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QUANTIZATION EFFECTS FOR A VARIANT OF THE GINZBURG-LANDAU TYPE SYSTEM

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Abstract. The author uses Pohoaev's identity to research the quantization for a Ginzburg-Landau type functional. Under the logarithmic growth condition which is different assumption from that of in [2], the author obtain the analogous quantization results.

1. Introduction

In [2] and [5], the authors have studied the quantization effects for the system

$$-\Delta u = u(1-|u|^2) \quad \text{in } \mathbb{R}^2,$$

which is associated with the Ginzburg-Landau functional

$$F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2]dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and $\varepsilon > 0$ is a small parameter [1]. Lassoued and Lefter have investigated the asymptotic behavior of minimizers $u_\varepsilon \in H^1_g(B_1, \mathbb{R}^2)$ to the Ginzburg-Landau type energy

$$E_\varepsilon(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_\Omega |u|^2(1-|u|^2)^2 dx,$$

when $\varepsilon \to 0$, where $g : \partial\Omega \to S^1$ is a smooth map [3]. In view of [3, (1.4)], the Euler-Lagrange system of the minimizer u_ε is

$$-\Delta u = \frac{1}{\varepsilon^2}u|u|^2(1-|u|^2) - \frac{1}{2\varepsilon^2}u(1-|u|^2)^2 \quad \text{in } \Omega.$$

Let $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. Then we have

$$-\Delta u = u|u|^2(1-|u|^2) - \frac{1}{2}u(1-|u|^2)^2 \tag{1.1}$$

in Ω_ε. In a natural way, we shall study the system (1.1) in $\mathbb{R}^2$. In view of [3, Propositions 2.1 and 2.2], we have

$$|u| \le 1, \quad \text{in } \mathbb{R}^2; \tag{1.2}$$

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} < +\infty. \tag{1.3}$$

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Regarding the boundary condition $u_\varepsilon|_{\partial B_1} = g$, we assume that

$$|u(x)| \to 1, \quad as \quad |x| \to \infty. \tag{1.4}$$

Then, $\deg(u, \partial B_r)$ is well defined for r large [2]. We denote $d = |\deg(u, \partial B_r)|$. In virtue of (1.4), we see that there exists $R_0 > 0$, such that

$$|u(x)| \geq \sqrt{\frac{2}{3}}, \quad \text{for } |x| = R \geq R_0. \tag{1.5}$$

Thus, there is a smooth single-valued function $\psi(x)$, defined for $|x| \geq R_0$, such that

$$u(x) = \varrho(x) e^{i(d\theta + \psi(x))}, \tag{1.6}$$

where $\varrho = |u|$. If denote $\phi(x) = d\theta + \psi$, then ϕ is well defined and smooth locally on the set $|x| \geq R_0$.

In this paper, we investigate the quantization of the energy functional $E_\varepsilon(u, \Omega)$, by an argument as in [2] for the systems (1.1).

Theorem 1.1. *Assume that u solves* (1.1)*. If u satisfies* (1.4)*, and there exists an absolute constant $C > 0$, such that for any $r > 1$,*

$$\int_{B_r} |\nabla u|^2 dx + \int_{B_r} |u|^2 (1 - |u|^2)^2 dx \leq C(\ln r + 1). \tag{1.7}$$

Then

$$\int_{\mathbb{R}^2} |u|^2 (1 - |u|^2)^2 dx = 2\pi d^2. \tag{1.8}$$

If u is a solution of (1.1), and under the assumption

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx < +\infty, \tag{1.9}$$

instead of (1.2)-(1.4) and (1.7), then there holds the following stronger conclusion.

Theorem 1.2. *Assume u solves* (1.1) *and satisfies* (1.9)*, then either $u(x) \equiv 0$ or $u \equiv C$ with $|C| = 1$ on $\mathbb{R}^2$.*

2. Preliminaries

Proposition 2.1 (Pohozaev identity)**.** *If u solves* (1.1)*. Then for any $r > 0$, there holds*

$$\int_{B_r} |u|^2 (1 - |u|^2)^2 dx = \frac{1}{2} \int_{\partial B_r} |u|^2 (1 - |u|^2)^2 |x| ds + \int_{\partial B_r} |x| (|\partial_\tau u|^2 - |\partial_\nu u|^2) ds. \tag{2.1}$$

Proof. Multiply (1.1) with $(x \cdot \nabla u)$, and integrate over a bounded domain Ω with smooth boundary. Noting

$$\begin{aligned}
\int_\Omega (x \cdot \nabla u) \Delta u dx &= \int_{\partial\Omega} \partial_\nu u (x \cdot \nabla u) ds - \int_\Omega \nabla (x \cdot \nabla u) \nabla u \, dx \\
&= \int_{\partial\Omega} (x \cdot \nu) ||\partial_\nu u|^2 ds - \frac{1}{2} \int_\Omega x \cdot \nabla (|\nabla u|^2) dx - \int_\Omega |\nabla u|^2 dx \quad (2.2) \\
&= \int_{\partial\Omega} (x \cdot \nu) |\partial_\nu u|^2 ds - \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) |\nabla u|^2 ds,
\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega (x\cdot\nabla u)u|u|^2(1-|u|^2)dx - \frac{1}{2}\int_\Omega (x\cdot\nabla u)u(1-|u|^2)^2dx\\&= \frac{1}{2}\int_\Omega |u|^2(1-|u|^2)^2dx - \frac{1}{4}\int_\Omega \operatorname{div}[x|u|^2(1-|u|^2)^2]dx\\&= \frac{1}{2}\int_\Omega |u|^2(1-|u|^2)^2dxdy - \frac{1}{4}\int_{\partial\Omega} |u|^2(1-|u|^2)^2(x\cdot\nu)ds,\end{aligned}$$

we obtain

$$\begin{aligned}&\int_\Omega |u|^2(1-|u|^2)^2dx\\&= \frac{1}{2}\int_{\partial\Omega} |u|^2(1-|u|^2)^2(x\cdot\nu)ds + \int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2ds - 2\int_{\partial\Omega}(x\cdot\nu)|\partial_\nu u|^2ds.\end{aligned} \tag{2.3}$$

Thus, (2.1) can be seen by taking $\Omega = B_r$ in the identity above. The proof is complete. □

3. Proof of Theorem 1.1

Proposition 3.1. *Assume u solves* (1.1). *If u satisfies* (1.4) *and* (1.7), *then*

$$\int_{\mathbb{R}^2}(1-|u|^2)^2dx < +\infty. \tag{3.1}$$

Proof. Denote $f(t) = \int_{\partial B_t}[|\nabla u|^2 + |u|^2(1-|u|^2)^2]ds$. Applying [4, Proposition 2.2], from (1.7) we are led to

$$\frac{1}{2}\inf\{tf(t); t\in[\sqrt{r}, r]\ln r \le \int_{\sqrt{r}}^r \frac{tf(t)}{t}dt \le E(u, B_r) \le C\ln r,$$

which implies $\inf\{tf(t); t\in[\sqrt{r}, r] \le C$. Thus, there exists $t_m\to\infty$ such that

$$t_m f(t_m) \le O(1). \tag{3.2}$$

Taking $r = t_j \to\infty$ in (2.1), and substituting (3.2) into it, we obtain

$$\int_{\mathbb{R}^2}|u|^2(1-|u|^2)^2dx < +\infty. \tag{3.3}$$

Noting (1.5) we can see the conclusion of the proposition. □

Substituting (1.6) into (1.1) yields

$$-\Delta\varrho + \varrho|\nabla\phi|^2 = \varrho^3(1-\varrho^2) - \frac{1}{2}\varrho(1-\varrho^2)^2, \quad \text{in } \mathbb{R}^2\setminus B_{R_0}, \tag{3.4}$$

$$-\operatorname{div}(\varrho^2\nabla\phi) = 0 \quad \text{in } \mathbb{R}^2\setminus B_{R_0}. \tag{3.5}$$

By an analogous argument of Steps 1 and 2 in the proof of [2, Proposition 1], we also derive from (3.5) that

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\psi|^2dx < +\infty. \tag{3.6}$$

In addition, we also deduce the following proposition.

Proposition 3.2. *Under the assumption of Proposition 3.1, we have*

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\varrho|^2dx < +\infty. \tag{3.7}$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^2, [0,1])$ satisfy $\eta(x) = 1$ for $|x| \le 1/2$, and $\eta(x) = 0$ for $|x| \ge 1$. Set $\eta_t(x) = \eta(\frac{x}{t})$ for $t < r$. Multiplying (3.4) by $(1-\varrho)\eta_t^2$ and integrating over $B_r \setminus B_{R_0}$, we obtain

$$\begin{aligned} &\int_{B_r\setminus B_{R_0}} |\nabla \varrho|^2 \eta_t^2 dx + \int_{B_r\setminus B_{R_0}} [\varrho^3(1-\varrho^2) - \frac{1}{2}\varrho(1-\varrho^2)^2](1-\varrho)\eta_t^2 dx \\ &= -\int_{\partial B_{R_0}} (1-\varrho)\eta_t^2 \partial_\nu \varrho ds - \frac{1}{2}\int_{B_r\setminus B_{R_0}} \nabla(1-\varrho)^2 \nabla \eta_t^2 dx \\ &\quad + \int_{B_r\setminus B_{R_0}} |\nabla\phi|^2 \varrho(1-\varrho)\eta_t^2 dx. \end{aligned} \tag{3.8}$$

Clearly, (1.3) leads to

$$\int_{\partial B_{R_0}} |\partial_\nu \varrho| ds \le C(R_0) = C. \tag{3.9}$$

In addition, in view of Proposition 3.1, it follows that

$$\begin{aligned} &\Big|\int_{B_r\setminus B_{R_0}} \nabla(1-\varrho)^2 \nabla\eta_t^2 dx\Big| \\ &\le \Big|\int_{\partial B_{R_0}} (1-\varrho)^2 \partial_\nu \eta_t^2 ds\Big| + \Big|\int_{B_r\setminus B_{R_0}} (1-\varrho)^2 \Delta\eta_t^2 dx\Big| \\ &\le C(R_0) + Ct^{-2}\Big|\int_{\mathbb{R}^2} (1-\varrho)^2 dx\Big| < +\infty, \quad \forall t > R_0. \end{aligned} \tag{3.10}$$

Using Hölder's inequality, from (3.1) and (3.6), we deduce that

$$\begin{aligned} \int_{B_r\setminus B_{R_0}} |\nabla\phi|^2 \varrho(1-\varrho)\eta_t^2 dx &\le \Big(\int_{B_r\setminus B_{R_0}} \frac{d^4}{|x|^4} dx\Big)^{1/2} \Big(\int_{\mathbb{R}^2} (1-\varrho)^2 dx\Big)^{1/2} \\ &\quad + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\psi|^2 dx < +\infty. \end{aligned} \tag{3.11}$$

At last, (1.5) implies

$$\int_{B_r\setminus B_{R_0}} [\varrho^3(1-\varrho^2) - \frac{1}{2}\varrho(1-\varrho^2)^2](1-\varrho)\eta_t^2 dx \ge 0. \tag{3.12}$$

Substituting (3.9)-(3.12) into (3.8), and letting $t \to \infty$, we can deduce (3.7). The proof is complete. $\square$

Proof of Theorem 1.1. First, we have

$$\begin{aligned} |\partial_\tau u|^2 &= |\partial_\tau \varrho|^2 + \varrho^2(\frac{d}{|x|} + \partial_\tau\psi)^2 \\ &= \frac{d^2}{|x|^2} + |\partial_\tau\varrho|^2 + (\varrho^2 - 1)\frac{d^2}{|x|^2} + 2\varrho^2\frac{d}{|x|}\partial_\tau\psi + \varrho^2|\partial_\tau\psi|^2, \end{aligned} \tag{3.13}$$

Obviously, (3.1), (3.3), (3.6) and (3.7) imply

$$\begin{aligned} &\int_{B_r\setminus B_{R_0}} [|u|^2(1-|u|^2)^2 + |\partial_\tau\varrho|^2 + (1-\varrho^2)\frac{d^2}{|x|^2} \\ &+ 2\varrho^2\frac{d}{|x|}|\partial_\tau\psi| + \varrho^2|\partial_\tau\psi|^2 + |\partial_\nu u|^2] dx \le C, \end{aligned}$$

where C is independent of r. Similar to the derivation of (3.2), by using [4, Proposition 2.2], it also follows that

$$\inf\{F(r_j); r_j \in [\sqrt{r}, r]\} \leq C(\ln r)^{-1},$$

where

$$F(r_j) := r_j \int_{\partial(B_{r_j}\setminus B_{R_0})} [|u|^2(1-|u|^2)^2 + |\partial_\tau \varrho|^2 + (1-\varrho^2)\frac{d^2}{|x|^2} + 2\varrho^2 \frac{d}{|x|}|\partial_\tau \psi| + \varrho^2|\partial_\tau\psi|^2 + |\partial_\nu u|^2]ds.$$

Thus, we see that there exists $r_j \to \infty$, such that $F(r_j) \leq o(1)$. Combining this with (3.13), we can see (1.8) since

$$\int_{\partial B_r} |x|\frac{d^2}{|x|^2}ds = 2\pi d^2.$$

The proof is complete. □

4. Proof of Theorem 1.2

First, we shall prove (1.2). Similar to the derivation of (3.8) in [2], we also have

$$\Delta h \geq |u|(1+|u|)h(3|u|^2-1)/2, \quad h = (|u|-1)^+.$$

Write $G = \{x \in \mathbb{R}^2; |u(x)| > \sqrt{1/3}\}$. In the argument of Step 1 in the proof of [2, Theorem 2], we replace $\mathbb{R}^2$ by G to be the integral domain. Applying (1.9) we also deduce that

$$|u|h(3|u|^2-1) \equiv 0, \quad \text{on } G.$$

This implies (1.2). Next, (1.1) leads to

$$\Delta|u|^2 = 2|\nabla u|^2 + |u|^2(|u|^2-1)(3|u|^2-1), \quad \text{on } B_r. \tag{4.1}$$

Multiplying this equality by η_t and integrating over B_r, we have

$$\begin{aligned} &\int_{B_r} |u|^2(1-|u|^2)(3|u|^2-1)\eta_t dx \\ &= 2\int_{B_r} |\nabla u|^2\eta_t dx - \int_{\partial B_r} \eta_t \partial_\nu |u|^2 ds + 2\int_{B_r} u\nabla u \nabla\eta_t dx. \end{aligned} \tag{4.2}$$

From (4.2) with $t < r$ (which implies $\eta_t = 0$ on ∂B_r) and (1.9), it is not difficult to deduce that

$$\int_{B_r} |u|^2(1-|u|^2)\eta_t dx \leq C.$$

Letting $t \to \infty$, we can see that

$$\int_{\mathbb{R}^2} |u|^2(1-|u|^2)dx < \infty. \tag{4.3}$$

Similar to the calculation in the proof of (2.2), we have that, for $t < r$,

$$\int_{B_r} \Delta u(x\cdot\nabla u)\eta_t dx = -\int_{B_r} (x\cdot\nabla u)\nabla u\nabla\eta_t dx. \tag{4.4}$$

Take $\sqrt{r} < t < r$ and let $r \to \infty$, then by [4, Proposition 2.3], (1.9) leads to

$$\Big|\int_{B_r} (x\cdot\nabla u)\nabla u\nabla\eta_t dx\Big| \leq C\int_{t/2\leq|x|\leq t} |\nabla u|^2 \leq o(1). \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain that as $r \to \infty$,

$$\Big| \int_{B_r} \Delta u (x \cdot \nabla u) \eta_t dx \Big| \le o(1). \tag{4.6}$$

By (1.1), we obtain that for $t < r$,

$$\begin{aligned} \int_{B_r} \Delta u (x \cdot \nabla u) \eta_t dx &= \frac{1}{4} \int_{B_r} \operatorname{div}[x|u|^2(|u|^2-1)^2] \eta_t dx - \frac{1}{2} \int_{B_r} |u|^2 (1-|u|^2)^2 \eta_t dx \\ &= -\frac{1}{4} \int_{B_r} |u|^2(|u|^2-1)^2 x \cdot \nabla \eta_t dx - \frac{1}{2} \int_{B_r} |u|^2(|u|^2-1)^2 \eta_t dx. \end{aligned} \tag{4.7}$$

Using [4, Proposition 2.3], from (4.3) we have

$$\Big| \int_{B_r} |u|^2(|u|^2-1)^2 x \cdot \nabla \eta_t dx \Big| \le o(1),$$

when $r \to \infty$. Substituting this and (4.6) into (4.7), leads to

$$\int_{\mathbb{R}^2} |u|^2 (1-|u|^2)^2 dx = 0.$$

This implies either $|u| \equiv 0$ or $|u| \equiv 1$ on $\mathbb{R}^2$.

Assume $|u| \equiv 1$ on $\mathbb{R}^2$. Integrating by parts over B_r, we can deduce that, for $t \in (\sqrt{r}, r)$,

$$\int_{B_r} \eta_t \Delta |u|^2 dx = -\int_{B_r} \nabla \eta_t \nabla |u|^2 dx.$$

Then there holds

$$\Big| \int_{B_r} \eta_t \Delta |u|^2 dx \Big| = \Big| \int_{B_r} \nabla \eta_t \nabla |u|^2 dx \Big| \le \frac{C}{t} \int_{t/2 \le |x| \le t} |\nabla |u|^2| dx.$$

Letting $t \to \infty$, from (1.9) we see that

$$\Big| \int_{B_r} \Delta |u|^2 dx \Big| \le o(1). \tag{4.8}$$

By (4.1), it follows

$$\int_{B_r} \Delta |u|^2 dx = 2 \int_{B_r} [|\nabla u|^2 + |u|^2(|u|^2-1)(3|u|^2-1)] dx.$$

Substituting (4.8) and $|u| \equiv 1$ into it, we obtain $\int_{\mathbb{R}^2} |\nabla u|^2 dx = 0$. Then, $u \equiv C$ with $|C| = 1$ on $\mathbb{R}^2$. The proof is complete.

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