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PROPERTIES OF SOLUTIONS TO NONLINEAR DYNAMIC INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. The main objective of the present paper is to study some basic qualitative properties of solutions of a certain dynamic integral equation on time scales. The tools employed in the analysis are based on the applications of the Banach fixed point theorem and a certain inequality with explicit estimate on time scale.

1. Introduction

Stefan Hilger in his doctoral dissertation, that resulted in his seminal paper [6] in 1990, initiated the study of time scales in order to unify continuous and discrete analysis. During the previous three decades many authors have studied various aspects of dynamic equations on time scales by using different techniques. An excellent account on time scales and dynamic equations on time scales can be found in the two recent books [2, 3] by Bohner and Peterson.

In the study of dynamic equations on time scales, most often the analysis turns to that of a related integral equation on time scales. It seems integral equations on time scales have an enormous potential for rich and diverse applications and thus they are most worthy of attention. In this paper we consider a general nonlinear dynamic integral equation

$$x(t) = f\Big(t, x(t), \int_{t_0}^{t} g(t,\tau,x(\tau))\Delta\tau\Big), \tag{1.1}$$

where x is the unknown function to be found, $g : I_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$, $f : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, t is from a time scale $\mathbb{T}$, which is a known nonempty closed subset of $\mathbb{R}$, the set of real numbers, $\tau \le t$ and $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty]$ be the given subset of $\mathbb{R}$, $\mathbb{R}^n$ the real n-dimensional Euclidean space with appropriate norm defined by $|\cdot|$, x_0 is a given constant in $\mathbb{R}^n$ and the integral sign represents a very general type of operation, known as the delta integral. For more details, see [2]. The aim of this paper is to study some fundamental qualitative properties of solutions of (1.1) under some suitable conditions on the functions involved therein. The well known Banach fixed point theorem (see [5, p.37]) coupled with Bielecki type norm (see [1]) and the time scale analogue of a certain integral inequality with explicit estimate

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are used to establish the results. Here, our approach is elementary and provide some useful results for future reference.

2. Preliminaries

In this section we give some preliminaries and basic lemmas used in our subsequent discussion. We assume that any time scale has the topology that it inherits from the standard topology on $\mathbb{R}$. Since a time scale may or may not be connected, we need the concept of jump operators. We denote two jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right scattered, while if $\rho(t) < t$, we say that t is left scattered. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point in $\mathbb{T}$. The set of all rd-continuous functions is denoted by C_{rd}. If $\mathbb{T}$ has a left scattered maximum m, then

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - m & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

We define the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at the point $t \in I_{\mathbb{T}}$, denoted by $f^{\Delta}(t)$ as for given $\epsilon > 0$ there exists a neighborhood N of t with

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in N$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta} = f(t)$ holds for all $t \in I_{\mathbb{T}}$. In this case we define the integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$$

where $s, t \in \mathbb{T}$. For $p \in R$, we define (see [7]) the exponential function $e_p(., t_0)$ on time scale $\mathbb{T}$ (for each fixed $t_0 \in \mathbb{T}$) as the unique solution to the scalar initial value problem

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1.$$

For more on the basic theory and recent developments of time scales, see [2, 3].

Following [7], we first construct the appropriate metric space with $I_{\mathbb{T}} := [t_0, \infty)_{\mathbb{T}}$ for our analysis. Let $\beta > 0$ be a constant and consider the space of continuous functions $C([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ such that $\sup_{t \in [t_0,\infty)_T} \frac{x(t)}{e_\beta(t,t_0)} < \infty$ and denote this special space by $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$. We couple the linear space $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ with a suitable metric; namely,

$$\mathrm{d}_\beta^\infty(x, y) = \sup_{t \in [t_0,\infty)_{\mathbb{T}}} \frac{|x(t) - y(t)|}{e_\beta(t, t_0)},$$

with norm defined by

$$|x|_\beta^\infty = \sup_{t \in [t_0,\infty)_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, t_0)}.$$

For some important properties of d_β^∞ and $|\cdot|_\beta^\infty$ see [7].

We use the following fundamental result proved in Bohner and Peterson [2].

Lemma 2.1. *Let $t_0 \in \mathbb{T}^k$ and assume that $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > t_0$. Also assume that $k(t,.)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\epsilon > 0$ there exists a neighborhood N of t independent of $t \in [t_0, \sigma(t)]$ such that*

$$|k(\sigma(t),\tau) - k(s,\tau) - k^{\Delta}(t,\tau)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for $s \in N$, where k^{Δ} denotes the Δ derivative of k with respect to the first variable. Then

$$g(t) = \int_{t_0}^{t} k(t,\tau)\Delta\tau,$$

for $t \in I_{\mathbb{T}}$, implies

$$g^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau + k(\sigma(t),t),$$

for $t \in I_{\mathbb{T}}$.

The following Lemma proved in [4, Corollary 3.11 p. 8] is useful in our main results.

Lemma 2.2. *Assume that $u \in c_{rd}$ and $u \ge 0$ and $c \ge 0$ is a real constant. Let $k(t,s)$ be defined as in Lemma 1 such that $k(\sigma(t),t) \ge 0$ and $k^{\Delta}(t,s) \ge 0$ for $s,t \in \mathbb{T}$ with $s \le t$. Then*

$$u(t) \le c + \int_{t_0}^{t} k(t,\tau)u(\tau)\Delta\tau, \tag{2.1}$$

implies

$$u(t) \le c e_A(t,t_0), \tag{2.2}$$

for all $t \in \mathbb{T}$, where

$$A(t) = k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau. \tag{2.3}$$

3. Existence and Uniqueness

In this section we present our result on the existence and uniqueness of solutions of (1.1).

Theorem 3.1. *Let $L > 0$, $\beta > 0$, $M \ge 0$, $\gamma > 1$ be constants with $\beta = L\gamma$. Suppose that the functions f, g in (1.1) are rd-continuous and satisfy the conditions*

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le M[|u - \bar{u}| + |v - \bar{v}|], \tag{3.1}$$

$$|g(t,s,u) - g(t,s,v)| \le L|u - v|, \tag{3.2}$$

$$d_1 = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,\alpha)}\Big|f\Big(t,0,\int_{t_0}^{t} g(t,\tau,0)\Delta\tau\Big)\Big| < \infty. \tag{3.3}$$

If $M(1+\frac{1}{\gamma}) < 1$, then (1.1) has a unique solution $x \in C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$.

Proof. Let $x \in C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$ and define the operator F by

$$\begin{aligned}(Fx)(t) &= f\Big(t,x(t),\int_{t_0}^{t} g(t,\tau,x(\tau))\Delta\tau\Big) - f\Big(t,0,\int_{t_0}^{t} g(t,\tau,0)\Delta\tau\Big) \\ &\quad + f\Big(t,0,\int_{t_0}^{t} g(t,\tau,0)\Delta\tau\Big).\end{aligned} \tag{3.4}$$

Now, we show that F maps $C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$ into itself. Let $x \in C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$. From (3.4) and using the hypotheses, we have

$$
\begin{aligned}
|Fx|_\beta^\infty &= \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{|F(x)(t)|}{e_\beta(t,\alpha)}\\
&\le \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\Big|f\Big(t,x(t),\int_{t_0}^t g(t,\tau,x(t))\Delta\tau)\\
&\quad -f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)+f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big|\\
&\le \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\Big|f\Big(t,x(t),\int_{t_0}^t g(t,\tau,x(\tau))\Delta\tau\Big)-f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big|\\
&\quad + \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\Big|f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big|\\
&\le d_1 + \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} M[|x(t)|+\int_{t_0}^t L|x(\tau)|\Delta\tau]\\
&= d_1 + M\Big[\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t,t_0)}\Big] + L\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\int_{t_0}^t e_\beta(\tau,t_0)\frac{|x(\tau)|}{e_\beta(\tau,t_0)}\Delta\tau\Big]\\
&\le d_1 + M\Big[|x|_\beta^\infty + L|x|_\beta^\infty \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\int_{t_0}^t e_\beta(\tau,t_0)\Delta\tau\Big]\\
&= d_1 + M|x|_\beta^\infty\Big[1+L\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}(\frac{e_\beta(t,t_0)-1}{\beta})\Big]\\
&= d_1 + M|x|_\beta^\infty[1+\frac{L}{\beta}]\\
&= d_1 + |x|_\beta^\infty M\big(1+\frac{1}{\gamma}\big) < \infty.
\end{aligned}
$$

This proves that the operator F maps $C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$ into itself.

Next we verify that F is a contraction mapping. Let $u, v \in C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$. From (3.4) and by using the hypotheses, we have

$$
\begin{aligned}
&d_\beta^\infty(Fu,Fv)\\
&= \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{|(Fu)(t)-(Fv)(t)|}{e_\beta(t,t_0)}\\
&= \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\Big|f\Big(t,u(t),\int_{t_0}^t g(t,\tau,u(\tau))\Delta\tau\Big)-f\Big(t,v(t),\int_{t_0}^t g(t,\tau,v(\tau))\Delta\tau\Big)\Big|\\
&\le \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} M\Big[|u(t)-v(t)|+\int_{t_0}^t L|u(\tau)-v(\tau)|\Delta\tau\Big]\\
&= M\Big[\sup_{t\in[t_0,\infty)_T} \frac{|u(t)-v(t)|}{e_\beta(t,t_0)} + \sup_{t\in[t_0,\infty)_T} \frac{1}{e_\beta(t,t_0)} L\int_{t_0}^t e_\beta(\tau,t_0)\frac{|u(\tau)-v(\tau)|}{e_\beta(\tau,t_0)}\Delta\tau\Big]\\
&\le M\Big[d_\beta^\infty(u,v) + Ld_\beta^\infty(u,v)\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\int_{t_0}^t e_\beta(\tau,t_0)\Delta\tau\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= M d_\beta^\infty(u,v)\Big[1 + L \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)}\big(\frac{e_\beta(t,t_0)-1}{\beta}\big)\Big] \\
&= M d_\beta^\infty(u,v)[1+\frac{L}{\beta}] \\
&= M(1+\frac{1}{\gamma}) d_\beta^\infty(u,v).
\end{aligned}
$$

Since $M(1+\frac{1}{\gamma}) < 1$, it follows from the Banach fixed point theorem [5, p. 37] that F has a unique fixed point in $C_\beta([t_0,\infty)_{\mathbb{T}};\mathbb{R}^n)$. The fixed point of F is however a solution of (1.1). The proof is complete. □

We note that the norm $|\cdot|_\beta^\infty$ used in the proof of Theorem 1 is a variant of Bielecki's norm [1], first used in 1956 while proving existence and uniqueness of solutions of ordinary differential equations (see also [10]).

4. Estimates on the solution

The following theorem provides an estimate on the solution of (1.1).

Theorem 4.1. *Suppose that the functions f, g in (1.1) are rd-continuous and satisfy the conditions*

$$
|f(t,u,v) - f(t,\bar{u},\bar{v})| \le N[|u-\bar{u}| + |v-\bar{v}|], \tag{4.1}
$$

$$
|g(t,\tau,u) - g(t,\tau,v)| \le k(t,\tau)|u-v| \tag{4.2}
$$

where $0 \le N < 1$ is a constant, $k(t,\tau)$ be defined as in Lemma 1 such that $k(\sigma(t),t) \ge 0$ and $k^\Delta(t,\tau) \ge 0$ for $\tau, t \in \mathbb{T}$ with $\tau \le t$. Let

$$
c_1 = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \Big| f\Big(t, 0, \int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big| < \infty\,. \tag{4.3}
$$

If $x(t)$, $t \in I_{\mathbb{T}}$ is any solution of (1.1), then

$$
|x(t)| \le \big(\frac{c_1}{1-N}\big) e_B(t,t_0), \tag{4.4}
$$

for $t \in I_{\mathbb{T}}$, where

$$
B(t) = \frac{N}{1-N}\Big[k(\sigma(t),t) + \int_{t_0}^t k^\Delta(t,\tau)\Delta\tau\Big]. \tag{4.5}
$$

Proof. By using the fact that $x(t)$ is a solution of (1.1) and hypotheses, we have

$$
\begin{aligned}
|x(t)| &\le \Big| f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big| \\
&\quad + \Big| f\Big(t,x(t),\int_{t_0}^t g(t,\tau,x(\tau))\Delta\tau\Big) - f\Big(t,0,\int_{t_0}^t g(t,\tau,0)\Delta\tau\Big)\Big| \\
&\le c_1 + N\Big[|x(t)| + \int_{t_0}^t k(t,\tau)|x(\tau)|\Delta\tau\Big].
\end{aligned}
$$

From the above inequality and using the assumption $0 \le N < 1$, we observe that

$$
|x(t)| \le \frac{c_1}{1-N} + \int_{t_0}^t \frac{N}{1-N} k(t,\tau)|x(\tau)|\Delta\tau.
$$

Now applying Lemma 2 to the above inequality, we obtain (4.4). □

We note that the estimate obtained in (4.4) yields bound on the solution of (1.1). If the bound on the right hand side in (4.4) is bounded then the solution of (1.1) is bounded.

5. Continuous dependence

In this section we study the continuous dependence of solutions of (1.1) on the functions involved therein and also the continuous dependence of solutions of equations of the form (1.1).

Consider (1.1) and the corresponding equation

$$y(t) = \bar{f}(t, y(t), \int_{t_0}^{t} \bar{g}(t, \tau, y(\tau))\Delta\tau), \tag{5.1}$$

for $t \in I_{\mathbb{T}}$, $\tau \le t$, where $\bar{g} : I_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$, $\bar{f} : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and y_0 is a given constant in $\mathbb{R}^n$.

The following theorem deals with the continuous dependence of solutions of (1.1) on the functions involved therein.

Theorem 5.1. *Suppose that the functions f, g in* (1.1) *are rd-continuous and satisfy the conditions* (4.1) *and* (4.2). *Furthermore, suppose that*

$$\Big| f\Big(t, y(t), \int_{t_0}^{t} g(t, \tau, y(\tau))\Delta\tau\Big) - \bar{f}\Big(t, y(t), \int_{t_0}^{t} \bar{g}(t, \tau, y(\tau))\Delta\tau\Big)\Big| \le \epsilon_1 \tag{5.2}$$

where f, g and $\bar{f}$, $\bar{g}$ are the functions involved in (1.1) *and* (5.1), *$\epsilon_1 > 0$ is an arbitrary small constant and $y(t)$ is a given solution of* (5.1). *Then the solution $x(t)$, $t \in I_{\mathbb{T}}$ of* (1.1) *depends continuously on the functions involved on the right hand side of* (1.1).

Proof. Let $u(t) = |x(t) - y(t)|$, $t \in I_{\mathbb{T}}$. Using the facts that $x(t)$ and $y(t)$ are the solutions of (1.1) and (5.1) respectively and the hypotheses, we have

$$\begin{aligned}
u(t) &\le \Big| f\Big(t, x(t), \int_{t_0}^{t} g(t, \tau, x(\tau))\Delta\tau\Big) - f\Big(t, y(t), \int_{t_0}^{t} g(t, \tau, y(\tau))\Delta\tau\Big))\Big| \\
&\quad + \Big| f\Big(t, y(t), \int_{t_0}^{t} g(t, \tau, y(\tau))\Delta\tau\Big) - \bar{f}\Big(t, y(t), \int_{t_0}^{t} \bar{g}(t, \tau, y(\tau))\Delta\tau\Big)\Big| \\
&\le \epsilon_1 + N\Big[u(t) + \int_{t_0}^{t} k(t, \tau)u(\tau)\Delta\tau\Big].
\end{aligned} \tag{5.3}$$

From the above inequality and using the assumption that $0 \le N < 1$, we observe that

$$u(t) \le \frac{\epsilon_1}{1 - N} + \frac{N}{1 - N} \int_{t_0}^{t} k(t, \tau)u(\tau)\Delta\tau. \tag{5.4}$$

Now an application of Lemma 2 to (5.4) yields

$$|x(t) - y(t)| \le (\frac{\epsilon_1}{1 - N})e_B(t, t_0), \tag{5.5}$$

where $B(t)$ is given by (4.5). From (5.5) it follows that the solution of (1.1) depends continuously on the functions involved on right hand side of (1.1). □

Now we consider the dynamic integral equations on time scales:

$$z(t) = h(t, z(t), \int_{t_0}^{t} g(t, \tau, z(\tau))\Delta\tau, \mu), \tag{5.6}$$

$$z(t) = h(t, z(t), \int_{t_0}^{t} g(t, \tau, z(\tau))\Delta\tau, \mu_0), \tag{5.7}$$

for $t \in I_{\mathbb{T}}$, $\tau \le t$ where $g : I_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$, $h : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of (5.6), (5.7) on parameters.

Theorem 5.2. *Suppose that the function h in* (5.6), (5.7) *is rd-continuous and satisfy the conditions*

$$|h(t, u, v, \mu) - h(t, \bar{u}, \bar{v}, \mu)| \le \bar{N}[|u - \bar{u}| + |v - \bar{v}|], \tag{5.8}$$

$$|h(t, u, v, \mu) - h(t, u, v, \mu_0)| \le q(t)|\mu - \mu_0|, \tag{5.9}$$

where $0 \le \bar{N} < 1$ is a constant, $q : I_{\mathbb{T}} \to R_+$, $q \in C_{rd}$ such that $q(t) \le Q < \infty$, Q is a constant and the function g in (5.6), (5.7) *satisfies the condition* (4.2). *Let $z_1(t)$ and $z_2(t)$ be the solutions of* (5.6) *and* (5.7) *respectively. Then*

$$|z_1(t) - z_2(t)| \le \frac{Q|\mu - \mu_0|}{1 - \bar{N}} e_{\bar{B}}(t, s),$$

for $t \in I_{\mathbb{T}}$, where

$$\bar{B}(t) = \frac{\bar{N}}{1 - \bar{N}} \Big[k(\sigma(t), t) + \int_{t_0}^{t} k^{\Delta}(t, \tau)\Delta\tau\Big]. \tag{5.10}$$

Proof. Let $z(t) = |z_1(t) - z_2(t)|$, $t \in I_{\mathbb{T}}$. Using the fact that $z_1(t)$ and $z_2(t)$ are the solutions of (5.6) and (5.7) and hypotheses, we have

$$\begin{aligned} z(t) &\le \Big|h\Big(t, z_1(t), \int_{t_0}^{t} g(t, \tau, z_1(\tau))\Delta\tau, \mu\Big) - h\Big(t, z_2(t), \int_{t_0}^{t} g(t, \tau, z_2(\tau))\Delta\tau, \mu\Big)\Big| \\ &\quad + \Big|h\Big(t, z_2(t), \int_{t_0}^{t} g(t, \tau, z_2(\tau))\Delta\tau, \mu\Big) - h\Big(t, z_2(t), \int_{t_0}^{t} g(t, \tau, z_2(\tau))\Delta\tau, \mu_0\Big)\Big| \\ &\le \bar{N}\Big[z(t) + \int_{t_0}^{t} k(t, \tau)z(\tau)\Delta\tau\Big] + Q|\mu - \mu_0|. \end{aligned}$$

From this inequality and using the assumption $0 \le \bar{N} < 1$, we observe that

$$z(t) \le \frac{Q|\mu - \mu_0|}{1 - \bar{N}} + \frac{\bar{N}}{1 - \bar{N}} \int_{t_0}^{t} k(t, \tau)z(\tau)\Delta\tau.$$

Now an application of Lemma 2 to the above inequality yields (5.2), which shows the dependency of solutions of (5.6) and (5.7) on parameters. □

We note that, if $f(t, x, y)$ in (1.1) is of the form $f(t, x, y) = h(t, x) + y$, then (1.1) reduces to

$$x(t) = h(t, x(t)) + \int_{t_0}^{t} g(t, \tau, x(\tau))\Delta\tau,$$

which in turn contains as a special case the equation studied by Kulik and Tisdell [7]. Furthermore, we note that the idea used in this paper can be very easily extended to study the general dynamic integrodifferential equation of the form

$$x^{\Delta}(t) = f\Big(t, x(t), \int_{t_0}^{t} g(t, \tau, x(\tau))\Delta\tau\Big), \quad x(t_0) = x_0 \tag{5.11}$$

under some suitable conditions on the functions involved in (5.11). We leave the details to the reader to fill in where needed.

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