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LONG TERM BEHAVIOR OF SOLUTIONS FOR RICCATI INITIAL-VALUE PROBLEMS

SARAH Y. BAHK, NADEJDA E. DYAKEVICH, STEFAN C. JOHNSON

ABSTRACT. The Riccati equation has been known since the early 1700s. Numerous papers have been written on the solvability of its special cases. However, to the best of our knowledge, there are no papers that investigate the exact (equation specific) conditions for unbounded growth in finite time of solutions for Riccati initial-value problems. In this paper, we first derive conditions that are necessary and sufficient for the solutions of Riccati problems with constant coefficients to grow unbounded in finite time. Then we use a comparison method to extend these results to Riccati problems with variable coefficients.

1. Introduction

Count Jacopolo Francesco Riccati (May 28, 1676 - April 15, 1754) is famous for introducing and researching the solvability of the equation that now bears his name:

$$y'(t) = a(t)y^2 + b(t)y + c(t). \tag{1.1}$$

The matrix form of this equation is very important in modern times since it is used extensively in design problems in filtering and control [1, 3]. Even though the Riccati equation (1.1) is not solvable in general, numerous methods have been developed for finding solutions for many of its special cases [2], [6]-[10].

In Section 2, we consider real solutions of the initial-value problem

$$\begin{gathered} y'(t) = ay^2 + by + c, \\ y(0) = d, \end{gathered} \tag{1.2}$$

where a, b, c, and d are real numbers and $t \geq 0$ represents time. We determine conditions on the constants a, b, c, and d that are necessary and sufficient for $y(t)$ to approach either $+\infty$ or $-\infty$ as t approaches some finite value t_b. We provide exact values for the time t_b for the cases when $4ac - b^2$ is positive, negative, or zero. In particular, we are interested in the first occurrence of blow-up. We do not consider behavior of $y(t)$ for $t > t_b$.

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In Section 3, we use a comparison theorem to extend the results to the more general initial-value problem

$$\begin{gathered} y'(t) = a(t)y^2 + b(t)y + c(t), \\ y(0) = d, \end{gathered}$$

where $a(t)$, $b(t)$, $c(t)$ are continuous and differentiable functions for $t \geq 0$, and d is a real number.

2. Riccati Problems with Constant Coefficients

Theorem 2.1. *The following is true for the solution $y(t)$ of (1.2):*

(1) *Let $4ac - b^2 > 0$. If $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$.*
(2) *Let $4ac - b^2 = 0$. If $a > 0$ and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$. If $a < 0$ and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. If $d = -\frac{b}{2a}$, then $y(t) \equiv d$.*
(3) *Let $4ac - b^2 < 0$. If $a > 0$ and $d > \frac{-b+\sqrt{b^2-4ac}}{2a}$, then $y(t) \to +\infty$. If $a < 0$ and $d < \frac{-b+\sqrt{b^2-4ac}}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. If $d = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, then $y(t) \equiv d$.*
(4) *If $a = 0$, then $y(t)$ is bounded for all $t > 0$.*

Proof. (1) Let $4ac - b^2 > 0$. The solution of the initial value problem (1.2) can be found using separation of variables:

$$y(t) = \frac{\sqrt{4ac-b^2}}{2a} \tan\Big[\frac{t\sqrt{4ac-b^2}}{2} + \arctan\Big(\frac{b+2ad}{\sqrt{4ac-b^2}}\Big)\Big] - \frac{b}{2a}. \tag{2.1}$$

We can find the blow-up time t_b by solving for t in equation

$$\begin{gathered} \frac{t\sqrt{4ac-b^2}}{2} + \arctan\Big(\frac{b+2ad}{\sqrt{4ac-b^2}}\Big) = \frac{\pi}{2}, \\ t_b = \frac{\pi}{\sqrt{4ac-b^2}} - \frac{2}{\sqrt{4ac-b^2}} \arctan\big[\frac{b+2ad}{\sqrt{4ac-b^2}}\big]. \end{gathered}$$

Also,

$$-\frac{\pi}{2} < \arctan\big[\frac{b+2ad}{\sqrt{4ac-b^2}}\big] < \frac{\pi}{2}.$$

This implies that t_b is always positive and that the solution $y(t)$ of (1.2) is guaranteed to blow-up as t approaches t_b. Also, from equation (2.1) we notice that if $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$. Changing the initial value d cannot prevent blow-up from occurring. However, d influences the blow-up time t_b. For example, if $a < 0$, then decreasing d will accelerate the blow-up. If $a > 0$, then increasing d will accelerate the blow-up.

(2) Let $4ac - b^2 = 0$. Using separation of variables, we obtain

$$\int \frac{dy}{a(y+\frac{b}{2a})^2} = \int dt.$$

Integration leads to the solution:

$$y(t) = \frac{2ad+b}{a(2-2adt-bt)} - \frac{b}{2a}. \tag{2.2}$$

To find the blow-up time, we set the denominator of the first term in (2.2) equal to 0 and solve for t in $2 - 2adt - bt = 0$,

$$t_b = \frac{2}{2ad + b}.$$

From the inequality $t_b > 0$, and from (2.2) we obtain the following: If $a > 0$ and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$, while if $a < 0$ and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$. Initial value d is very important since certain values can prevent blow-up from occurring. Also, d influences the blow-up time t_b. If blow-up occurs for some value d, then decreasing d (if $a < 0$) or increasing d (if $a > 0$) will accelerate the blow-up. If $d = -\frac{b}{2a}$, then $y(t) \equiv d$ satisfies the initial-value problem (1.2). Therefore, in this special case, $y(t)$ is bounded for all finite $t > 0$.

(3) Let $4ac - b^2 < 0$. We notice that if

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

then $y \equiv d$ is the solution of the initial-value problem (1.2). Therefore, in this case $y(t)$ is bounded for all $t > 0$. Now let us consider the case when

$$d \neq \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Using separation of variables, we have

$$\int \frac{dy}{ay^2 + by + c} = \int dt,$$

$$\frac{1}{\sqrt{b^2 - 4ac}} \ln \Big| \frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \Big| = t + C_1, \tag{2.3}$$

where C_1 is a constant of integration. We can find C_1 by substituting the initial condition $y(0) = d$ into equation (2.3). Thus,

$$C_1 = \frac{1}{\sqrt{b^2 - 4ac}} \ln \Big| \frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} \Big|. \tag{2.4}$$

We will consider the two possible cases:

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} > 0 \tag{2.5}$$

and

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} < 0. \tag{2.6}$$

We now substitute (2.4) into (2.3) and solve for $y(t)$. In case (2.5), we can omit the absolute value symbol:

$$\frac{1}{\sqrt{b^2 - 4ac}} \ln \Big(\frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \Big) = t + \frac{1}{\sqrt{b^2 - 4ac}} \ln \Big(\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} \Big).$$

In the case (2.6), we have

$$\begin{aligned} &\frac{1}{\sqrt{b^2 - 4ac}} \ln \Big(- \frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \Big) \\ &= t + \frac{1}{\sqrt{b^2 - 4ac}} \ln \Big(- \frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} \Big). \end{aligned}$$

In both cases (2.5) and (2.6), the solution of (1.2) is given by the formula

$$y(t) = \frac{-bd + d\sqrt{b^2-4ac} - 2c + (bd + d\sqrt{b^2-4ac} + 2c)e^{t\sqrt{b^2-4ac}}}{2ad + b + \sqrt{b^2-4ac} - (2ad + b - \sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}}}. \tag{2.7}$$

To find the blow-up time we set the denominator equal to 0,

$$2ad + b + \sqrt{b^2-4ac} - (2ad + b - \sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}} = 0 \tag{2.8}$$

and solve for t to obtain

$$t_b = \frac{1}{\sqrt{b^2-4ac}} \ln\Big(\frac{2ad + b + \sqrt{b^2-4ac}}{2ad + b - \sqrt{b^2-4ac}}\Big).$$

Since the blow-up time t_b must be positive, we have

$$\frac{2ad + b + \sqrt{b^2-4ac}}{2ad + b - \sqrt{b^2-4ac}} > 1. \tag{2.9}$$

Let us observe that if (2.6) holds, then (2.8) can never be satisfied, thus, there is no blowup. On the other hand, if (2.5) holds, then there are two possibilities: either

$$2ad + b - \sqrt{b^2-4ac} > 0 \quad \text{and} \quad 2ad + b + \sqrt{b^2-4ac} > 0 \tag{2.10}$$

or

$$2ad + b - \sqrt{b^2-4ac} < 0 \quad \text{and} \quad 2ad + b + \sqrt{b^2-4ac} < 0. \tag{2.11}$$

Solving (2.10) and (2.9) simultaneously, we obtain the conditions on d that lead to blow-up in finite time:

$$d > \frac{-b + \sqrt{b^2-4ac}}{2a}, \quad \text{if } a > 0,$$

$$d < \frac{-b + \sqrt{b^2-4ac}}{2a}, \quad \text{if } a < 0.$$

Solving (2.11) and (2.9) simultaneously, we obtain a contradiction which implies that there is no blow-up in this case. We notice from (2.7) that if $a > 0$ and $d > \frac{-b+\sqrt{b^2-4ac}}{2a}$, then $y(t) \to +\infty$. If $a < 0$ and $d < \frac{-b+\sqrt{b^2-4ac}}{2a}$, then $y(t) \to -\infty$. Thus, initial value d is very important since certain values can prevent blow-up from occurring. Also, d influences the value of t_b at which blow-up occurs.

(4) If $a = 0$, then the equation is linear. Using separation of variables, we obtain

$$\int \frac{dy}{by + c} = \int dt,$$

$$y(t) = \frac{(bd + c)e^{bt} - c}{b}.$$

Here we find that $y(t)$ is bounded for any finite time $t > 0$. □

Example. Let us investigate the blow-up property of the solution for the initial-value problem

$$\begin{gathered} y'(t) = -4y^2 + 5y - 1, \\ y(0) = d \end{gathered} \tag{2.12}$$

with three different values of d as indicated below.

First, we note that $a = -4 < 0$, $b = 5$, $c = -1$, and $4ac - b^2 = -9 < 0$. Also, $(-b + \sqrt{b^2 - 4ac})/(2a) = 0.25$. According to Theorem 2.1, we expect that the solution of problem (2.12) blows up for any $d < 0.25$ and is bounded otherwise. The solution of problem (2.12) is given by formula (2.7):

$$y(t) = \frac{-2d + 2 + (8d - 2)e^{3t}}{-8d + 8 - (-8d + 2)e^{3t}}.$$

Let us notice that if we differentiate function y with respect to d, then the corresponding derivative is as follows:

$$\frac{9e^{3t}}{(-4d + 4 + 4de^{3t} - e^{3t})^2}.$$

Therefore, the function y and its derivative with respect to d are both discontinuous when

$$t = \frac{1}{3} \ln \big(\frac{4d - 4}{4d - 1} \big),$$

which holds only for $d < 0.25$.

If $d = 2$, we have

$$y(t) = \frac{-2 + 14e^{3t}}{-8 + 14e^{3t}}.$$

This function is bounded for any finite time $t > 0$.

If $d = 0.5$, we have

$$y(t) = \frac{1 + 2e^{3t}}{4 + 2e^{3t}}.$$

This function is bounded for any finite time $t > 0$.

If $d = 0$, we have

$$y(t) = \frac{1 - e^{3t}}{4 - e^{3t}}.$$

Here $y(t) \to -\infty$ when $t_b = \ln(4)/3$.

3. Riccati Problems with Variable Coefficients

Let $a(t)$, $b(t)$, and $c(t)$ be continuous and differentiable functions for $t \geq 0$. We consider the initial-value problem

$$\begin{gathered} y'(t) = a(t)y^2 + b(t)y + c(t), \\ y(0) = d. \end{gathered} \tag{3.1}$$

We notice that $\varphi(t) = \exp\big(\int_0^t b(\hat{t})d\hat{t}\big)$ is the unique solution of the initial-value problem

$$\begin{gathered} \varphi'(t) = b(t)\varphi(t), \\ \varphi(0) = 1. \end{gathered}$$

Also, $\varphi(t)$ is bounded for any finite $t \geq 0$. Let $\psi(t)$ satisfy the initial-value problem

$$\begin{gathered} \psi'(t) = a(t)\psi^2(t)\varphi(t) + \frac{c(t)}{\varphi(t)}, \\ \psi(0) = d. \end{gathered} \tag{3.2}$$

Then $y(t) = \varphi(t)\psi(t)$ satisfies the Riccati initial-value problem (3.1). We now investigate conditions on $a(t)$, $c(t)$, and d that lead to unbounded growth of $\psi(t)$, and therefore, of $y(t)$.

Theorem 3.1. *Let $a(t)c(t) \geq 0$. Then the following is true for the solution $y(t)$ of (3.1):*

(1) *If $a(t)\exp\big(\int_0^t b(\hat{t})d\hat{t}\big) \geq k_1 > 0$ and $c(t)\exp\big(-\int_0^t b(\hat{t})d\hat{t}\big) \geq k_2 > 0$ for all $t > 0$, then $y(t) \to +\infty$ for any initial condition d.*

(2) *If $a(t)\exp\big(\int_0^t b(\hat{t})d\hat{t}\big) \leq k_3 < 0$ and $c(t)\exp\big(-\int_0^t b(\hat{t})d\hat{t}\big) \leq k_4 < 0$ for all $t > 0$, then $y(t) \to -\infty$ for any initial condition d.*

(3) *Let $a(t)c(t)$ have zeroes at $t = t_1, t_2, t_3, \dots$ and let $g_1(t)$ and $g_2(t)$ be non-trivial, continuous, and differentiable functions such that for all $t \geq 0$,*

$$
\begin{gathered}
g_1(t) \leq \min\big\{a(t)e^{\int_0^t b(\hat{t})d\hat{t}}, c(t)e^{-\int_0^t b(\hat{t})d\hat{t}}\big\},\\
g_2(t) \geq \max\big\{a(t)e^{\int_0^t b(\hat{t})d\hat{t}}, c(t)e^{-\int_0^t b(\hat{t})d\hat{t}}\big\}.
\end{gathered}
$$

Then one of the following three statements is true:

(a) *$y(t) \to +\infty$ if for some $t_{b_1} > 0$,*

$$
\int_0^{t_{b_1}} g_1(t)dt = \frac{\pi}{2} - \arctan(d). \tag{3.3}
$$

(b) *$y(t) \to -\infty$ if for some $t_{b_2} > 0$,*

$$
\int_0^{t_{b_2}} g_2(t)dt = -\frac{\pi}{2} - \arctan(d). \tag{3.4}
$$

(c) *$y(t)$ is bounded for all $t > 0$ if the following two inequalities hold simultaneously:*

$$
\int_0^t g_2(\hat{t})d\hat{t} < \frac{\pi}{2} - \arctan(d), \quad \int_0^t g_1(\hat{t})d\hat{t} > -\frac{\pi}{2} - \arctan(d).
$$

Proof. (1) By the comparison theorem [4, pp. 221-223], the solution $\bar{\psi}(t)$ of the initial-value problem

$$
\begin{gathered}
\bar{\psi}'(t) = k_1\bar{\psi}^2(t) + k_2,\\
\bar{\psi}(0) = \bar{d},
\end{gathered}
$$

where $\bar{d} \leq d$, is a lower solution for (3.2). By Theorem 2.1, $\bar{\psi}(t)$ approaches $+\infty$ as t approaches $\bar{t}_b$. Therefore, $\psi(t) \geq \bar{\psi}(t)$ also approaches $+\infty$ as t approaches some $t_b \leq \bar{t}_b$.

(2) By the comparison theorem, solution $\tilde{\psi}(t)$ of the initial-value problem

$$
\begin{gathered}
\tilde{\psi}'(t) = k_3\tilde{\psi}^2(t) + k_4,\\
\tilde{\psi}(0) = \tilde{d},
\end{gathered}
$$

where $\tilde{d} \geq d$, is an upper solution for (3.2). By Theorem 2.1, $\tilde{\psi}(t)$ approaches $-\infty$ as t approaches $\tilde{t}_b$. Therefore, $\psi(t) \leq \tilde{\psi}(t)$ also approaches $-\infty$ as t approaches $t_b \leq \tilde{t}_b$.

(3)(a). By the comparison theorem, the solution $\hat{\psi}(t)$ of the initial-value problem

$$
\begin{gathered}
\hat{\psi}'(t) = g_1(t)\hat{\psi}^2(t) + g_1(t),\\
\hat{\psi}(0) = \hat{d},
\end{gathered}
$$

where $\hat{d} \geq d$, is a lower solution for (3.2). Using the separation of variables method, we obtain that $\hat{\psi}(t)$ approaches $+\infty$ as t approaches $\hat{t}_b$ provided that (3.3) holds.

Therefore, $\psi(t) \geq \hat{\psi}(t)$ also approaches $+\infty$ as t approaches $t_{b_1} \leq \hat{t}_b$. If condition (3.3) does not hold, then $\hat{\psi}(t)$ and $\psi(t)$ are bounded from above for all $t > 0$.

(3)(b) By the comparison theorem, the solution $\breve{\psi}(t)$ of the initial-value problem

$$\begin{gathered}\breve{\psi}'(t) = g_2(t)\breve{\psi}^2(t) + g_2(t),\\ \breve{\psi}(0) = \breve{d},\end{gathered}$$

where $\breve{d} \geq d$, is an upper solution for (3.2). Using the separation of variables method, we obtain that $\breve{\psi}(t)$ approaches $-\infty$ as t approaches $\breve{t}_b$ provided that (3.4) holds. Therefore, $\psi(t) \leq \breve{\psi}(t)$ also approaches $-\infty$ as t approaches $t_{b_2} \leq \breve{t}_b$. If condition (3.4) does not hold, then $\breve{\psi}(t)$ and $\psi(t)$ are bounded from below for all $t > 0$.

(3)(c) The proof of this part follows directly from the proofs described in parts (a) and (b) above. □

Theorem 3.2. *Let $c(t) = 0$. One of the following three statements is true for the solution $y(t)$ of problem* (3.1)*:*

1. *$y(t) \to +\infty$ provided $d > 0$ and*
$$\int_0^{t_{b_1}} a(t)e^{\int_0^t b(\hat{t})d\hat{t}}dt = \frac{1}{d} \tag{3.5}$$
for some $t_{b_1} > 0$.
2. *$y(t) \to -\infty$ provided $d < 0$ and*
$$\int_0^{t_{b_2}} a(t)e^{\int_0^t b(\hat{t})d\hat{t}}dt = \frac{1}{d} \tag{3.6}$$
for some $t_{b_2} > 0$.
3. *If*
$$-|\frac{1}{d}| < \int_0^t a(\bar{t})e^{\int_0^{\bar{t}} b(\hat{t})d\hat{t}}d\bar{t} < |\frac{1}{d}|$$
for all $t \geq 0$, then $y(t)$ is bounded for all $t \geq 0$.

Proof. (1) Applying the separation of variables method to the problem

$$\begin{gathered}\psi'(t) = a(t)\psi^2(t),\\ \psi(0) = d,\end{gathered}$$

we obtain that $\psi(t)$ approaches $+\infty$ as t approaches t_{b_1} provided that $d > 0$, and (3.5) holds. Otherwise, $\psi(t)$ is bounded from above for all $t > 0$.

(2) Similarly, we obtain that $\psi(t)$ approaches $-\infty$ as t approaches t_{b_2} provided that $d < 0$, and (3.6) holds. Otherwise, $\psi(t)$ is bounded from below for all $t > 0$.

(3) The proof of this part follows directly from the proofs of parts 1 and 2 above. □

Theorem 3.3. *For the case $a(t)c(t) < 0$, one of the following two statements is true for the solution $y(t)$ of* (3.1)*:*

1. *If $a(t)\exp\big(\int_0^t b(\hat{t})d\hat{t}\big) \geq k_5 > 0$ and $0 > c(t)\exp\big(-\int_0^t b(\hat{t})d\hat{t}\big) \geq k_6$ for all $t > 0$, then $y(t) \to +\infty$ for any $d > \sqrt{|k_6/k_5|}$.*
2. *If $a(t)\exp\big(\int_0^t b(\hat{t})d\hat{t}\big) \leq k_7 < 0$ and $0 < c(t)\exp\big(-\int_0^t b(\hat{t})d\hat{t}\big) \leq k_8$ for all $t > 0$, then $y(t) \to -\infty$ for any $d < -\sqrt{|k_8/k_7|}$.*

Proof. (1) By the comparison theorem [4, pp. 221-223], the solution $\bar{\psi}(t)$ of the initial-value problem

$$\begin{gathered}\bar{\psi}'(t) = k_5\bar{\psi}^2(t) + k_6,\\ \bar{\psi}(0) = \bar{d},\end{gathered}$$

where $\sqrt{|k_6/k_5|} < \bar{d} \le d$, is a lower solution for (3.2). By Theorem 2.1, $\bar{\psi}(t)$ approaches $+\infty$ as t approaches $\bar{t}_b$. Therefore, $\psi(t) \ge \bar{\psi}(t)$ also approaches $+\infty$ as t approaches some $t_b \le \bar{t}_b$.

(2) By the comparison theorem, the solution $\tilde{\psi}(t)$ of the initial-value problem

$$\begin{gathered}\tilde{\psi}'(t) = k_7\tilde{\psi}^2(t) + k_8,\\ \tilde{\psi}(0) = \tilde{d},\end{gathered}$$

where $-\sqrt{|k_8/k_7|} > \tilde{d} \ge d$ is an upper solution for (3.2). By Theorem 2.1, $\tilde{\psi}(t)$ approaches $-\infty$ as t approaches $\tilde{t}_b$. Therefore, $\psi(t) \le \tilde{\psi}(t)$ also approaches $-\infty$ as t approaches $t_b \le \tilde{t}_b$. □

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Department of Mathematics, California State University San Bernardino, 5500 University Parkway, San Bernardino, CA 92407-2397, USA

E-mail address, Sarah Y. Bahk: `sbahk@csusb.edu`
E-mail address, Nadejda E. Dyakevich: `dyakevic@csusb.edu`
E-mail address, Stefan C. Johnson: `steve@sevej.us`