|_B$, for all $f \in B$.

For a given $f \in B$, let $x = T^{-1}f$ be the corresponding solution $x \in D$ of (1.1). We have $\|x\|_D = \|x\|_B + \|x' - Ax\|_B = \|x\|_B + \|f\|_B \le \|T^{-1}\|\|f\|_B$. It follows that $\|x\|_B \le K_0\|f\|_B$, where $K_0 = \|T^{-1}\| - 1$, which is equivalent with (1.4).

Step 3. The end of the proof. Let $T_1 < 0 < T_2$ be fixed points but arbitrarily and let $f : \mathbb{R} \to \mathbb{R}^d$ be a continuous and Ψ-bounded function which vanishes on $(-\infty, T_1] \cup [T_2, +\infty)$.

It is easy to see that the function $x : \mathbb{R} \to \mathbb{R}^d$ defined by

$$x(t) = \begin{cases} -\int_{T_1}^{0} Y(t)P_0Y^{-1}(s)f(s)ds - \int_{T_1}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds, & t < T_1 \\ \int_{T_1}^{t} Y(t)P_-Y^{-1}(s)f(s)ds + \int_{0}^{t} Y(t)P_0Y^{-1}(s)f(s)ds & \\ -\int_{t}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds, & T_1 \le t \le T_2 \\ \int_{T_1}^{T_2} Y(t)P_-Y^{-1}(s)f(s)ds + \int_{0}^{T_2} Y(t)P_0Y^{-1}(s)f(s)ds, & t > T_2 \end{cases}$$

is the solution in D of the system (1.1). Putting

$$G(t,s) = \begin{cases} Y(t)P_-Y^{-1}(s), & t > 0, s \le 0 \\ Y(t)(P_0+P_-)Y^{-1}(s), & t > 0, s > 0, s < t \\ -Y(t)P_+Y^{-1}(s), & t > 0, s > 0, s \ge t \\ Y(t)P_-Y^{-1}(s), & t \le 0, s < t \\ -Y(t)(P_0+P_+)Y^{-1}(s), & t \le 0, s \ge t, s < 0 \\ -Y(t)P_+Y^{-1}(s), & t \le 0, s \ge t, s \ge 0 \end{cases}$$

we have that $x(t) = \int_{T_1}^{T_2} G(t,s)f(s)ds$, $t \in \mathbb{R}$. Indeed,

- for $t > T_2$, we have

$$\begin{aligned} \int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{0} Y(t)P_-Y^{-1}(s)f(s)ds + \int_{0}^{T_2} Y(t)(P_0 + P_-)Y^{-1}(s)f(s)ds \\ &= \int_{T_1}^{T_2} Y(t)P_-Y^{-1}(s)f(s)ds + \int_{0}^{T_2} Y(t)P_0Y^{-1}(s)f(s)ds = x(t), \end{aligned}$$

• for $t \in (0, T_2]$, we have

$$\begin{aligned}\int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{0} Y(t)P_- Y^{-1}(s)f(s)ds + \int_0^t Y(t)(P_0 + P_-)Y^{-1}(s)f(s)ds \\ &\quad - \int_t^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds \\ &= \int_{T_1}^{t} Y(t)P_- Y^{-1}(s)f(s)ds + \int_0^t Y(t)P_0 Y^{-1}(s)f(s)ds \\ &\quad - \int_t^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds = x(t),\end{aligned}$$

• for $t \in [T_1, 0]$, we have

$$\begin{aligned}\int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{t} Y(t)P_- Y^{-1}(s)f(s)ds - \int_t^0 Y(t)(P_0 + P_+)Y^{-1}(s)f(s)ds \\ &\quad - \int_0^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds \\ &= \int_{T_1}^{t} Y(t)P_- Y^{-1}(s)f(s)ds + \int_0^t Y(t)P_0 Y^{-1}(s)f(s)ds \\ &\quad - \int_t^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds = x(t),\end{aligned}$$

• for $t < T_1$, we have

$$\begin{aligned}&\int_{T_1}^{T_2} G(t,s)f(s)ds \\ &= -\int_{T_1}^{0} Y(t)(P_0 + P_+)Y^{-1}(s)f(s)ds - \int_0^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds \\ &= -\int_{T_1}^{0} Y(t)P_0 Y^{-1}(s)f(s)ds - \int_{T_1}^{T_2} Y(t)P_+ Y^{-1}(s)f(s)ds = x(t).\end{aligned}$$

Now, putting $\Psi(t)G(t,s)\Psi^{-1}(s) = (G_{ij}(t,s))$, inequality (1.4) becomes

$$\Big| \int_{T_1}^{T_2} \sum_{k=1}^{d} G_{ik}(t,s)\Psi_k(s)f_k(s)\, ds\Big| \le K_0 \sup_{t\in\mathbb{R}} \max_{1\le i\le d} |\Psi_i(t)f_i(t)|, \quad t \in \mathbb{R},$$

$i = 1, 2, \dots, d$, for every $f = (f_1, f_2, \dots, f_d) : \mathbb{R} \to \mathbb{R}^d$, continuous and Ψ-bounded, which vanishes on $(-\infty, T_1] \cup [T_2, +\infty)$.

For a fixed i and t, we consider the function f such that

$$f_k(s) = \begin{cases} \Psi_k^{-1}(s)\operatorname{sgn} G_{ik}(t,s), & T_1 \le s \le T_2 \\ 0, & \text{elsewhere} \end{cases}$$

The function $\Psi_k(s)f_k(s)$ is pointwise limit of a sequence of continuous functions having the same supremum 1. The above inequality continues to hold for the functions of this sequence. By the dominated convergence Theorem, we get

$$\int_{T_1}^{T_2} \sum_{k=1}^{d} |G_{ik}(t,s)|ds \le K_0, t \in \mathbb{R}, \quad i = 1, 2, \dots, d.$$

Since $|\Psi(t)G(t,s)\Psi^{-1}(s)| \le \sum_{i,k=1}^{d} |G_{ik}(t,s)|$, it follows that

$$\int_{T_1}^{T_2} |\Psi(t)G(t,s)\Psi^{-1}(s)|ds \le dK_0.$$

This holds for any $T_1 < 0$ and $T_2 > 0$. Hence, $|\Psi(t)G(t,s)\Psi^{-1}(s)|$ is integrable over $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} |\Psi(t)G(t,s)\Psi^{-1}(s)|ds \le dK_0, \quad forall t \in \mathbb{R}.$$

By the Definition of $\Psi(t)G(t,s)\Psi^{-1}(s)$, this is equivalent to (1.3), with $K = dK_0$.

Now, we prove the "if" part. Suppose that the fundamental matrix Y of (1.2) satisfies the conditions (1.3) for some $K > 0$. For a continuous and Ψ-bounded function $f : \mathbb{R} \to \mathbb{R}^d$, we consider the function $u : \mathbb{R} \to \mathbb{R}^d$, defined by

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} Y(t)P_-Y^{-1}(s)f(s)ds \\ &\quad + \int_{0}^{t} Y(t)P_0Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_+Y^{-1}(s)f(s)ds. \end{aligned} \tag{1.5}$$

Step 4. The function u is well-defined on $\mathbb{R}$. For v $\ge t$, we have

$$\begin{aligned} &\int_{t}^{v} \|Y(t)P_+Y^{-1}(s)f(s)\|ds \\ &= \int_{t}^{v} \|\Psi^{-1}(t)\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)\|ds \\ &\le |\Psi^{-1}(t)| \int_{t}^{v} |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\ &\le |\Psi^{-1}(t)| \sup_{s\in\mathbb{R}} \|\Psi(s)f(s)\| \int_{t}^{v} |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|ds. \end{aligned}$$

This shows that the integral $\int_{t}^{\infty} Y(t)P_+Y^{-1}(s)f(s)ds$ is absolutely convergent.

Similarly, the integral $\int_{-\infty}^{t} Y(t)P_-Y^{-1}(s)f(s)ds$ is absolutely convergent. Thus, the function u is continuously differentiable on $\mathbb{R}$.

Step 5. The function u is a solution of the equation (1.1). For $t \in \mathbb{R}$, we have

$$\begin{aligned} u'(t) &= \int_{-\infty}^{t} A(t)Y(t)P_-Y^{-1}(s)f(s)ds + Y(t)P_-Y^{-1}(t)f(t) \\ &\quad + \int_{0}^{t} A(t)Y(t)P_0Y^{-1}(s)f(s)ds + Y(t)P_0Y^{-1}(t)f(t) \\ &\quad - \int_{t}^{\infty} A(t)Y(t)P_+Y^{-1}(s)f(s)ds + Y(t)P_+Y^{-1}(t)f(t) \\ &= A(t)u(t) + Y(t)(P_- + P_0 + P_+)Y^{-1}(t)f(t) \\ &= A(t)u(t) + f(t), \end{aligned}$$

which shows that u is a solution of (1.1) on $\mathbb{R}$.

Step 6. The solution u is Ψ-bounded on $\mathbb{R}$. For $t \geq 0$, we have

$$\begin{aligned}
\Psi(t)u(t) &= \int_{-\infty}^{t} \Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad + \int_{0}^{t} \Psi(t)Y(t)P_{0}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_{t}^{\infty} \Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&= \int_{-\infty}^{0} \Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad + \int_{0}^{t} \Psi(t)Y(t)(P_{0}+P_{-})Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_{t}^{\infty} \Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\,.
\end{aligned}$$

Then

$$\|\Psi(t)u(t)\| \leq K \sup_{t\in\mathbb{R}} \|\Psi(t)f(t)\|.$$

For t < 0, we have

$$\begin{aligned}
\Psi(t)u(t) &= \int_{-\infty}^{t} \Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad + \int_{0}^{t} \Psi(t)Y(t)P_{0}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_{t}^{\infty} \Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&= \int_{-\infty}^{t} \Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_{t}^{0} \Psi(t)Y(t)(P_{0}+P_{+})Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_{0}^{\infty} \Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\,.
\end{aligned}$$

Then

$$\|\Psi(t)u(t)\| \leq K \sup_{t\in\mathbb{R}} \|\Psi(t)f(t)\|.$$

Hence,

$$\sup_{t\in\mathbb{R}} \|\Psi(t)u(t)\| \leq K \sup_{t\in\mathbb{R}} \|\Psi(t)f(t)\|,$$

which shows that u is a Ψ-bounded solution on $\mathbb{R}$ of (1.1). The proof is now complete. □

As a particular case, we have the following result.

Theorem 1.2. *If the homogeneous equation* (1.2) *has no nontrivial Ψ-bounded solution on $\mathbb{R}$, then, the equation* (1.1) *has a unique Ψ-bounded solution on $\mathbb{R}$ for* *every continuous and* Ψ*-bounded function* $f : \mathbb{R} \to \mathbb{R}^d$ *if and only if there exists a positive constant* K *such that for* $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} |\Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)|ds + \int_{t}^{\infty} |\Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)|ds \le K \tag{1.6}$$

Proof. Indeed, in this case, $P_0 = 0$. Now, the Proof goes in the same way as before. We prove finally a theorem in which we will see that the asymptotic behavior of the solutions of (1.1) is determined completely by the asymptotic behavior of f as $t \to \pm\infty$. □

Theorem 1.3. *Suppose that:*

1. *The fundamental matrix* $Y(t)$ *of* (1.2) *satisfies:*
   - *(a) conditions* (1.3) *for some* $K > 0$*;*
   - *(b) the condition* $\lim_{t\to\pm\infty} |\Psi(t)Y(t)P_0| = 0$*;*
2. *the continuous and* Ψ*-bounded function* $f : \mathbb{R} \to \mathbb{R}^d$ *is such that*
$$\lim_{t\to\pm\infty} \|\Psi(t)f(t)\| = 0.$$

Then, every Ψ*-bounded solution* x *of* (1.1) *satisfies*

$$\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 0.$$

Proof. By Theorem 1.1, for every continuous and Ψ-bounded function $f : \mathbb{R} \to \mathbb{R}^d$, the equation (1.1) has at least one Ψ-bounded solution. Let x be a Ψ-bounded solution of (1.1). Let u be defined by (1.5). This function is a Ψ-bounded solution of (1.1).

Now, let the function $y(t) = x(t) - Y(t)P_0x(0) - u(t)$, $t \in \mathbb{R}$. Obviously, y is a Ψ-bounded solution on $\mathbb{R}$ of (1.2). Thus, $y(0) \in X_0$. On the other hand,

$$\begin{aligned}
y(0) &= x(0) - Y(0)P_0x(0) - u(0) \\
&= (I - P_0)x(0) - P_{-}\int_{-\infty}^{0} Y^{-1}(s)f(s)ds + P_{+}\int_{0}^{\infty} Y^{-1}(s)f(s)ds \\
&= P_{-}(x(0) - \int_{-\infty}^{0} Y^{-1}(s)f(s)ds) \\
&\quad + P_{+}(x(0) + \int_{0}^{\infty} Y^{-1}(s)f(s)ds) \in X_{-} \oplus X_{+}.
\end{aligned}$$

Therefore, $y(0) \in X_0 \cap (X_{-} \oplus X_{+}) = \{0\}$ and then, $y = 0$. It follows that

$$x(t) = Y(t)P_0x(0) + u(t), \quad t \in \mathbb{R}.$$

We prove that $\lim_{t\to\pm\infty} \|\Psi(t)u(t)\| = 0$. For a given $\varepsilon > 0$, there exists $t_1 > 0$ such that $\|\Psi(t)f(t)\| < \frac{\varepsilon}{3K}$, for all $t \ge t_1$. For $t > 0$, write

$$\begin{aligned}
\Psi(t)u(t) = &\int_{-\infty}^{0} \Psi(t)Y(t)P_{-}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&+ \int_{0}^{t} \Psi(t)Y(t)(P_0 + P_{-})Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&- \int_{t}^{\infty} \Psi(t)Y(t)P_{+}Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds.
\end{aligned}$$

From the hypothesis (1)(a), it follows that

$$\int_0^t |\Psi(t)Y(t)(P_0+P_-)Y^{-1}(s)\Psi^{-1}(s)|ds \le K, t \ge 0.$$

From the [8, Lemma 1], it follows that

$$\lim_{t\to+\infty} |\Psi(t)Y(t)(P_0+P_-)| = 0.$$

From this and from hypothesis (1)(b), it follows that $\lim_{t\to+\infty} |\Psi(t)Y(t)P_-| = 0$. Thus, there exists $t_2 \ge t_1$ such that, for all $t \ge t_2$,

$$|\Psi(t)Y(t)P_-| < \frac{\varepsilon}{3\big(1+\int_{-\infty}^0 \|P_-Y^{-1}(s)f(s)\|ds\big)},$$
$$|\Psi(t)Y(t)(P_0+P_-)| < \frac{\varepsilon}{3\big(1+\int_0^{t_1} \|Y^{-1}(s)f(s)\|ds\big)}.$$

Then, for $t \ge t_2$, we have

$$\begin{aligned}
\|\Psi(t)u(t)\| &\le |\Psi(t)Y(t)P_-| \int_{-\infty}^0 \|P_-Y^{-1}(s)f(s)\|ds \\
&\quad + |\Psi(t)Y(t)(P_0+P_-)| \int_0^{t_1} \|Y^{-1}(s)f(s)\|ds \\
&\quad + \int_{t_1}^t |\Psi(t)Y(t)(P_0+P_-)Y^{-1}(s)\Psi^{-1}(s)| \|\Psi(s)f(s)\|ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \|\Psi(s)f(s)\|ds \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3K} \int_{t_1}^t |\Psi(t)Y(t)(P_0+P_-)Y^{-1}(s)\Psi^{-1}(s)|ds \\
&\quad + \frac{\varepsilon}{3K} \int_t^\infty |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \|\Psi(s)f(s)\|ds \\
&\le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3K} K = \varepsilon.
\end{aligned}$$

This shows that $\lim_{t\to+\infty} \|\Psi(t)u(t)\| = 0$.

Now, from hypothesis (1)(b) it follows that $\lim_{t\to+\infty} \|\Psi(t)Y(t)P_0x(0)\| = 0$ and then, $\lim_{t\to+\infty} \|\Psi(t)x(t)\| = 0$. Similarly, $\lim_{t\to-\infty} \|\Psi(t)x(t)\| = 0$. The proof is now complete. □

Corollary 1.4. *Suppose that:*

1. *The homogeneous equation* (1.2) *has no nontrivial* Ψ*-bounded solution on* $\mathbb{R}$*;*
2. *the fundamental matrix* Y *of* (1.2) *satisfies the condition* (1.6) *for some* $K > 0$*;*
3. *the continuous and* Ψ*-bounded function* $f : \mathbb{R} \to \mathbb{R}^d$ *is such that*

$$\lim_{t\to\pm\infty} \|\Psi(t)f(t)\| = 0.$$

Then, the equation (1.1) *has a unique solution* x *on* $\mathbb{R}$ *such that*

$$\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 0.$$

The above result follows from the Theorems 1.2 and 1.3. Furthermore, this unique solution of (1.1) is

$$u(t) = \int_{-\infty}^{t} Y(t)P_{-}Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_{+}Y^{-1}(s)f(s)ds.$$

Remark 1.5. If we do not have $\lim_{t\to\pm\infty} \|\Psi(t)f(t)\| = 0$, then the solution x may be such that $\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| \neq 0$. This is shown by the next example: Consider the linear system (1.1) with

$$A(t) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \quad f(t) = \begin{pmatrix} e^{3t} \\ e^{-4t} \end{pmatrix}$$

A fundamental matrix for the homogeneous system (1.2) is

$$Y(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}$$

Consider

$$\Psi(t) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{4t} \end{pmatrix}.$$

Then, we have $\|\Psi(t)f(t)\| = 1$ for all $t \in \mathbb{R}$. The first condition of Theorem 1.3 is satisfied with $K = 2$ and

$$P_{-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The solutions of the system (1.1) are

$$x(t) = \begin{pmatrix} c_1e^{2t}+e^{3t} \\ c_2e^{-3t} - e^{-4t} \end{pmatrix}$$

with $c_1, c_2 \in \mathbb{R}$ and $t \in \mathbb{R}$. There exists a unique Ψ-bounded solution on $\mathbb{R}$,

$$x(t) = \begin{pmatrix} e^{3t} \\ -e^{-4t} \end{pmatrix},$$

but $\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 1$.

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