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CONSTRUCTION OF ENTIRE SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS

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Abstract. Entire solutions of parabolic equations (those which are defined for all time) are typically rather rare. For example, the heat equation has exactly one entire solution – the trivial solution. While solutions to the heat equation exist for all forward time, they cannot be extended backwards in time. Nonlinearities exasperate the situation somewhat, in that solutions may form singularities in both backward and forward time. However, semilinear parabolic equations can also support nontrivial entire solutions. This article shows how nontrivial entire solutions can be constructed for a semilinear equation that has at least two distinct equilibrium solutions. The resulting entire solution is a heteroclinic orbit which connects the two given equilibria.

1. Introduction

Consider the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sum_{i=0}^{N} a_i(x) u^i(t,x), \tag{1.1}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and the a_i are bounded and smooth. In this article, we consider *entire* solutions, those classical solutions u which satisfy (1.1) for all time $t \in \mathbb{R}$.

This kind of equation provides a simple model for a number of physical phenomena. First, choosing the right side to be $\Delta u - u^2 + a_1 u$ results in an equation which can represent a model of the population of a single species with diffusion and a spatially-varying carrying capacity, $a_1(x)$. As a second application, this equation is a very simple model of combustion. If a_1 is a positive constant, then the equation supports travelling waves. Such travelling waves can model the propagation of a flame through a fuel source.

Equations of the form (1.1) have been of interest to researchers for quite some time. Existence and uniqueness of solutions on short time intervals (on strips $(-t_0, t_0) \times \mathbb{R}^n$) can been shown using semigroup methods and are entirely standard [17]. However, there are obstructions to the existence of entire solutions. Aside from the typical loss of regularity due to solving the backwards heat equation, there is

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also a blow-up phenomenon which can spoil existence in the forward-time solution to (1.1). Blow-up phenonmena in the forward time Cauchy problem (where one does not consider $t < 0$) have been studied by a number of authors [6, 5, 15, 11, 2, 19, 20]. More recently, Zhang *et al.* ([18, 14, 16]) studied the existence of global for the forward Cauchy problem for

$$\frac{\partial u}{\partial t} = \Delta u + u^p - V(x)u$$

for positive u, V. Du and Ma studied a related problem in [4] under more restricted conditions on the coefficients but they obtained stronger existence results. In fact, they found that all of the solutions which were defined for all $t > 0$ tended to equilibrium solutions.

Entire solutions to (1.1) are rather rare. Most works which describe blow-up make the assumption that the solution is positive. Unfortunately, blow-up is much more difficult to characterize in the general situation, and understanding exactly what kind of initial conditions are responsible for blow-up in the Cauchy problem for (1.1) is an important part of the question.

As an aside, the boundary value problem that results from taking $x \in \Omega \subset \mathbb{R}^n$ for some bounded Ω (instead of $x \in \mathbb{R}^n$) has also been discussed extensively in the literature [9, 10, 3]. For the boundary value problem, all bounded forward Cauchy problem solutions tend to limits as $|t| \to \infty$, and these limits are equilibrium solutions.

The existence of entire solutions is a difficult problem, because the backward-time Cauchy problem is well known to be ill-posed. Obviously, equilibrium solutions are trivial examples of such entire solutions, and in [12] it was shown that they can exist. It is not at all clear that there are other entire solutions, and indeed there may not be. In this article we assume the existence of a pair of nonintersecting equilibrium solutions and construct a heteroclinic orbit which connects them. (A *heteroclinic orbit* is a special kind of entire solution, whose limits as $t \to \pm\infty$ are equilibria.)

For simplicity and concreteness, we will work with the more limited equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \tag{1.2}$$

where ϕ is a smooth function which decays to zero. We follow the general technique for constructing sub-super solutions largely as appears in [7] and [8]. It should be emphasized that the technique examined in this article can handle the problem for (1.1) in full generality (under mild decay assumptions for the coefficients a_i), though this complicates the exposition needlessly. Previous work by other authors typically require other technical or restrictive conditions on the class of solutions that are extended. They often require some kind of monotonicity of time-slices, for instance. The improvement in our method comes from the fact that uniform bounds on spatial derivatives are obtained.

This simpler model still provides insight into applications, as it is still a model of the population of a single species, with a spatially-varying carrying capacity, ϕ. Indeed, one easily finds that under certain conditions the behavior of solutions to (1.2) is reminiscent of the growth and (admittedly tenuous) control of invasive species [1]. It is the control of invasive species that is of most interest, and it is also what the structure of the space of heteroclines describes. In one of the examples given in [12], there is one "more stable" equilibrium, and several other "less stable" ones. The more stable equilibrium can be thought of as the situation where an invasive species dominates. The task, then, is to try to perturb the system so that it no longer is attracted to that equilibrium. An optimal control approach is to perturb the system so that it barely crosses the boundary of the stable manifold of the the undesired equilibrium, and thereby the invasive species is eventually brought under control with minimal disturbance to the rest of the environment. Such an optimal control approach, though, is beyond the scope of this article.

2. Equilibrium solutions

We choose $\phi(x) = (x^2 - 0.4)e^{-x^2/2}$. It has been shown in [12], that in this situation, there exists a pair of equilibrium solutions f_+, f_- with the following properties:

1. f_+ and f_- are smooth and bounded,
2. f_+ and f_- have bounded first and second derivatives,
3. f_+ and f_- are asymptotic to $6/x^2$ for large x, and so both belong to $L^1(\mathbb{R})$,
4. $f_+(x) > f_-(x)$ for all x,
5. there is no equilibrium solution f_2 with $f_+(x) > f_2(x) > f_-(x)$ for all x.

Additionally, there exists a one-parameter family g_c of solutions to

$$0 = g_c''(x) - g_c^2(x) + \phi_c(x) \tag{2.1}$$

with

1. $c \in [0, 1)$,
2. $g_0 = f_-$ and $\phi_0 = \phi$,
3. $\phi_a(x) < \phi_b(x)$ and $g_a(x) > g_b(x)$ for all x if $a > b$.

The latter set of properties can occur as a consequence of the specific structure of f_-. For instance, consider the following result.

Proposition 2.1. *Suppose $f_- \in C^{2,\alpha}(\mathbb{R})$ satisfies the above conditions and additionally, there is a compact $K \subset \mathbb{R}$ with nonempty interior such that f_- is negative on the interior of K and is nonnegative on the complement of K. Then such a family g_c above exists.*

Proof. (Sketch) Work in $T_{f_-}C^{2,\alpha}(\mathbb{R})$, the tangent space at f_-. Then (2.1) becomes its linearization (for h_c, say), namely

$$0 = h_c''(x) - 2f_-(x)h_c(x) + (\phi_c - \phi). \tag{2.2}$$

Consider the slightly different problem,

$$0 = y''(x) - 2f_-(x)y(x) + v(x)y(x), \tag{2.3}$$

where v is a smooth function to be determined. If we can find a $v \leq 0$ such that $y > 0$ and $y \to 0$ as $|x| \to \infty$, then we are done, because we simply let $vy = \phi_c - \phi$ in (2.2). In that case, $h_c = y$ has the required properties. We sketch why such a v exists:

- If $v \equiv 0$, then $y \equiv 0$ is a solution, giving $g_c = f_-$ as a base case.
- If $v(x) = -2\|u\|_\infty \beta(x)$ for β is a smooth bump function with compact support and $\beta|K = 1$, then the Sturm-Liouville comparison theorem implies that y has no sign changes. We can take y strictly positive. However, in this case, the Sturm-Liouville theorem imples that there are no critical points of y either, so y may not tend to zero as $|x| \to \infty$.

- Hence there should exist an s with $0 < s < 2\|u\|_\infty$ such that if $v(x) = -s\beta(x)$, then y has no sign changes, one critical point, and tends to zero as $|x| \to \infty$. This choice of v is what is required. (The precise details of this argument fall under standard Sturm-Liouville theory, which are omitted here.)

□

In what follows, we shall not be concerned with the exact form of ϕ, but rather we shall assume that the above properties of the equilibria hold. Many other choices of ϕ will allow a similar construction.

Lemma 2.2. *The set*

$$W = \{v \in C^2(\mathbb{R}) : f_-(x) < v(x) < f_+(x) \text{ for all } x\} \tag{2.4}$$

is a forward invariant set for (1.2)*. That is, if u is a solution to* (1.2) *and $u(t_0) \in W$, then $u(t) \in W$ for all $t > t_0$.*

Proof. We show that the flow of (1.2) is inward whenever a timeslice is tangent to either f_- or f_+. To this end, define the set

$$\begin{aligned} B = \big\{ & v \in C^2(\mathbb{R}) : f_-(x) \le v(x) \le f_+(x) \text{ for all } x, \text{ and there exists an } x_0 \\ & \text{such that } v(x_0) = f_+(x) \text{ or } v(x_0) = f_-(x) \big\}. \end{aligned}$$

Without loss of generality, consider a $v \in B$ with a single point of tangency, $v(x_0) = f_-(x_0)$. At such a point x_0, the smoothness of v and f_- implies that $\Delta v(x_0) \ge \Delta f_-(x_0)$ using the maximum principle. Then, if u is a solution to (1.2) with $u(0, x) = v(x)$, we have that

$$\begin{aligned} \frac{\partial u(0, x_0)}{\partial t} &= \Delta v(x_0) - v^2(x_0) + \phi(x_0) \\ &\ge \Delta f_-(x_0) - f_-^2(x_0) + \phi(x_0) = 0, \end{aligned}$$

hence the flow is inward. One can repeat the above argument for each point of tangency, and for tangency with f_+ as well. □

Lemma 2.3. *Solutions to the Cauchy problem*

$$\begin{gathered} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \\ u(0,x) = U(x) \in W_c \end{gathered} \tag{2.5}$$

where

$$W_c = \{v \in C^2(\mathbb{R}) : g_c(x) < v(x) < f_+(x) \text{ for all } x\}$$

for $c \in [0, 1)$ have the property that they lie in $L^1 \cap L^\infty(\mathbb{R})$ for all $t > 0$. We shall assume that U has bounded first and second derivatives.

Additionally, when $c \in (0, 1)$, solutions to (2.5) *cannot have f_- as a limit as $t \to \infty$.*

Proof. The fact that solutions lie in $L^1 \cap L^\infty(\mathbb{R})$ is immediate from Lemma 2.2 and the asymptotic behavior of f_+, f_- (Section 4 of [12]). Observe that for each $c \in [0, 1)$, W_c is forward invariant, and that $W_a \subset W_b$ if $a > b$. Since f_- is not in W_c for c strictly larger than 0, the proof is completed. □

The following is an outline for the rest of the article. We show that all solutions to (2.5) have bounded first and second spatial derivatives. This implies that all of their first partial derivatives are bounded (the time derivative is controlled by (1.2)). Using the fact that (1.2) is autonomous in time, time translations of solutions are also solutions. We therefore construct a sequence of solutions $\{u_k\}$ to Cauchy problems started at $t = 0, T_1, T_2, \dots$ which tend to f_+ as $t \to +\infty$, but their initial conditions tend to f_- as $k \to \infty$. By Ascoli's theorem, this sequence converges uniformly on compact subsets to a continuous entire solution.

3. Integral equation formulation

In order to estimate the derivatives of a solution to (2.5), it is more convenient to work with an integral equation formulation of (2.5). This is obtained in the usual way.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - u^2 + \phi \\
\big(\frac{\partial}{\partial t} - \Delta\big) u &= -u^2 + \phi \\
u &= \big(\frac{\partial}{\partial t} - \Delta\big)^{-1} (\phi - u^2),
\end{aligned}
$$

$$
\begin{aligned}
u(t,x) = &\int_{-\infty}^{\infty} H(t, x-y) U(y) dy \\
&+ \int_0^t \int_{-\infty}^{\infty} H(t-s, x-y)\big(\phi(y) - u^2(s,y)\big) dy\, ds,
\end{aligned}
\tag{3.1}
$$

where $H(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the usual heat kernel.

Calculation 3.1. We begin by estimating the first derivative of u for a short time. Let $T > 0$ be given, and consider $0 \le t \le T$. The key fact is that $\int H(t,x)dx = 1$ for all t. Using (3.1)

$$
\begin{aligned}
\|\frac{\partial u}{\partial x}\|_\infty &\le \|\frac{\partial U}{\partial x}\|_\infty + \Big| \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial x} H(t-s, x-y)\big(\phi(y) - u^2(s,y)\big) dy\, ds \Big| \\
&\le \|\frac{\partial U}{\partial x}\|_\infty + \int_0^t \int_{-\infty}^{\infty} \Big| \frac{\partial}{\partial y}(H(t-s, x-y))\big(\phi(y) - u^2(s,y)\big)\Big| dy\, ds \\
&\le \|\frac{\partial U}{\partial x}\|_\infty + \int_0^t \int_{-\infty}^{\infty} \Big| H(t-s, x-y)\big(\frac{\partial \phi}{\partial y} - 2u\frac{\partial u}{\partial y}\big)\Big| dy\, ds \\
&\le \|\frac{\partial U}{\partial x}\|_\infty + T\|\frac{\partial \phi}{\partial x}\|_\infty + 2\|u\|_\infty \int_0^t \|\frac{\partial u}{\partial x}\|_\infty ds.
\end{aligned}
$$

This integral equation fence is easily solved to give

$$
\begin{aligned}
\|\frac{\partial u}{\partial x}\|_\infty &\le \Big(\|\frac{\partial U}{\partial x}\|_\infty + T\|\frac{\phi}{\partial x}\|_\infty\Big) e^{2t \max\{\|f_+\|_\infty, \|f_-\|_\infty\}} \\
&\le K_1 e^{K_2 T}.
\end{aligned}
$$

Calculation 3.2. With the same choice of T as above, we find a bound for the second derivative in the same way:

$$\begin{aligned}
\|\frac{\partial^2 u}{\partial x^2}\|_\infty &\le \|\frac{\partial^2 U}{\partial x^2}\|_\infty + T\|\frac{\partial^2 \phi}{\partial x^2}\|_\infty + \int_0^t \|\frac{\partial}{\partial y}\big(2u\frac{\partial u}{\partial y}\big)\|_\infty ds \\
&\le \|\frac{\partial^2 U}{\partial x^2}\|_\infty + T\|\frac{\partial^2 \phi}{\partial x^2}\|_\infty + \int_0^t 2\|\frac{\partial u}{\partial x}\|_\infty^2 + 2\|u\|_\infty\|\frac{\partial^2 u}{\partial x^2}\|_\infty ds \\
&\le K_3 e^{K_2 T}
\end{aligned}$$

for some K_3 which depends on U, ϕ, and T.

Calculation 3.3. Now, we extend Calculation 3.2 to handle $t > T$,

$$\begin{aligned}
\|\frac{\partial^2 u}{\partial x^2}\|_\infty &\le \|\frac{\partial^2 U}{\partial x^2}\|_\infty + \big|\frac{\partial^2}{\partial x^2}\int_0^T\int_{-\infty}^\infty H(t-s,x-y)(\phi(y)-u^2(s,y))dy\,ds\big| \\
&\quad + \big|\frac{\partial^2}{\partial x^2}\int_T^t\int_{-\infty}^\infty H(t-s,x-y)(\phi(y)-u^2(s,y))dy\,ds\big| \\
&\le K_3e^{K_2T} + \int_T^t \|\frac{\partial^2}{\partial x^2}H(t-s,x)\|_\infty(\|\phi\|_1 + \|u^2\|_1)ds \\
&\le K_3e^{K_2T} + K_4\int_T^t \frac{1}{s\sqrt{s}}ds + K_4'\int_T^t \frac{1}{s^2\sqrt{s}}ds \\
&\le K_3e^{K_2T} + K_5\big(\frac{1}{\sqrt{T}} - \frac{1}{\sqrt{t}}\big) + K_5'\big(\frac{1}{T\sqrt{T}} - \frac{1}{t\sqrt{t}}\big) \\
&\le K_3e^{K_2T} + K_6,
\end{aligned}$$

hence there is a uniform upper bound on $\|\frac{\partial^2 u}{\partial x^2}\|_\infty$ which depends only on the initial conditions, ϕ, and T.

Lemma 3.4. *Let $f \in C^2(\mathbb{R})$ be a bounded function with a bounded second derivative. Then the first derivative of f is also bounded, and the bound depends only on $\|f\|_\infty$ and $\|f''\|_\infty$.*

Proof. The proof is elementary. The key fact is that at its maxima and minima, f has a horizontal tangent. From a horizontal tangent, the quickest f' can grow is at a rate of $\|f''\|_\infty$. However, since f is bounded, there is a maximum amount that this growth of f' can accrue. Indeed, a sharp estimate is

$$\|f'\|_\infty \le \sqrt{2\|f\|_\infty\|f''\|_\infty}.$$

□

Using the fact that u is bounded, Lemma 3.4 implies that the first spatial derivative of u is bounded. By (1.2), it is clear that the first time derivative of u is also bounded.

Lemma 3.5. *As an immediate consequence of Lemmas 2.3 and 3.4, the action integral*

$$A(u(t)) = \int_{-\infty}^\infty \frac{1}{2}\big|\frac{\partial u}{\partial x}\big|^2 + \frac{1}{3}u^3(t,x) - u(t,x)\phi(x)dx$$

is bounded. Therefore, the solutions to the Cauchy problem (2.5) *all tend to limits as $t \to \infty$* [13, Corollary 7]*. By Lemma 2.3, we conclude that they all tend to the common limit of f_+ when $c > 0$.*

Proof. The latter two terms are bounded due to the fact that u lies in $L^1 \cap L^\infty(\mathbb{R})$ for all t. The bound on the first term comes from combining the fact that u and its first two spatial derivatives are bounded with the asymptotic decay of $f_\pm$, and is otherwise straightforward (use L'Hôpital's rule). □

4. Construction of an entire solution

Let

$$U_k(x) = (1 - 2^{-k-1})g_{1/k}(x) + 2^{-k-1}f_+(x), \quad \text{for } k \geq 0$$

noting that $U_k \to f_-$ as $k \to \infty$. Since U_k is a convex combination of f_+ and $g_{1/k}$, it follows that $U_k \in W_{1/k}$ for all k. Also, since f_+ and f_- have bounded first and second derivatives, the $\{U_k\}$ have a common bound for their first and second derivatives.

Now consider solutions to the set of Cauchy problems

$$\begin{gathered} \frac{\partial u_k(t,x)}{\partial t} = \frac{\partial^2 u_k(t,x)}{\partial x^2} - u_k^2(t,x) + \phi(x), \\ u_k(T_k, x) = U_k(x). \end{gathered} \tag{4.1}$$

We choose T_k so that for all $k > 0$, $u_k(0,0) = u_0(0,0)$. We can do this using the continuity of the solution and Lemma 3.5. As $k \to \infty$, solutions are started nearer and nearer to the equilibrium f_-, so we are forced to choose $T_k \to -\infty$ as $k \to \infty$.

It is clear that each solution u_k is defined for only $t > T_k$. However, for each compact set $S \subset \mathbb{R}^2$, there are infinitely many elements of $\{u_k\}$ which are defined on it. The results of the previous section imply that $\{u_k\}$ is a bounded, equicontinous family. As a result, Ascoli's theorem implies that $\{u_k\}$ converges uniformly on compact subsets to a continuous u, which is an entire solution to (1.2).

Our constructed entire solution will have the value $u(0,0) = u_0(0,0)$, which is strictly between f_+ and f_-. As a result, the entire solution we have constructed is not an equilibrium solution. By Lemma 3.5, it is a finite energy solution, so it must be a heteroclinic orbit connecting f_- to f_+.

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