Electronic Journal of Differential Equations, Vol. 2008(2008), No. 164, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

THEORETICAL ANALYSIS AND CONTROL RESULTS FOR THE FITZHUGH-NAGUMO EQUATION

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Abstract. In this paper we are concerned with some theoretical questions for the FitzHugh-Nagumo equation. First, we recall the system, we briefly explain the meaning of the variables and we present a simple proof of the existence and uniqueness of strong solution. We also consider an optimal control problem for this system. In this context, the goal is to determine how can we act on the system in order to get good properties. We prove the existence of optimal state-control pairs and, as an application of the Dubovitski-Milyoutin formalism, we deduce the corresponding optimality system. We also connect the optimal control problem with a controllability question and we construct a sequence of controls that produce solutions that converge strongly to desired states. This provides a strategy to make the system behave as desired. Finally, we present some open questions related to the control of this equation.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$ ($N = 1, 2$ or 3) and let $T > 0$ be a finite number. We will set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $|\cdot|$ (resp. $(\cdot\,,\cdot)$) the usual norm (resp. scalar product) in $L^2(\Omega)$. In the sequel, C denotes a generic positive constant.

Let ψ_1, ψ_2 and ψ_3 be three given functions in $L^\infty(Q)$. We will consider the FitzHugh-Nagumo equation

$$
\begin{gathered}
u_t - \Delta u + v + F_0(x, t; u) = g, \\
v_t - \sigma u + \gamma v = 0, \\
u(x, t)|_\Sigma = 0, \\
u(x, 0) = u^0(x), \quad v(x, 0) = 0,
\end{gathered}
\tag{1.1}
$$

where $g \in L^2(Q)$, $\sigma > 0$ and $\gamma \geq 0$ are constants, $u^0 \in L^2(\Omega)$ (at least) and $F_0(x, t; u)$ is given by

$$
F_0(x, t, u) = (u + \psi_1(x, t))(u + \psi_2(x, t))(u + \psi_3(x, t)).
$$

In this system, g is the control, which is constraint to belong to a nonempty closed convex set $\mathcal{G}_{ad} \subset L^2(Q)$ and u and v are the state variables.

2000 *Mathematics Subject Classification.* 35B37, 49J20, 93B05.
Key words and phrases. Optimal control; controllability; FitzHugh-Nagumo equation; Dubovitski-Milyoutin.

Submitted November 13, 2007. Published December 23, 2008.

The FitzHugh-Nagumo system is a simplified version of the Hodgkin-Huxley model, which seems to reproduce most of its qualitative features. The variable u is the electrical potential across the axonal membrane; v is a recovery variable, associated to the permeability of the membrane to the principal ionic components of the transmembrane current; g is the medicine actuator (the control variable), see [13, 14] for more details. Taking into account the role that can be played by actuators in this context (by inhibiting in the case of calmant medicines and by exciting in the case of anti-depressive products), it is natural to consider control questions for this model.

This system has attracted a lot of interest, since it is relatively simple and, at the same time, describes appropriately *excitability* and *bistability* phenomena. For instance, it has also been used as the starting point for models of cardiac excitation [1, 17], labyrinth pattern formation in an activatorinhibitor system [10], etc. For more details, see [18] and the references therein.

An equivalent formulation to (1.1) is easily obtained by solving the second equation, which gives

$$v(x,t) = \sigma \int_0^t e^{-\gamma(t-s)} u(x,s)\, ds. \tag{1.2}$$

We obtain:

$$\begin{gathered} u_t - \Delta u + \sigma \int_0^t e^{-\gamma(t-s)} u(s)\, ds + F_0(x,t;u) = g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x). \end{gathered} \tag{1.3}$$

We could have started from a system more general than (1.1), including a nonzero right hand side in the second equation and nonzero initial values for v. It will be seen later that this does not incorporate any essential difficulty (see remark 2.3 in Section 2).

In the sequel, unless otherwise specified, we will always prefer this shorter formulation of the problem. Accordingly, we will work with couples (u,g) which *a posteriori* give the secondary variable v through (1.2).

This paper deals with several questions concerning systems (1.1) and (1.3). First, we will deal with existence, uniqueness and regularity results. In this context, we will provide a simple proof of a known result; a previous proof was given in [15].

The main result is the following:

Theorem 1.1. *Assume that $g \in L^2(Q)$ and $u^0 \in H_0^1(\Omega)$. Then (1.1) possesses exactly one solution (u,v), with*

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q), \tag{1.4}$$

$$v \in C^0([0,T];H^2(\Omega)), \quad v_t \in L^2(0,T;H^2(\Omega)). \tag{1.5}$$

In the sequel, $H^{1,2}(Q)$ stands for the Hilbert space

$$H^{1,2}(Q) = \{w \in L^2(0,T;H^2(\Omega)) : w = 0 \text{ on } \Sigma, \ \ w_t \in L^2(Q)\}.$$

In view of theorem 1.1, the mapping $g \mapsto u$ is well-defined from $L^2(Q)$ into $H^{1,2}(Q)$. Among other things, this means that the equations in (1.1) are satisfied a.e. (notice that g can be discontinuous). Furthermore, the regularity of u and v makes it possible to derive error estimates for standard numerical approximations.

Our second goal in this paper is to study an optimal control problem for (1.3). We will mainly deal with the cost functional

$$\mathcal{J}(u,g)=\frac{1}{2}\iint_Q |u-u_d|^2\,dx\,dt+\frac{a}{2}\iint_Q |g|^2\,dx\,dt, \tag{1.6}$$

where u_d is a *desired state,* $a>0$ and the control g is assumed to belong to a closed convex set $\mathcal{G}_{ad}\subset L^2(Q)$.

The fact that we choose this functional means that g is a "good" control if its associated state u is not too far from the desired state u_d and, furthermore, its $L^2(Q)$ norm is not too large.

We will deduce the optimality system for (1.3), (1.6) following the Dubovistky-Milyutin formalism (see [11]).

Definition 1.2. Let

$$\mathcal{Q}=\{(u,g)\in H^{1,2}(Q)\times L^2(Q):(1.3)\ \text{ is satisfied}\}. \tag{1.7}$$

Let $\mathcal{G}_{ad}\subset L^2(Q)$ be a closed convex set. Then the associated admissibility set for (1.3), (1.6) is

$$\mathcal{U}_{ad}=\{(u,g):(u,g)\in\mathcal{Q},\quad g\in\mathcal{G}_{ad}\}. \tag{1.8}$$

It will be said that $(\hat{u},\hat{g})$ is a (global) optimal state-control if $(\hat{u},\hat{g})\in\mathcal{U}_{ad}$ and

$$\mathcal{J}(\hat{u},\hat{g})\le\mathcal{J}(u,g)\quad\forall(u,g)\in\mathcal{U}_{ad}.$$

It will be said that $(\hat{u},\hat{g})$ is a local optimal state-control if $(\hat{u},\hat{g})\in\mathcal{U}_{ad}$ and there exists $\epsilon>0$ such that, whenever $(u,g)\in\mathcal{U}_{ad}$ and $\|u-\hat{u}\|_{H^{1,2}(Q)}+\|g-\hat{g}\|_{L^2(Q)}\le\epsilon$, one has

$$\mathcal{J}(\hat{u},\hat{g})\le\mathcal{J}(u,g).$$

Several particular choices of $\mathcal{G}_{ad}$ of practical interest are the following:

- $\mathcal{G}_{ad}=L^2(\omega\times(0,T))$, where $\omega\subset\Omega$ is a given non-empty open set. This is a non-realistic case in which it is assumed that we can act on the system (only on $\omega\times(0,T)$) with no restriction.
- $\mathcal{G}_{ad}=\{g\in L^2(Q):0\le g\le M \text{ a.e.}\}$. Now, we assume that the medicine actuator cannot exceed a fixed level M.
- $\mathcal{G}_{ad}=\{g:g=\sum_{i=1}^I g_i(x)\delta_{(t=t_i)},\quad g_i\in L^2(\Omega)\}$, for some t_i with $0<t_1<\cdots<t_I<T$. In fact, this choice is not covered by the previous definition (it will be out of the scope of the next result as well).
- $\mathcal{G}_{ad}=\{g:g=\sum_{i=1}^I g_i(x)1_{(t_i-\epsilon,t_i+\epsilon)}(t),\quad g_i\in L^2(\Omega)\}$. Obviously, this can be regarded as an approximation of the previous choice.
- $\mathcal{G}_{ad}=B(\mathcal{Z}_{ad})=\{B(f):f\in\mathcal{Z}_{ad}\}$, where $\mathcal{Z}_{ad}=\{f\in L^2(0,T):0\le f\le K\quad\text{a.e.}\}$ and $B:L^2(0,T)\mapsto L^2(Q)$ is a (nonlinear) C^1 mapping. This is an example of non-convex $\mathcal{G}_{ad}$.

The second main result in this paper is the following:

Theorem 1.3. *Assume that $u^0\in H_0^1(\Omega)$ and $\mathcal{G}_{ad}\subset L^2(Q)$ is a nonempty closed convex set. Then there exists at least one global optimal state-control $(\hat{u},\hat{g})$. Furthermore, if $(\hat{u},\hat{g})$ is a local optimal state-control of* (1.3), (1.6) *and $\mathcal{J}'(\hat{u},\hat{g})$ does not vanish, there exists $\hat{p}\in H^{1,2}(Q)$ such that the triplet $(\hat{u},\hat{p},\hat{g})$ satisfies* (1.3) *with*

g replaced by $\hat{g}$, the linear backwards system

$$
\begin{gathered}
-\hat{p}_t - \Delta \hat{p} + \sigma \int_t^T e^{-\gamma(s-t)} \hat{p}(s)\, ds + D_u F_0(x,t;\hat{u})\, \hat{p} = \hat{u} - u_d, \\
\hat{p}(x,t)|_\Sigma = 0, \\
\hat{p}(x,T) = 0
\end{gathered} \tag{1.9}
$$

and the additional inequalities

$$
\iint_Q (\hat{p} + a\hat{g})(g - \hat{g})\, dx\, dt \geq 0 \quad \forall g \in \mathcal{G}_{ad}, \quad \hat{g} \in \mathcal{G}_{ad}. \tag{1.10}
$$

To apply the Dubovistky-Milyutin formalism, we first reformulate the control problem in the form

$$
\begin{gathered}
\text{Minimize } \mathcal{J}(u,g) \\
\text{subject to } (u,g) \in \mathcal{Q},\ g \in \mathcal{G}_{ad},
\end{gathered} \tag{1.11}
$$

where $\mathcal{G}_{ad}$ is (as before) a nonempty closed convex subset of $L^2(Q)$ (the control constraint set) and $\mathcal{Q}$ is given by an equality constraint:

$$
\mathcal{Q} = \{(u,g) \in H^{1,2}(Q) \times L^2(Q) : M(u,g) = 0\}
$$

for a suitable operator M.

Assume that $(\hat{u},\hat{g})$ is a local minimizer of (1.11). Then we associate to $(\hat{u},\hat{g})$ the cone K_0 of decreasing directions of $\mathcal{J}$, the cone K_1 of feasible directions of $\mathcal{G}_{ad}$ and the tangent subspace K_2 to the constraint set $\mathcal{Q}$. These cones are respectively given by (3.2), (3.4) and (3.6). We have the following (geometrical) necessary condition of optimality:

$$
K_0 \cap K_1 \cap K_2 = \emptyset.
$$

Accordingly, there must exist continuous linear functionals Φ_0, Φ_1 and Φ_2, not simultaneously zero, such that $\Phi_i \in K_i^*$ for $i = 1,2,3$ and

$$
\Phi_0 + \Phi_1 + \Phi_2 = 0
$$

(this is the Euler-Lagrange equation for the previous extremal problem). From this equation we obtain the optimality system (1.3) (with g replaced by $\hat{g}$), (1.9), (1.10).

A large family of control problems involving partial differential equations can be solved by this method. In particular, several interesting generalizations and modified versions of (1.3), (1.6) can be considered: other non-quadratic functionals, control problems with constraints on the state, multi-objective control problems, etc.

Remark 1.4. When $\mathcal{J}'(\hat{u},\hat{g}) = (0,0)$, it is natural to look for second-order optimality conditions. This can be made for this and many other problems following the results in [2]. An analysis of this situation will be given in a next paper.

Let us apply theorem 1.3 to some specific choices of $\mathcal{G}_{ad}$ we have made before:

- When $\mathcal{G}_{ad} = L^2(\omega \times (0,T))$, (1.10) is equivalent to

$$
\iint_{\omega\times(0,T)} (\hat{p} + a\hat{g}) h\, dx\, dt = 0 \quad \forall h \in L^2(\omega \times (0,T)), \quad \hat{g} \in L^2(\omega \times (0,T)),
$$

that is to say,

$$
\hat{g} = -\frac{1}{a}\, \hat{p}|_{\omega\times(0,T)}.
$$

• When $\mathcal{G}_{ad} = \{g \in L^2(Q) : 0 \le g \le M \text{ a.e.}\}$, (1.10) is equivalent to

$$\hat{g} = P_{[0,M]}(-\frac{1}{a}\,\hat{p}),$$

where $P_{[0,M]}$ is the usual (pointwise) projector on the closed interval $[0, M]$.

• Finally, when $\mathcal{G}_{ad} = \{g : g = \sum_{i=1}^{I} g_i(x)1_{(t_i-\epsilon,t_i+\epsilon)}(t),\quad g_i \in L^2(\Omega)\}$, we see that

$$\hat{g}(x,t) = -\frac{1}{2a\epsilon}\sum_{i=1}^{I}\Big(\int_{t_i-\epsilon}^{t_i+\epsilon}\hat{p}(x,s)\,ds\Big)1_{(t_i-\epsilon,t_i+\epsilon)}(t) \text{ a.e.}$$

Our third goal in this paper is related to the behavior of the solutions to problems of the kind (1.3), (1.6) as $a \to 0^+$. It is well known that this is a way to pass from the optimal control to a controllability approach. More precisely, if $\mathcal{G}_{ad} = L^2(\Omega)$, it is expected that the solutions $(\hat{u}, \hat{g})$ of (1.3), (1.6) satisfy $\hat{u} \to u_d$ as $a \to 0^+$.

A result of this kind is established in our next theorem. In order to give the statement, we have to introduce a new function:

$$H_0(x,t;s) = \begin{cases} \frac{F_0(x,t;s)-F_0(x,t;0)}{s} & \text{if } s \neq 0,\\ D_uF_0(x,t;0) & \text{otherwise.}\end{cases}$$

Then we have the following result.

Theorem 1.5. *Assume that $u^0 = 0$ and $u_d \in L^r(Q)$, where $r \in [4, +\infty)$. For each $n = 1, 2, \dots,$ let (u^n, p^n, g^n) be a solution of the coupled problem*

$$\begin{gathered} u_t^n - \Delta u^n + \sigma\int_0^t e^{-\gamma(t-s)}u^n(s)\,ds + F_0(x,t;u^n) = g^n,\\ -p_t^n - \Delta p^n + \sigma\int_t^T e^{-\gamma(s-t)}p^n(s)\,ds + H_0(x,t;u^n)\,p^n = |u^n-u_d|^{r-2}(u^n-u_d),\\ u^n(x,t)|_\Sigma = p^n(x,t)|_\Sigma = 0,\\ u^n(x,0) = 0,\quad p^n(x,T) = 0,\\ p^n + \frac{1}{n}g^n = 0. \end{gathered} \tag{1.12}$$

Then $u^n \to u_d$ strongly in $L^r(Q)$ as $n \to \infty$.

In this way, for any target $u_d \in L^r(Q)$ we can construct a sequence of (possibly unbounded) controls g^n and associated states u^n that converge to u_d. For each n, the task is reduced to solve the coupled system (1.12), where the genuine unknowns are the state u^n and the adjoint state p^n.

The proof of this theorem will be given below. It relies on some estimates of the functions u^n in $L^r(Q)$ and the functions p^n in $L^2(Q)$.

Remark 1.6. This result is inspired by the ideas of J.-L. Lions in the context of the approximate controllability of linear parabolic equations; see [16, 12].

Remark 1.7. The equation satisfied by p^n in (1.12) is not exactly the same satisfied by $\hat{p}$ in (1.9). First, we have a different right hand side. This is motivated by the search of a good estimate for u^n. Indeed, it will be seen in Section 4 that the term $|u^n-u_d|^{r-2}(u^n-u_d)$ with $r \ge 4$ is needed to bound u^n in $L^r(Q)$ and then $H_0(\cdot;u^n)$ in $L^2(Q)$. The second difference is that the coefficient of p^n in (1.12) is $H_0(x,t;u^n)$ and not $D_uF_0(x,t;u^n)$. This is also needed to estimate u^n.

This paper is organized as follows. Sections 2, 3 and 4 are respectively devoted to the proofs of theorems 1.1, 1.3 and 1.5. Then, we present in Section 5 several additional remarks and open questions. Among other things, we will address some controllability questions. It will be seen there that, unfortunately, very few is known on the subject.

2. Existence, uniqueness and regularity results

Assume that $g \in L^2(Q)$ and $u^0 \in H_0^1(\Omega)$ in (1.3). Notice that (1.3) can be written in the form

$$\begin{gathered} u_t - \Delta u + G(u) + F(u) = g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x), \end{gathered} \tag{2.1}$$

where we have set

$$G(u)(x,t) = \sigma \int_0^t e^{-\gamma(t-s)} u(x,s)\, ds\,, \tag{2.2}$$

$$F(u)(x,t) = F_0(x,t;u(x,t)). \tag{2.3}$$

We will first prove that (2.1) possesses at least one solution $u \in H^{1,2}(Q)$ with the help of the Leray-Schauder's principle.

Thus, let us consider for each $\lambda \in [0,1]$ the auxiliary problem

$$\begin{gathered} u_t - \Delta u = \lambda(g - G(u) - F(u)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x). \end{gathered} \tag{2.4}$$

Also, let us introduce the mapping $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$, with $u = \Lambda(w,\lambda)$ if and only if u is the unique solution to

$$\begin{gathered} u_t - \Delta u = \lambda(g - G(w) - F(w)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x). \end{gathered} \tag{2.5}$$

We will prove the following results.

Lemma 2.1. *The mapping $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$ is well-defined, continuous and compact.*

Lemma 2.2. *All functions u such that $\Lambda(u,\lambda) = u$ for some λ are uniformly bounded in $L^6(Q)$.*

In view of the Leray-Schauder's principle, this will suffice to affirm that (1.3) possesses at least one solution.

Proof of lemma 2.1. First, notice that for any $w \in L^6(Q)$ we have $F(w) \in L^2(Q)$ and $G(w) \in L^\infty(0,T;L^6(\Omega))$. Furthermore, the mappings $w \mapsto F(w)$ and $w \mapsto G(w)$ are continuous. Consequently, it is obvious that $(w,\lambda) \mapsto \Lambda(w,\lambda)$ is well-defined and continuous from $L^6(Q) \times [0,1]$ into $L^6(Q)$.

The compactness of Λ is a consequence of parabolic regularity. Indeed, if $(w,\lambda) \in L^6(Q) \times [0,1]$ and $u = \Lambda(w,\lambda)$, we have

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q),$$

i.e. $u \in H^{1,2}(Q)$ (we are using here that $u^0 \in H_0^1(\Omega)$).

Moreover, the estimates we will prove in lemma 2.2 show that, whenever (w, λ) belongs to a bounded set of $L^6(Q) \times [0,1]$, the associated u belongs to a bounded set in $H^{1,2}(Q)$. Since this space is compactly embedded in $L^6(Q)$ for $N = 1, 2$ or 3, we deduce that $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$ is compact. □

Proof of lemma 2.2. Let us assume that $\lambda \in [0,1]$, $u \in L^6(Q)$ and $\Lambda(u,\lambda) = u$, i.e. u solves (2.4). We will prove that, for some constant $C > 0$ independent of λ and u, one has

$$\|u\|_{L^6(Q)} \le C. \tag{2.6}$$

In fact, we will directly prove more: that u is uniformly bounded in $H^{1,2}(Q)$. Let us rewrite (2.4) in the form

$$\begin{gathered} u_t - \Delta u + \lambda v + \lambda F(u) = \lambda g, \\ v_t + \gamma v - \sigma u = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x), \quad v(x,0) = 0. \end{gathered} \tag{2.7}$$

Then, by multiplying by u (resp. $\frac{\lambda}{\sigma} v$) the first equation (resp. the second equation), integrating in Ω and adding the resulting identities, we get:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{\lambda}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\lambda\gamma}{\sigma}|v|^2 + \lambda(F(u), u) = \lambda(g, u) \quad \text{in } (0,T). \tag{2.8}$$

in $(0,T)$. Notice that, for any $\epsilon > 0$, there exists C_ϵ such that

$$(F(u), u) \ge (1-\epsilon)\|u\|_{L^4}^4 - C_\epsilon. \tag{2.9}$$

Indeed, we have for instance

$$\Big| \int_\Omega \psi_j\, u^3\, dx \Big| \le C\|\psi_j\|_{L^\infty} \|u\|_{L^4}^3 \le \frac{\epsilon}{8}\|u\|_{L^4}^4 + C_\epsilon$$

for any $j = 1, 2, 3$. In view of (2.8) and (2.9), we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{\lambda}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\lambda\gamma}{\sigma}|v|^2 + \lambda(1-\epsilon)\|u\|_{L^4}^4 \le \frac{1}{2}|\nabla u|^2 + \lambda C_\epsilon.$$

Since $\lambda \in [0,1]$, $\gamma \ge 0$ and $\sigma > 0$, from Gronwall's lemma the following is obtained:

$$\begin{gathered} \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H_0^1(\Omega))} \le C, \\ \lambda\|v\|_{L^\infty(0,T;H_0^1(\Omega))} + \lambda\|u\|_{L^4(Q)} \le C. \end{gathered} \tag{2.10}$$

Let us now multiply by u_t the first equation in (2.7) and let us integrate in Ω. We get

$$\frac{1}{2}|u_t|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \lambda(v, u_t) + \lambda(F(u), u_t) = \lambda(g, u_t) \ \text{ in } (0,T). \tag{2.11}$$

Now, we have

$$(F(u), u_t) \ge \frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 - \epsilon|u_t|^2 - C_\epsilon(1 + |\nabla u|^2) - C_\epsilon\|u\|_{L^4}^4 \tag{2.12}$$

since, for instance,

$$\Big| \int_\Omega \psi_j\, u^2 u_t\, dx \Big| \le C\|\psi_j\|_{L^\infty} \|u\|_{L^4}^2 |u_t| \le \frac{\epsilon}{8}|u_t|^4 + C_\epsilon\|u\|_{L^4}^4$$

for any $j = 1, 2, 3$. On the other hand,

$$|(v, u_t)| \le \epsilon |u_t|^2 + C_\epsilon |v|^2 \le \epsilon |u_t|^2 + C_\epsilon. \tag{2.13}$$

From (2.11)–(2.13), we obtain the inequality

$$(1 - 2\epsilon)|u_t|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \frac{\lambda}{4}\frac{d}{dt}\|u\|_{L^4}^4 \le C_\epsilon(1 + |\nabla u|^2) + \lambda C_\epsilon \|u\|_{L^4}^4$$

and, from Gronwall's lemma and (2.10), we find:

$$\|u_t\|_{L^2(Q)} + \|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \lambda \|u\|_{L^\infty(0,T;L^4(\Omega))} \le C. \tag{2.14}$$

Note that we have used here again the fact that $u^0 \in H_0^1(\Omega)$.

In view of the first estimate in (2.10), u is uniformly bounded in $L^2(0, T; L^6(\Omega))$ and $F(u)$ is uniformly bounded in $L^2(Q)$. Therefore, taking into account (2.10), (2.14) and the identity

$$\Delta u = u_t + \lambda v + \lambda F(u) - \lambda g,$$

we see that Δu is uniformly bounded in $L^2(Q)$, that is,

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C.$$

This completes the proof. $\square$

Let us now see that the solution we have found is unique. Thus, let u^1 and u^2 be two solutions (in $H^{1,2}(Q)$) of (1.3) and let us set $u = u^1 - u^2$. Let us also introduce

$$v = v^1 - v^2 = \sigma \int_0^t e^{-\gamma(t-s)}(u^1(s) - u^2(s))\, ds.$$

Then the following holds:

$$\begin{gathered} u_t - \Delta u + v + F(u^1) - F(u^2) = 0, \\ v_t + \gamma v - \sigma u = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = 0, \quad v(x,0) = 0. \end{gathered}$$

Consequently, by multiplying the first and second equations respectively by u and $\frac{1}{\sigma} v$ and integrating in Ω, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 + (F(u^1) - F(u^2), u) = 0. \tag{2.15}$$

We have

$$\begin{aligned} &(F(u^1) - F(u^2), u) \\ &= \int_\Omega \big[(u^1 + \psi_1)(u^1 + \psi_2)(u^1 + \psi_3) - (u^2 + \psi_1)(u^2 + \psi_2)(u^2 + \psi_3)\big]\, u\, dx \\ &= I_0 + \sum_{j=1}^3 I_j + \sum_{1 \le j < k \le 3} I_{j,k}, \end{aligned}$$

where

$$I_0 = \int_\Omega \big((u^1)^3 - (u^2)^3\big)\,(u^1 - u^2)\, dx, \quad I_j = \int_\Omega \psi_j (u^1 + u^2)|u^1 - u^2|^2\, dx$$

for $1 \le j \le 3$ and

$$I_{j,k} = \int_\Omega \psi_j \psi_k |u^1 - u^2|^2\, dx$$

for $1 \le j < k \le 3$. Since $I_0 \ge 0$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2\\&\le C\int_\Omega (1 + |u^1| + |u^2|)\,|u|^2\,dx \le \|\beta(t)\|_{L^\infty}|u|^2,\end{aligned}$$

where the function β belongs to $L^2(0,T;L^\infty(\Omega))$. Since $u(x,0) \equiv 0$ and $v(x,0) \equiv 0$, we deduce that u vanishes identically, whence $u^1 = u^2$. Hence, (1.3) possesses exactly one solution in $H^{1,2}(Q)$.

Remark 2.3. Instead of (1.1), we could have started from the more general system

$$\begin{gathered}u_t - \Delta u + v + F_0(x,t;u) = g,\\ v_t - \sigma u + \gamma v = \tilde{g},\\ u(x,t)|_\Sigma = 0,\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x),\end{gathered}\tag{2.16}$$

where $\tilde{g} \in L^1(0,T;L^2(\Omega))$, $v^0 \in L^2(\Omega)$. Then, the problem is reduced again to a system of the form (1.3), with g replaced by

$$\overline{g} = g - v^0(x)e^{-\gamma t} - \int_0^t e^{-\gamma(t-s)}\tilde{g}(s)\,ds$$

(which again belongs to $L^2(Q)$). Indeed, the unique solution of (2.16) is (u,v), where u is the solution of (1.3) with g replaced by $\overline{g}$ (this is furnished by theorem 1.1) and

$$v = v^0(x)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\tilde{g}(s)\,ds + \sigma\int_0^t e^{-\gamma(t-s)}u(s)\,ds.$$

3. An optimal control problem. The Dubovitski-Milyoutin formalism

Let us consider the optimal control problem

$$\begin{gathered}\text{Minimize } \mathcal{J}(u,g)\\ \text{subject to } g \in \mathcal{G}_{ad}, \quad (u,g) \in \mathcal{Q},\end{gathered}\tag{3.1}$$

where $\mathcal{G}_{ad} \subset L^2(Q)$ is a nonempty closed convex set and $\mathcal{Q}$ is given by (1.7).

The proof of the existence of at least one (global) optimal state-control $(\hat{u},\hat{g})$ is completely standard. For completeness, let us sketch the argument.

Let $\{(u^n,g^n)\}$ be a minimizing sequence for (1.3), (1.6). This means that $(u^n,g^n) \in \mathcal{U}_{ad}$ for all n and

$$\lim_{n\to\infty}\mathcal{J}(u^n,g^n) = \mathcal{J}_* := \inf_{\mathcal{U}_{ad}}\mathcal{J}$$

($\mathcal{U}_{ad}$ is given by (1.8)). Then, it is immediate that g^n is uniformly bounded in $L^2(Q)$. Taking into account the estimates in Section 2, we see that u^n is uniformly bounded in $H^{1,2}(Q)$ and the sequence $\{u^n\}$ is relatively compact in $L^6(Q)$. Therefore, at least for a subsequence, we have

$$g^n \to \hat{g} \quad \text{weakly in } L^2(Q)$$

and

$$u^n \to \hat{u} \quad \text{weakly in } H^{1,2}(Q) \quad \text{and strongly in } L^6(Q),$$

for some $(\hat{u},\hat{g}) \in H^{1,2}(Q) \times L^2(Q)$. Obviously, $\hat{g} \in \mathcal{G}_{ad}$. Furthermore, in view of the strong convergence of u^n in $L^6(Q)$, we can take limits in the equation satisfied by u^n and deduce that $\hat{u}$ is the state associated to $\hat{g}$. This shows that $(\hat{u},\hat{g}) \in \mathcal{U}_{ad}$.

On the other hand,

$$\mathcal{J}(\hat{u},\hat{g}) \le \liminf_{n\to\infty} \mathcal{J}(u^n, g^n) = \mathcal{J}_*,$$

whence $(\hat{u},\hat{g})$ is an optimal state-control.

To our knowledge, the uniqueness of optimal control is an open question.

Now, let $(\hat{u},\hat{g})$ be a local optimal state-control. Let us prove that the optimality system (1.3), (1.9), (1.10) holds. For simplicity, we will assume that $\mathcal{G}_{ad}$ has nonempty interior; otherwise, it would suffice to argue as in [9].

As mentioned above, our approach will rely on the Dubovitskii-Milyoutin formalism. Thus let us introduce the cone K_0 of decreasing directions of $\mathcal{J}$ at $(\hat{u},\hat{g})$:

$$\begin{aligned} K_0 = \{(w,h) \in L^2(Q) \times L^2(Q) : \exists \delta_0 > 0 \text{ such that} \\ \mathcal{J}((\hat{u},\hat{g}) + \delta(w,h)) < \mathcal{J}(\hat{u},\hat{g}) \quad \text{for} \quad 0 < \delta \le \delta_0\}. \end{aligned} \tag{3.2}$$

Since $\mathcal{J}$ is Fréchet-differentiable at any point, it is immediate that

$$K_0 = \{(w,h) \in L^2(Q) \times L^2(Q) : \langle \mathcal{J}'(\hat{u},\hat{g}), (w,h)\rangle < 0\}. \tag{3.3}$$

Let us also introduce the cone of feasible directions of $\mathcal{G}_{ad}$ at $\hat{g}$. This is the set

$$\begin{aligned} K_1 = \{(w,h) \in L^2(Q) \times L^2(Q) : \exists \delta_1 > 0 \quad \text{such that} \\ \hat{g} + \delta h \in \mathcal{G}_{ad} \quad \text{for} \quad 0 < \delta \le \delta_1\}. \end{aligned} \tag{3.4}$$

Since $\mathcal{G}_{ad}$ has nonempty interior, it is clear that

$$K_1 = \{(w, \lambda(g - \hat{g})) : w \in L^2(Q),\ \lambda > 0,\ g \in \operatorname{int} \mathcal{G}_{ad}\}. \tag{3.5}$$

Finally, let us consider the cone K_2 of tangent directions of $\mathcal{Q}$ at $(\hat{u},\hat{g})$. This is given as follows:

$$\begin{aligned} K_2 = \{(w,h) \in H^{1,2}(Q) \times L^2(Q) : \exists \theta^n, (u^n, g^n) \text{ for } n = 1, 2, \dots \\ \text{with } \theta^n \to 0,\ (u^n, g^n) \in \mathcal{Q} \text{ and} \\ \lim_{n\to\infty} \frac{1}{\theta^n}\,[(u^n, g^n) - (\hat{u},\hat{g})] = (w,h)\}. \end{aligned} \tag{3.6}$$

To give a more explicit description of K_2, it is convenient to introduce the spaces

$$E_1 = H^{1,2}(Q) \times L^2(Q), \quad E_2 = L^2(Q) \times H^1_0(\Omega)$$

and the nonlinear mapping $M : E_1 \mapsto E_2$, with

$$M(u,g) = (u_t - \Delta u + G(u) + F(u) - g, u|_{t=0} - u^0) \quad \forall (u,g) \in E_1. \tag{3.7}$$

Let us also set

$$F'(u)(x,t) \equiv D_u F_0(x,t; u(x,t)).$$

Then we have the following result.

Lemma 3.1. *The mapping M is continuously differentiable in E_1 and*

$$\begin{aligned} M'(u,g)(w,h) = (w_t - \Delta w + G(w) + F'(u)w - h, w|_{t=0}) \\ \forall (u,g) \in E_1, \quad (w,h) \in E_1. \end{aligned} \tag{3.8}$$

Furthermore, for each $(u,g) \in E_1$ the linear operator $M'(u,g) : E_1 \mapsto E_2$ is onto.

Proof. There is only one nontrivial step in the proof of this lemma. Indeed, it is clear that $M : E_1 \mapsto E_2$ is well-defined and continuously differentiable. It is also clear that its F-derivative is given by (3.8).

To see that $M'(u,g)$ is an epimorphism, let (k, w^0) be given in $L^2(Q) \times H^1_0(\Omega)$ and let us consider the linear problem

$$\begin{gathered} w_t - \Delta w + G(w) + F'(u)w = k, \\ w(x,t)|_\Sigma = 0, \\ w(x,0) = w^0(x), \end{gathered} \tag{3.9}$$

All we have to do is to prove that (3.9) possesses at least one solution $w \in H^{1,2}(Q)$.

Note that, in this system, $F'(u) \in L^\infty(0,T;L^3(\Omega)) \cap L^1(0,T;L^\infty(\Omega)) \hookrightarrow L^4(Q)$. This is sufficient to prove the existence of a weak solution; i.e., a solution in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Indeed, to get energy estimates, we multiply the first equation in (3.9) by w and we integrate in Ω. All the terms can be estimated easily except possibly $(F'(u)w,w)$. But this one satisfies

$$|(F'(u)w,w)| \le C\|F'(u)\|_{L^4}\,|w|^{5/4}|\nabla w|^{3/4} \le \epsilon|\nabla w|^2 + C_\epsilon\|F'(u)\|^{8/5}_{L^4}|w|^2,$$

which leads to the the usual estimates for w.

Now, observe that w can be regarded as the solution of

$$\begin{gathered} w_t - \Delta w = k - G(w) - F'(u)w, \\ w(x,t)|_\Sigma = 0, \\ w(x,0) = w^0(x). \end{gathered} \tag{3.10}$$

Obviously, $G(w) \in L^\infty(0,T;H^1_0(\Omega))$.

On the other hand, since $F'(u) \in L^\infty(0,T;L^3(\Omega))$ and $w \in L^2(0,T;L^6(\Omega))$, we also have $F'(u)w \in L^2(Q)$. Consequently, the right hand side of (3.10) belongs to $L^2(Q)$ and, from the well known parabolic regularity theory, we deduce that $w \in H^{1,2}(Q)$. This completes the proof. $\square$

Notice that $\mathcal{Q}$ can be written in the form

$$\mathcal{Q} = \{(u,g) \in H^{1,2}(Q) \times L^2(Q) : M(u,g) = 0\}. \tag{3.11}$$

Therefore, in view of Lemma 3.1 and the results in [11], the tangent cone at $(\hat u, \hat g)$ is

$$K_2 = \{(w,h) \in H^{1,2}(Q) \times L^2(Q) : M'(\hat u,\hat g)(w,h) = 0\}. \tag{3.12}$$

In view of (3.3), (3.5) and (3.12), it is easy to determine the dual cones K_i^* for $i = 0,1,2$. Specifically, we have:

$$\begin{gathered} K_0^* = \{-\lambda \mathcal{J}'(\hat u,\hat g) : \lambda \ge 0\}, \qquad (3.13) \\ K_1^* = \{(0,f) : f \in L^2(Q) : \iint_Q f g\,dx\,dt \ge \iint_Q f\hat g\,dx\,dt \ \ \forall g \in \mathcal{G}_{ad}\}, \\ K_2^* = \{\Phi \in E_1' : \langle \Phi,(w,h)\rangle = 0 \ \ \forall (w,h) \in E_1 \ \text{ such that } \ M'(\hat u,\hat g)(w,h) = 0\}. \end{gathered}$$

We can now apply the main result in [11]. Thus, for some $(f_{01}, f_{02}) \in K_0^*$, $(0, f_{12}) \in K_1^*$ and $\Phi_2 \in K_2^*$ not vanishing simultaneously, one has:

$$\begin{gathered}\iint_Q (f_{01} w + f_{02} h)\, dx\, dt + \iint_Q f_{12} h\, dx\, dt + \langle \Phi_2, (w,h)\rangle = 0 \\ \forall (w,h) \in E_1 = H^{1,2}(Q) \times L^2(Q).\end{gathered} \tag{3.14}$$

Let us now see that (3.14) leads to (1.3), (1.9), (1.10). In view of (3.13), there exists $\lambda_0 \geq 0$ such that

$$(f_{01}, f_{02}) = -\lambda_0\, (\hat{u} - u_d, a\hat{g}).$$

Let us choose $(w,h) \in E_1$ such that $M'(\hat{u},\hat{g})(w,h) = 0$. Then

$$-\lambda_0 \iint_Q ((\hat{u} - u_d) w + a\hat{g} h)\, dx\, dt + \iint_Q f_{12} h\, dx\, dt = 0. \tag{3.15}$$

But this implies that $\lambda_0 > 0$; otherwise, we would have $(f_{01}, f_{02}) = (0,0)$, $f_{12} = 0$ (by (3.15)) and $\Phi_2 = 0$ (by (3.14)). Consequently, we can assume that $\lambda_0 = 1$ and

$$\begin{gathered}\iint_Q f_{12} h\, dx\, dt = \iint_Q ((\hat{u} - u_d) w + a\hat{g} h)\, dx\, dt \\ \forall (w,h) \in E_1 \quad \text{such that} \quad M'(\hat{u},\hat{g})(w,h) = 0.\end{gathered} \tag{3.16}$$

Let us introduce the adjoint system

$$\begin{gathered}-\hat{p}_t - \Delta \hat{p} + \sigma \int_t^T e^{-\gamma(s-t)} \hat{p}(s)\, ds + D_u F_0(x,t;\hat{u})\, \hat{p} = \hat{u} - u_d, \\ \hat{p}(x,t)|_\Sigma = 0, \\ \hat{p}(x,T) = 0.\end{gathered} \tag{3.17}$$

Then, for any $(w,h) \in E_1$ such that $M'(\hat{u},\hat{g})(w,h) = 0$ one has

$$\begin{aligned}&\iint_Q (\hat{u} - u_d) w\, dx\, dt \\ &= \iint_Q \Big(-\hat{p}_t - \Delta \hat{p} + \sigma \int_t^T e^{-\gamma(s-t)} \hat{p}(s)\, ds + D_u F_0(x,t;\hat{u})\, \hat{p} \Big) w\, dx\, dt \\ &= \iint_Q \hat{p} \Big(w_t - \Delta w + \sigma \int_0^t e^{-\gamma(t-s)} w(s)\, ds + D_u F_0(x,t;\hat{u})\, w \Big)\, dx\, ds \\ &= \iint_Q \hat{p} h\, dx\, dt.\end{aligned}$$

Hence,

$$\iint_Q f_{12} h\, dx\, dt = \iint_Q (\hat{p} + a\hat{g}) h\, dx\, dt \quad \forall h \in L^2(Q).$$

From the fact that $(0, f_{12}) \in K_1^*$, we also have

$$\iint_Q (\hat{p} + a\hat{g})(g - \hat{g})\, dx\, dt \geq 0 \quad \forall g \in \mathcal{G}_{ad}. \tag{3.18}$$

Thus, the triplet $(\hat{u}, \hat{p}, \hat{g})$ satisfies (1.3) (with g replaced by $\hat{g}$), (3.17) and (3.18) and this is what we wanted to prove.

4. A controllability question

In this section we prove theorem 1.5. Let us assume that $u^0 = 0$ and $u_d \in L^r(Q)$ with $r \geq 4$. For each $n \geq 1$, let us consider the coupled system (1.12). Notice that it can be written in the form

$$
\begin{gathered}
u_t^n - \Delta u^n + \sigma \int_0^t e^{-\gamma(t-s)} u^n(s)\,ds + F_0(x,t;u^n) = g^n, \\
-p_t^n - \Delta p^n + \sigma \int_t^T e^{-\gamma(s-t)} p^n(s)\,ds + H_0(x,t;u^n)p^n = |u^n - u_d|^{r-2}(u^n - u_d), \\
u^n(x,t)|_\Sigma = p^n(x,t)|_\Sigma = 0, \\
u^n(x,0) = 0, \quad p^n(x,T) = 0,
\end{gathered}
\tag{4.1}
$$

with $g^n = -np^n$.

Let us first show that, for each $n \geq 1$, there exists at least one solution of (4.1), with

$$
\begin{gathered}
u^n \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t^n \in L^2(Q), \\
p^n \in L^{r'}(0,T;W^{2,r'}(\Omega)), \quad p_t^n \in L^{r'}(Q).
\end{gathered}
\tag{4.2}
$$

For this end, we can argue as in the proof of theorem 1.1. Thus, let us set

$$
H(u)(x,t) \equiv H_0(x,t;u(x,t))
$$

and let us introduce the space $E = L^6(Q) \times L^2(Q) \times L^2(Q)$ and the mapping $\Xi : E \times [0,1] \mapsto E$, with $(u,p,g) = \Xi(w,q,h,\lambda)$ if and only if u is the unique solution to

$$
\begin{gathered}
u_t - \Delta u = \lambda\Big(h - \sigma \int_0^t e^{-\gamma(t-s)} w(s)\,ds - F(w)\Big), \\
u(x,t)|_\Sigma = 0, \\
u(x,0) = 0
\end{gathered}
\tag{4.3}
$$

and $g = -np$, where p is the unique solution to

$$
\begin{gathered}
-p_t - \Delta p = \lambda\Big(|w - u_d|^{r-2}(w - u_d) - \sigma \int_t^T e^{-\gamma(s-t)} q(s)\,ds - H(w)q\Big), \\
p(x,t)|_\Sigma = 0, \\
p(x,T) = 0
\end{gathered}
\tag{4.4}
$$

Then we have the following results.

Lemma 4.1. *The mapping $\Xi : E \times [0,1] \mapsto E$ is well-defined, continuous and compact.*

Lemma 4.2. *All (u,p,g) such that $\Xi(u,p,g,\lambda) = (u,p,g)$ for some λ are uniformly bounded in E.*

In view of the Leray-Schauder's principle, this yields the desired existence result for (4.1).

Proof of lemma 4.1. It is very similar to the proof of Lemma 2.1. If $(u,p,g) \in E$ and $\lambda \in [0,1]$, then the solution of (4.3) is well defined and satisfies

$$
u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q).
$$

On the other hand, since $H(w) \in L^3(Q)$ (and consequently $H(w)q \in L^{6/5}(Q)$) and $|w-u_d|^{r-2}(w-u_d) \in L^{r'}(Q)$, (4.4) possesses exactly one solution p, with

$$p \in L^m(0,T;W^{2,m}(\Omega)), \quad p_t \in L^m(Q), \tag{4.5}$$

where $m = \min(r', 6/5)$. Notice that the space of functions satisfying (4.5) is compactly embedded in $L^2(Q)$. Therefore, g is also well defined through the equality $g = -np$. Obviously, this construction shows that the mapping $(w,q,h,\lambda) \mapsto (u,p,g)$ is continuous and compact. □

Proof of lemma 4.2. Assume $\lambda \in [0,1]$, $(u,p,g) \in E$ and $\Xi(u,p,g,\lambda) = (u,p,g)$. This implies that u and p solve the problem

$$\begin{gathered} u_t - \Delta u = \lambda\Big(-np - \sigma\int_0^t e^{-\gamma(t-s)}u(s)\,ds - F(u)\Big),\\ -p_t - \Delta p = \lambda\Big(|u-u_d|^{r-2}(u-u_d) - \sigma\int_t^T e^{-\gamma(s-t)}p(s)\,ds - H(u)p\Big),\\ u(x,t)|_\Sigma = p(x,t)|_\Sigma = 0,\\ u(x,0) = 0, \quad p(x,T) = 0 \end{gathered} \tag{4.6}$$

and $g = -np$.

Let us prove that u (resp. p) is bounded in $L^6(Q)$ (resp. $L^2(Q)$) by a constant that can depend on n but is independent of λ. This will suffice to prove the lemma. Obviously, if $\lambda = 0$, then $u \equiv 0$ and $p \equiv 0$. Consequently, it can be assumed that $\lambda > 0$.

Let us multiply the first (resp. the second) equation in (4.6) by p (resp. u). Let us sum the resulting identities and let us integrate with respect to x and t in Q. After some short computations, in view of the definition of $H(u)$, and the fact that $u(x,0) = p(x,T) = 0$ in Ω, the following is found:

$$\lambda\iint_Q |u-u_d|^{r-2}(u-u_d)u\,dx\,dt + \lambda n\iint_Q |p|^2\,dx\,dt = -\lambda\iint_Q F(0)\,p\,dx\,dt$$

(note that $H(u)pu = F(u)p - F(0)p$). Consequently,

$$\begin{aligned} &\iint_Q |u-u_d|^r\,dx\,dt + n\iint_Q |p|^2\,dx\,dt\\ &= -\iint_Q |u-u_d|^{r-2}(u-u_d)u_d\,dx\,dt - \iint_Q F(0)\,p\,dx\,dt. \end{aligned} \tag{4.7}$$

Observe that, in view of Hölder's and Young's inequalities, the hand side in (4.7) is bounded by

$$\frac{1}{2}\iint_Q |u-u_d|^r\,dx\,dt + \frac{n}{2}\iint_Q |p|^2\,dx\,dt + C\left(\|u_d\|^r_{L^r(Q)} + \|F(0)\|^2_{L^2(Q)}\right).$$

Hence,

$$\iint_Q |u-u_d|^r\,dx\,dt + n\iint_Q |p|^2\,dx\,dt \le C, \tag{4.8}$$

where the constant C is independent of λ and n.

From (4.8), arguing as in the proof of lemma 2.2, we deduce that u is in fact bounded in $L^6(Q)$ by a constant that can depend on n. Obviously, we also obtain from (4.8) that the norm of p in $L^2(Q)$ is uniformly bounded. Then, arguing as in the proof of Lemma 3.1, the same is found for p. This completes the proof. □

Let us now complete the proof of theorem 1.5. For each n, let (u^n, p^n, g^n) be a solution of (1.12). Then, the identity (4.7) and the estimate (4.8) hold for (u^n, p^n):

$$\begin{aligned} &\iint_Q |u^n - u_d|^r \,dx\,dt + n \iint_Q |p^n|^2 \,dx\,dt \\ &= -\iint_Q |u^n - u_d|^{r-2}(u^n - u_d)u_d \,dx\,dt - \iint_Q F(0)\, p^n \,dx\,dt \end{aligned} \tag{4.9}$$

and

$$\iint_Q |u^n - u_d|^r \,dx\,dt + n \iint_Q |p^n|^2 \,dx\,dt \le C. \tag{4.10}$$

Accordingly, u^n is uniformly bounded in $L^r(Q)$ and $p^n \to 0$ strongly in $L^2(Q)$ as $n \to +\infty$.

Let us look at the equation satisfied by p^n in Q:

$$-p^n_t - \Delta p^n + \sigma \int_t^T e^{-\gamma(s-t)} p^n(s)\,ds + H(u^n)\,p^n = |u^n - u_d|^{r-2}(u^n - u_d).$$

In the left-hand side, the first three terms converge to zero in the distribution sense. This is also the case for the fourth one, since $H(u^n)$ is uniformly bounded in $L^2(Q)$ (it is just at this point where we use that $r \ge 4$). Consequently, the right hand side also converges to zero. Since it is bounded in $L^{r'}(Q)$, it converges weakly to zero in this space (r' is the conjugate exponent of r). But this implies that u^n converges strongly to u_d in $L^r(Q)$. Indeed, from (4.9), the weak convergence of $|u^n - u_d|^{r-2}(u^n - u_d)$ and the fact that $u_d \in L^r(Q)$, we see that

$$\iint_Q |u^n - u_d|^r \,dx\,dt + n \iint_Q |p^n|^2 \,dx\,dt \to 0.$$

This completes the proof.

5. Final remarks and open problems

This Section is devoted to discuss some additional facts concerning the control of (1.3). Some of them lead to open problems that, in our opinion, are of considerable interest.

5.1. **Other optimal control problems.** There are many other optimal control problems that can be considered for systems of the kind (1.3). Let us mention one of them. Thus, consider the new cost functional $\mathcal{K}$, where

$$\mathcal{K}(u,g) = \frac{1}{2}\int_\Omega \left|u(x,T) - u^1(x)\right|^2 dx + \frac{a}{2}\iint_Q |g|^2 \,dx\,dt \tag{5.1}$$

and $u^1 \in L^2(\Omega)$ is a given function. The following result holds.

Theorem 5.1. *Assume that $u^0 \in H^1_0(\Omega)$ and $\mathcal{G}_{ad} \subset L^2(Q)$ is a nonempty closed convex set. Then there exists at least one global optimal state-control $(\hat u, \hat g)$ of (1.3), (5.1). Furthermore, if $(\hat u, \hat g)$ is a local optimal state-control, $\mathcal{G}_{ad}$ has nonempty interior and $\mathcal{K}'(\hat u, \hat g)$ does not vanish, there exists $\hat p \in H^{1,2}(Q)$ such that the triplet*

$(\hat u,\hat p,\hat g)$ satisfies (1.3) with g replaced by $\hat g$, the linear backwards system

$$\begin{gathered} -\hat p_t-\Delta\hat p+\sigma\int_t^T e^{-\gamma(s-t)}\hat p(s)\,ds+D_uF_0(x,t;\hat u)\,\hat p=0,\\ \hat p(x,t)|_\Sigma=0,\\ \hat p(x,T)=\hat u(x,T)-u^1(x) \end{gathered}\tag{5.2}$$

and the additional inequalities

$$\iint_Q(\hat p+a\hat g)(g-\hat g)\,dx\,dt\ge 0\quad\forall g\in\mathcal{G}_{ad},\quad \hat g\in\mathcal{G}_{ad}.\tag{5.3}$$

The optimal control problem (1.3), (5.1) can be viewed as a first step towards the solution of a controllability problem for (1.3); see the next paragraphs. Contrarily to what was considered before, we are now accepting g as a "good" control if it drives the solution u to a final state $u(\cdot,T)$ reasonably close to u^1 and, moreover, its norm is not too large.

We can get a result similar to theorem 1.5 that provides a sequence of controls g_n and associated states u^n that converge globally in Q to a desired state u_d and, simultaneously, converge at $t=T$ to a desired final state u^1. More precisely, we have the following result.

Theorem 5.2. *Assume that $u^0=0$, $u_d\in L^r(Q)$ with $r\in[4,+\infty)$ and $u^1\in L^2(\Omega)$. For each $n=1,2,\dots$, let (u^n,p^n,g^n) be a solution of the coupled problem*

$$\begin{gathered} u^n_t-\Delta u^n+\sigma\int_0^t e^{-\gamma(t-s)}u^n(s)\,ds+F_0(x,t;u^n)=g^n,\\ -p^n_t-\Delta p^n+\sigma\int_t^T e^{-\gamma(s-t)}p^n(s)\,ds+H_0(x,t;u^n)\,p^n=|u^n-u_d|^{r-2}(u^n-u_d),\\ u^n(x,t)|_\Sigma=p^n(x,t)|_\Sigma=0,\\ u^n(x,0)=0,\quad p^n(x,T)=u^n(x,T)-u^1(x),\\ p^n+\frac1n g^n=0. \end{gathered}\tag{5.4}$$

Then $u^n\to u_d$ strongly in $L^r(Q)$ and $u^n(\cdot,T)\to u^1$ strongly in $L^2(\Omega)$) as $n\to\infty$.

The proof is very similar to the proof of theorem 1.5. This time, instead of (4.7), we find that

$$\begin{aligned} &\iint_Q|u-u_d|^r\,dx\,dt+n\iint_Q|p|^2\,dx\,dt+|u(\cdot,T)-u^1|^2\\ &=-\iint_Q|u-u_d|^{r-2}(u-u_d)u_d\,dx\,dt-\iint_Q F(0)\,p\,dx\,dt-(u(\cdot,T)-u^1,u^1). \end{aligned}\tag{5.5}$$

For this it suffices to prove a lemma similar to lemma 4.2. On the other hand, if (u^n,p^n,g^n) solves (5.4), from the identity (5.5) with u and p respectively replaced by u^n and p^n, we easily deduce that

$$\iint_Q|u^n-u_d|^r\,dx\,dt+n\iint_Q|p^n|^2\,dx\,dt+|u^n(\cdot,T)-u^1|^2\le C.\tag{5.6}$$

Thus, arguing as in the final part of the proof of theorem 1.5, we find that

$$\iint_Q |u^n - u_d|^r \,dx\,dt + n \iint_Q |p^n|^2 \,dx\,dt + |u^n(\cdot,T) - u^1|^2 \to 0.$$

5.2. **Further comments on controllability.** In general terms, the controllability problem for an evolution partial differential equation or system consists in trying to drive the system from a prescribed initial state at time $t = 0$ (u^0 in our case) to a *desired* final state (or, at least "near" a desired final state) at time $t = T$. In the interesting case, the control is supported by a set of the form $\omega \times (0,T)$, where $\omega \subset \Omega$ is a nonempty (small) open set.

Nowadays, controllability problems are relatively well understood for linear and semilinear parabolic equations; see for instance [6, 8, 5]. Unfortunately, this is not the case for the integro-differential system (1.3), not even for simplified (linearized) similar problems. For instance, consider the linear system

$$\begin{gathered} u_t - \Delta u + \sigma \int_0^t e^{-\gamma(t-s)} u(s)\,ds + \alpha(x,t)u = g1_\omega, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x), \end{gathered} \tag{5.7}$$

where $\alpha \in L^\infty(Q)$ and 1_ω is the characteristic function of ω.

It is said that this system is *approximately controllable* in $L^2(\Omega)$ at time T if, for any $u^1 \in L^2(\Omega)$ and any $\epsilon > 0$, there exists $g \in L^2(\omega \times (0,T))$ such that the corresponding solution satisfies

$$|u(\cdot,T) - u^1| \le \epsilon.$$

To our knowledge, it is unknown whether (5.7) is approximately controllable. Observe that (5.7) can be equivalently written in the form

$$\begin{gathered} u_t - \Delta u + v + \alpha(x,t)u = g1_\omega, \\ v_t - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u^0(x), \quad v(x,0) = 0. \end{gathered} \tag{5.8}$$

Hence, it can be regarded as the singular limit of the family of reaction-diffusion systems

$$\begin{gathered} u_t - \Delta u + v + \alpha(x,t)u = g1_\omega, \\ v_t - k\Delta v - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \quad \frac{\partial}{\partial n} v(x,t)|_\Sigma = 0 \\ u(x,0) = u^0(x), \quad v(x,0) = 0 \end{gathered} \tag{5.9}$$

as $k \to 0^+$.

Indeed, it is not difficult to prove that, for any $g \in L^2(Q)$, any $u^0 \in L^2(\Omega)$ and any $k > 0$, (5.9) possesses exactly one solution (u^k, v^k), with

$$u^k, v^k \in C^0([0,T]; L^2(\Omega)). \tag{5.10}$$

It is also easy to show that u^k and v^k are uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, respectively. As a consequence, as $k \to 0^+$, (u^k, v^k) converges in an appropriate sense to the unique solution (u, v) of (5.8).

The standard results concerning the approximate controllability of parabolic equations and systems can be applied to (5.9); see [6, 8]. In particular, for any $u^0, u^1 \in L^2(\Omega)$ and any $\epsilon > 0$, there exist controls $g^k \in L^2(\omega \times (0,T))$ such that the associated solutions of (5.9) satisfy

$$|u^k(\cdot,T) - u^1| \le \epsilon.$$

Obviously, in order to establish an approximate controllability result for (5.8), it suffices to prove that, for some controls g^k with these properties, one has

$$\|g^k\|_{L^2(\omega\times(0,T))} \le C.$$

But, at the present, this is unknown. Of course, the approximate controllability of the nonlinear system (1.3) with g replaced by $g1_\omega$ is completely open.

5.3. **Time-independent coefficients.** If, in (5.7), the coefficient α is independent of t, the approximate controllability property is satisfied. A sketch of the proof of this fact is as follows (see [3] and [4] for some related results). Let us consider the adjoint system

$$\begin{gathered}
-h_t - \Delta h + \sigma \int_t^T e^{-\gamma(s-t)} h(s)\,ds + \alpha(x) h = 0,\\
h(x,t)|_\Sigma = 0,\\
h(x,T) = h^0(x),
\end{gathered} \tag{5.11}$$

From classical results, we know that what we have to prove is the following *unique continuation property:*

> Let $h^0 \in L^2(\Omega)$ be given, let h be the associated solution of (5.11) and let us assume that $h = 0$ in $\omega \times (0,T)$. Then $h \equiv 0$.

The function $t \mapsto h(\cdot,t)$, regarded as a mapping from $(-\infty,T)$ into $L^2(\Omega)$, is analytic. This is because $h(\cdot,t)$ can be written as the sum of a series that converges normally and uniformly on any compact set in $(-\infty,T)$ and each term of the series is analytic in t.

Indeed, let us denote by (θ_n,λ_n) the n-th eigenfunction-eigenvalue pair for the elliptic operator $-\Delta + \alpha(x)$ with Dirichlet boundary conditions and let us set $h_{0n} = (h^0,\theta_n)$ for each n. We have

$$h(\cdot,t) = \sum_{n\ge1} \frac{h_{0n}}{\zeta_n}\,\theta_n(x)\,\left(\mu_n^+ e^{\mu_n^+(T-t)} - \mu_n^- e^{\mu_n^-(T-t)}\right)$$

for some ζ_n, μ_n^+ and μ_n^- satisfying

$$\zeta_n \sim \lambda_n, \quad \mu_n^+ \sim -C, \quad \mu_n^- \sim -\lambda_n \quad \text{as } n \to +\infty$$

(recall that $\lambda_n \sim n^{2/N}$). Therefore, the L^2-norm of the n-th term is bounded in each compact set $S \subset (-\infty,T)$ by a constant times

$$\frac{|h_{0n}|}{\lambda_n} + |h_{0n}| e^{\mu_n^- a}$$

where a depends on S. This proves that $t \mapsto h(\cdot,t)$ is analytic.

As a consequence, $t \mapsto h(\cdot,t)|_\omega$, regarded as a mapping from $(-\infty,T)$ into $L^2(\omega)$, is also analytic. Since it vanishes on $(0,T)$, it vanishes everywhere in $(-\infty,T)$.

Let us set $F(t) \equiv h(\cdot, T-t)$ and let $\tilde{F}(z)$ be the Laplace transform of F (a meromorphic $L^2(\Omega)$-valued function). Then $z \mapsto \tilde{F}(z)|_\omega$ is a meromorphic $L^2(\omega)$-valued function with poles at the $\mu_n^{\pm}$. But this function vanishes identically, since $F(t)|_\omega \equiv 0$. Consequently, all the residues vanish and this easily implies that $h_{0n} = 0$ for all n.

Acknowledgements. The authors are indebted to the anonymous referee for his/her comments which led to a substantial improvement of the first manuscript.

A. J. V. Brandão was partially supported by the Instituto do Milenio-IM-AGIMB (Brazil). E. Fernandez-Cara was supported by grant MTM2007-07932 from the D.G.E.S. (Spain) and grant 2005/50705-4 from the FAPESP-Brazil. P. M. D. Magalhães was supported by the Instituto do Milenio-IM-AGIMB (Brazil). M A. Rojas-Medar was partially supported by the National Fund for Science and Technology (Chile), grant 1080628.

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Addendum posted on July 8, 2009.

Following suggestions by the anonymous referee, the authors want to clarify the controllability result.

In the final part of the paper, where we prove approximate controllability, we use that the solution to the adjoint system

$$
\begin{gathered}
-h_t - \Delta h + \sigma \int_t^T e^{-\gamma(s-t)} h(s)\, ds + \alpha(x) h = 0, \\
h(x,t)\big|_{\Sigma} = 0, \\
h(x,T) = h^0(x)
\end{gathered}
$$

is analytic, regarded as a mapping from $(-\infty, T)$ to $L^2(\Omega)$. It was stated that this is true becasue

> $h(\cdot, t)$ can be written as the sum of a series that converges normally and uniformly on any compact set in $(-\infty, T)$ and each term of the series is analytic in t.

It should have been said that this is true because

> $h(\cdot, t)$ can be extended to a function in $G_T = \{z \in \mathbb{C} : \operatorname{Re} z < T\}$ that is the sum of a series that converges normally and uniformly on any compact set in G_T and each term of the series is analytic in z.

The argument and estimates needed to prove this last assertion are actually depicted in the paper: For all $z \in G_T$, we set

$$
h(\cdot, z) = \sum_{n \geq 1} \frac{h_{0n}}{\zeta_n} \theta_n(x) \big(\mu_n^+ e^{\mu_n^+(T-z)} - \mu_n^- e^{\mu_n^-(T-z)} \big)
$$

and note that the L^2-norm of the n-th term is bounded in each compact set $S \subset G_T$ by a constant times $\frac{|h_{0n}|}{\lambda_n} + |h_{0n}| e^{\mu_n^- a}$, where a depends on S. This proves normal and uniform convergence on the compact subsets of G_T and, consequently, that $z \mapsto h(\cdot, z)$ is analytic.

End of addendum.

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