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EXISTENCE RESULTS FOR IMPULSIVE EVOLUTION DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract. We study the existence of mild solution for impulsive evolution abstract differential equations with state-dependent delay. A concrete application to partial delayed differential equations is considered.

1. Introduction

In this work we discuss the existence of mild solutions for impulsive functional differential equations, with state-dependent delay, of the form

$$x'(t) = A(t)x(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \dots, n, \tag{1.3}$$

where $A(t) : \mathcal{D} \subset X \to X$, $t \in I$, is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in a Banach space $(X, \|\cdot\|)$; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$, $I_i : \mathcal{B} \to X$, $i = 1, 2, \dots, n$, are appropriate functions; $0 < t_1 < \dots . t_n < a$ are prefixed points and the symbol $\Delta\xi(t)$ represents the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

Various evolutionary processes from fields as diverse as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the total duration of the process, such changes can be reasonably well-approximated as being instantaneous changes of state, or in the form of impulses. These process tend to more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more details on this theory and on its applications we refer to the monographs of Lakshmikantham et

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al. [17], and Samoilenko and Perestyuk [25] for the case of ordinary impulsive system and [18, 23, 24, 15, 16] for partial differential and partial functional differential equations with impulses.

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. There exists a extensive literature for ordinary state-dependent delay equations, see among another works, [2, 1, 3, 4, 6, 7, 8]. The study of partial differential equations with state dependent delay have been initiated recently, and concerning this matter we cite the pioneer works Rezounenko et al. [21], Hernández el al. [11] and the papers [10, 12, 13, 14, 22].

To the best of our knowledge, the study of the existence of solutions for systems described in the abstract form (1.1)–(1.2) is a untreated problem, and this fact, is the main motivation of this paper.

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space, $\{A(t) : t \in \mathbb{R}\}$ is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in X, and we assume that the linear non-autonomous system

$$\begin{aligned} u'(t) &= A(t)u(t), \quad s \le t \le a, \\ u(s) &= x \in X, \end{aligned} \tag{1.4}$$

has an associated evolution family of operators $\{U(t,s) : a \ge t \ge s \ge 0\}$. In the next definition, $\mathcal{L}(X)$ is the space of bounded linear operator from X into X endowed with the uniform convergence topology.

Definition 1.1. A family of linear operators $\{U(t,s) : a \ge t \ge s \ge 0\} \subset \mathcal{L}(X)$ is called an evolution family of operators for (1.4) if the following conditions hold:

(a) $U(t,s)U(s,r) = U(t,r)$ and $U(r,r)x = x$ for every $r \le s \le t$ and all $x \in X$;
(b) For each $x \in X$ the function $(t,s) \to U(t,s)x$ is continuous and $U(t,s) \in \mathcal{L}(X)$ for every $t \ge s$; and
(c) For $s \le t \le a$, the function $(s,t] \to \mathcal{L}(X)$, $t \to U(t,s)$ is differentiable with $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$.

In the sequel, $\widetilde{M}$ is a positive constant such that $\|U(t,s)\| \le \widetilde{M}$ for every $t \ge s$, and we always assume that $U(t,s)$ is a compact operator for every $t > s$. We refer the reader to [20] for additional details on evolution operator families.

To consider the impulsive condition (1.3), it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma,\tau] \to X$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau];X)$ the space formed by the normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0,a] \to X$ such that u is continuous at $t \ne t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1,\dots,n$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $\|u\|_{\mathcal{PC}} = \sup_{s\in I}\|u(s)\|$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0,\ t_{n+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i \in C([t_i,t_{i+1}];X)$, $i = 0,1,\dots,n$, the function given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i,t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases} \tag{1.5}$$

Moreover, for $B \subseteq \mathcal{PC}$ we denote by $\widetilde{B}_i$, $i = 0, 1, \dots, n$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

Lemma 1.2. *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$, for every $i = 0, 1, \dots, n$.*

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [9]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following conditions:

- (A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b] : X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$,
  - (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.
- (B) The space $\mathcal{B}$ is complete.

Example 1.3. Phase spaces $\mathcal{PC}_h(X)$, $\mathcal{PC}^0_g(X)$. As usual, we say that $\psi : (-\infty, 0] \to X$ is normalized piecewise continuous, if ψ is left continuous and the restriction of ψ to any interval $[-r, 0]$ is piecewise continuous.

Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous, nondecreasing function with $g(0) = 1$, which satisfies the conditions (g-1), (g-2) of [9]. This means that $\lim_{\theta\to-\infty} g(\theta) = \infty$ and that the function $\Lambda(t) := \sup_{-\infty<\theta\le-t} \frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t \ge 0$. Next, we modify slightly the definition of the spaces C_g, C^0_g in [9]. We denote by $\mathcal{PC}_g(X)$ the space formed by the normalized piecewise continuous functions ψ such that $\frac{\psi}{g}$ is bounded on $(-\infty, 0]$ and by $\mathcal{PC}^0_g(X)$ the subspace of $\mathcal{PC}_g(X)$ formed by the functions ψ such that $\frac{\psi(\theta)}{g(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_g(X)$ and $\mathcal{PC}^0_g(X)$ endowed with the norm $\|\psi\|_{\mathcal{B}} := \sup_{\theta\le0} \frac{\|\psi(\theta)\|}{g(\theta)}$, are phase spaces in the sense considered in this work. Moreover, in these cases $K \equiv 1$.

Example 1.4. Phase space $\mathcal{PC}_r \times L^2(g, X)$. Let $1 \le p < \infty$, $0 \le r < \infty$ and $g(\cdot)$ be a Borel nonnegative measurable function on $(-\infty, r)$ which satisfies the conditions (g-5)-(g-6) in the terminology of [9]. Briefly, this means that $g(\cdot)$ is locally integrable on $(-\infty, -r)$ and that there exists a nonnegative and locally bounded function Λ on $(-\infty, 0]$ such that $g(\xi + \theta) \le \Lambda(\xi)g(\theta)$ for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0.

Let $\mathcal{B} := \mathcal{PC}_r \times L^p(g; X)$, $r \ge 0, p > 1$, be the space formed of all classes of functions $\psi : (-\infty, 0] \to X$ such that $\psi|_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$, $\psi(\cdot)$ is Lebesgue-measurable on $(-\infty, -r]$ and $g|\psi|^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\psi\|_{\mathcal{B}} := \sup_{\theta\in[-r,0]} \|\psi(\theta)\| + \Big(\int_{-\infty}^{-r} g(\theta)\|\psi(\theta)\|^p d\theta\Big)^{1/p}.$$

Proceeding as in the proof of [9, Theorem 1.3.8] it follows that $\mathcal{B}$ is a phase space which satisfies the axioms **A** and **B**. Moreover, for $r = 0$ and $p = 2$ this space coincides with $C_0 \times L^2(g, X)$, $H = 1$; $M(t) = \Lambda(-t)^{1/2}$ and $K(t) = 1 + \big(\int_{-t}^0 g(\tau)d\tau\big)^{1/2}$ for $t \ge 0$.

Remark 1.5. In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see for instance [9]. Due to the impulsive effect, this property is not satisfied in impulsive delay systems and, for this reason, has been eliminated in our abstract description of $\mathcal{B}$.

The terminology and notations are those generally used in functional analysis. In particular, for Banach a space $(Z, \|\cdot\|_Z)$, the notation $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z.

To prove some of our results, we use a fixed point Theorem which is referred in the Literature as Leray Schauder Alternative Theorem, see [5, Theorem 6.5.4].

Theorem 1.6. *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then the map G has a fixed point in D or the set $\{x \in D : x = \lambda G(x),\ 0 < \lambda < 1\}$ is unbounded.*

In the next section we study the existence of mild solutions for the abstract system (1.1)–(1.2). In the last section an application is discussed.

2. Existence Results

To prove our results on the existence of mild solutions for the abstract Cauchy problem (1.1)–(1.2), we always assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is continuous. In addition, we introduce the following conditions.

- (H0) Let $\mathcal{BPC}(\varphi) = \{u : (-\infty, a] \to X; u_0 = \varphi, u|_I \in \mathcal{PC}\}$. The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s, x_s) : \rho(s, x_s) \leq 0,\ x \in \mathcal{BPC}(\varphi), s \in [0, a]\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.
- (H1) The function $f : I \times \mathcal{B} \to X$ satisfies the following properties.
  - (a) The function $f(\cdot, \psi) : I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.
  - (b) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for each $t \in I$.
  - (c) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that $\|f(t, \psi)\| \leq m(t)W(\|\psi\|_{\mathcal{B}})$, for every $(t, \psi) \in I \times \mathcal{B}$.
- (H2) The maps I_i are completely continuous and there are positive constants c_i^j, $j = 1, 2$, such that $\|I_i(\psi)\| \leq c_i^1\|\psi\|_{\mathcal{B}} + c_i^2$, $i = 1, 2, \dots, n$, for every $\psi \in \mathcal{B}$.
- (H3) The function $I_i : \mathcal{B} \to X$ is continuous and there are positive constants L_i, $i = 1, 2, \dots, n$, such that $\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_i\|\psi_1 - \psi_2\|_{\mathcal{B}}$, for every $\psi_j \in \mathcal{B}$, $j = 1, 2$, $i = 1, 2, \dots, n$.

Remark 2.1. The condition (H0), is frequently verified by functions continuous and bounded. If, for instance, the space $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [9], then there exists a constant $\mathrm{L} > 0$ such that $\|\varphi\|_{\mathcal{B}} \leq \mathrm{L}\sup_{\theta\leq 0}\|\varphi(\theta)\|$ for every $\varphi \in \mathcal{B}$ continuous and bounded, see [9, Proposition 7.1.1] for details. Consequently, $\|\varphi_t\|_{\mathcal{B}} \leq L\frac{\sup_{\theta\leq 0}\|\varphi(\theta)\|}{\|\varphi\|_{\mathcal{B}}}\|\varphi\|_{\mathcal{B}}$ for every continuous and bounded function $\varphi \in \mathcal{B}\setminus\{0\}$ and every $t \leq 0$. We note that the spaces $C_r \times L^p(g; X)$, $C_g^0(X)$ verify axiom C_2, see [9, p.10] and [9, p.16] for details.

Remark 2.2. Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t + \theta)$. Consequently, if the function $x(\cdot)$ in axiom **A** is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We also note that, in general, $\varphi_t \notin \mathcal{B}$. Consider for example the characteristic function $\mathcal{X}_{[-r,0]}$, $r > 0$, in the space $\mathbf{C_r} \times \mathbf{L^p(g; X)}$.

In this paper, we adopt the following concept of mild solution.

Definition 2.3. A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem (1.1)–(1.2) if $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in I$ and

$$x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s, x_{\rho(s,x_s)})ds + \sum_{0<t_i<t} U(t,t_i)I_i(x_{t_i}), \quad t \in I.$$

The next result is a consequence of the phase space axioms.

Lemma 2.4. *If $x : (-\infty, a] \to X$ is a function such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}(I : X)$, then*

$$\|x_s\|_{\mathcal{B}} \le (M_a + J^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a \sup\{\|x(\theta)\|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup I,$$

where $J^{\varphi} = \sup_{t\in\mathcal{R}(\rho^-)} J^{\varphi}(t)$, $M_a = \sup_{t\in I} M(t)$ and $K_a = \sup_{t\in I} K(t)$.

Remark 2.5. In the rest of this work, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \varphi$ and $y(t) = U(t,0)\varphi(0)$ for $t \in I$.

Now, we can prove our first existence result.

Theorem 2.6. *Let conditions* (H0)–(H3) *be satisfied and assume that*

$$1 > K_a\widetilde{M}\Big(\liminf_{\xi\to\infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s)ds + \sum_{i=1}^n L_i\Big). \tag{2.1}$$

Then there exists a mild solution of (1.1)–(1.2).

Proof. On the space $Y = \{u \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence norm ($\|\cdot\|_\infty$), we define the operator $\Gamma : Y \to Y$ defined by

$$\Gamma x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds + \sum_{0<t_i<t} U(t,t_i)I_i(\bar{x}_{t_i}), \quad t \in I,$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From our assumptions, it is easy to see that $\Gamma x(\cdot) \in Y$.

Let $\bar{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I and $\widetilde{J}^{\varphi} = \sup\{J^{\varphi}(s) : s \in \mathcal{R}(\rho^-)\}$. By using Lemma 2.4, for $r > 0$ and

$x^r \in B_r(\bar{\varphi}|_I, Y)$ we obtain

$$\begin{aligned}
&\|\Gamma x^r - \varphi(0)\| \\
&\le (\widetilde{M}+1)H\|\varphi\|_{\mathcal{B}} + \widetilde{M}\int_0^a m(s)W(\|\overline{x^r}_{\rho(s,\overline{x^r_s})}\|_{\mathcal{B}})ds \\
&\quad + \widetilde{M}\sum_{i=1}^n (L_i\|\overline{x}_{t_i}\|_{\mathcal{B}} + \|I_i(0)\|) \\
&\le (\widetilde{M}+1)H\|\varphi\|_{\mathcal{B}} + \widetilde{M}\int_0^a m(s)W\Big((M_a+\widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a \sup_{\theta\in[0,a]}\|\overline{x^r}(\theta)\|\Big)ds \\
&\quad + \widetilde{M}\sum_{i=1}^n L_i\,(\|\overline{x}_{t_i} - \varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}} + \|I_i(0)\|) \\
&\le (\widetilde{M}+1)H\|\varphi\|_{\mathcal{B}} + \widetilde{M}W\Big((M_a+\widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a(r+\|\varphi(0)\|)\Big)\int_0^a m(s)ds, \\
&\quad + \ \widetilde{M}\sum_{i=1}^n L_i\,(K_a r + \|\varphi\|_{\mathcal{B}} + \|I_i(0)\|)
\end{aligned}$$

which from (2.1) implies that $\|\Gamma x^r - \varphi(0)\|_\infty \le r$ for r large enough.

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi}|_I, Y)) \subset B_r(\bar{\varphi}|_I, Y)$. Next, we will prove that $\Gamma(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_I, Y)$ into $B_r(\bar{\varphi}|_I, Y)$. To this end, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where $(\Gamma_1 x)_0 = \varphi$, $(\Gamma_2 x)_0 = 0$, and

$$\begin{gathered}
\Gamma_1 x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds, \quad t \in I \\
\Gamma_2 x(t) = \sum_{0<t_i<t} U(t,t_i)I_i(\bar{x}_{t_i}), \quad t \in I.
\end{gathered}$$

To begin, we prove that the set $\Gamma_1(B_r(\bar{\varphi}|_I, Y))(t) = \{\Gamma_1 x(t) : x \in B_r(\bar{\varphi}|_I, Y)\}$ is relatively compact in X for every $t \in I$.

The case $t = 0$ is obvious. Let $0 < \varepsilon < t \le a$. If $x \in B_r(\bar{\varphi}|_I, Y)$, from Lemma 2.4 follows that $\|\bar{x}_{\rho(t,\bar{x}_t)}\|_{\mathcal{B}} \le r^* := (M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|\varphi(0)\|)$ which implies

$$\Big\|\int_0^\tau U(\tau,s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds\Big\| \le r^{**} := \widetilde{M}W(r^*)\int_0^a m(s)ds, \quad \tau \in I. \tag{2.2}$$

From the above inequality, we find that

$$\begin{aligned}
\Gamma_1 x(t) &= U(t,0)\varphi(0) + U(t,t-\varepsilon)\int_0^{t-\varepsilon} U(t-\varepsilon,s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds \\
&\quad + \int_{t-\varepsilon}^t U(t,s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds \\
&\in \{U(t,0)\varphi(0)\} + U(t,t-\varepsilon)B_{r^{**}}(0,X) + C_\varepsilon,
\end{aligned}$$

where $\mathrm{diam}(C_\varepsilon) \le 2\widetilde{M}W(r^*)\int_{t-\varepsilon}^t m(s)ds \to 0$ as $\varepsilon \to 0$, which allows us to conclude that $\Gamma_1(B_r(\bar{\varphi}|_I, Y))(t)$ is relatively compact in X.

Now, we prove that $\Gamma_1(B_r(\bar{\varphi}|_I, Y))$ is equicontinuous on I. Let $0 < t < a$ and $\varepsilon > 0$. Since the set $\Gamma_1(B_r(\bar{\varphi}|_I, Y))(t)$ is relatively compact compact in X, from the properties of the evolution family $U(t,s)$, there exists $0 < \delta \le a - t$ such that

$\|U(t+h,t)x - x\| < \varepsilon$, for every $x \in \Gamma_1(B_r(\bar{\varphi}|_I, Y))(t)$ and all $0 < h < \delta$. Under these conditions, for $x \in B_r(\bar{\varphi}|_I, Y)$ and $0 < h < \delta$ we obtain

$$\begin{aligned}
\|\Gamma_1 x(t+h) - \Gamma_1 x(t)\| &\le \|U(t+h,0)\varphi(0) - U(t,0)\varphi(0)\| \\
&\quad + \|(U(t+h,t) - I)\int_0^t U(t,s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds\| \\
&\quad + \widetilde{M} \int_t^{t+h} m(s) W(r^*) ds \\
&\le 2\varepsilon + \widetilde{M} W(r^*) \int_t^{t+h} m(s) ds,
\end{aligned}$$

which proves that $\Gamma_1(B_r(\bar{\varphi}|_I, Y))$ is right equicontinuous at $t \in (0,a)$. A similar procedure shows that $\Gamma_1(B_r(\bar{\varphi}|_I, Y))$ is right equicontinuous at zero and left equicontinuous at $t \in (0,a]$. Thus, the set $\Gamma_1(B_r(\bar{\varphi}|_I, Y))$ is equicontinuos on I.

Using the same arguments as in [11, Theorem 2.2], it follows that Γ_1 is a continuous map, which complete the proof that Γ_1 is completely continuous. On the other hand, from the assumptions and the phase space axioms it follows that

$$\|\Gamma_2 x - \Gamma_2 y\|_\infty \le K_a \widetilde{M} \sum_{i=1}^n L_i \|x - y\|_\infty$$

which proves that Γ_2 is a contraction on $B_r(\bar{\varphi}|_I, Y)$ and that Γ is a condensing map on $B_r(\bar{\varphi}|_I, Y)$.

Finally, the existence of a mild solutions is a consequence of [19, Theorem 4.3.2]. The proof is complete. $\square$

In the next result, $\mathcal{BPC}(\varphi)$ is the set introduced in assumption (H0).

Theorem 2.7. *Let* (H0)–(H2) *be satisfied. If* $\rho(t, x_t) \le t$ *for every* $(t,x) \in I \times \mathcal{BPC}(\varphi)$, $\mu = 1 - K_a \widetilde{M} \sum_{i=1}^n c_i > 0$ *and*

$$K_a \widetilde{M} \int_0^a m(s) ds < \int_C^\infty \frac{ds}{W(s)},$$

where

$$C = (M_a + J^\varphi + \widetilde{M} H K_a)\|\varphi\|_{\mathcal{B}} + \frac{\widetilde{M} K_a}{\mu} \sum_{i=1}^n \Big[c_i^1 (M_a + \widetilde{M} H K_a)\|\varphi\|_{\mathcal{B}} + c_i^2 \Big]$$

then there exists a mild solution of (1.1)–(1.2).

Proof. On the space $\mathcal{BPC} = \{u : (-\infty, a] \to X; u_0 = 0, u|_I \in \mathcal{PC}\}$ provided with the sup-norm $\|\cdot\|_\infty$, we define the operator $\Gamma : \mathcal{BPC} \to \mathcal{BPC}$ by $(\Gamma u)_0 = 0$ and

$$\Gamma x(t) = \int_0^t U(t,s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds + \sum_{0 < t_i < t} U(t, t_i) I_i(\bar{x}_{t_i}), \quad t \in I,$$

where $\bar{x} = x + y$ on $(-\infty, a]$ and $y(\cdot)$ is the function defined in Remark 2.5. To use Theorem 1.6, we establish *a priori* estimates for the solutions of the integral equation $z = \lambda \Gamma z, \lambda \in (0,1)$. Let x^λ be a solution of $z = \lambda \Gamma z, \lambda \in (0,1)$. By using Lemma 2.4, the notation $\alpha^\lambda(s) = \sup_{\theta \in [0,s]} \|x^\lambda(\theta)\|$, and the fact that $\rho(s, \overline{(x^\lambda)}_s) \le$

s, for each $s \in I$, we find that

$$\|x^\lambda(t)\| \le \widetilde{M}\int_0^t m(s)W\Big((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(s)\Big)ds + \widetilde{M}\sum_{0<t_i\le t} c_i^1\Big[(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(t)\Big] + \widetilde{M}\sum_{i=1}^n c_i^2,$$

and so,

$$\alpha^\lambda(t) \le \widetilde{M}\sum_{i=1}^n \Big[c_i^1(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + c_i^2\Big] + K_a\widetilde{M}\sum_{0<t_i\le t} c_i^1\alpha^\lambda(t) + \widetilde{M}\int_0^t m(s)W\Big((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(s)\Big)ds,$$

which implies

$$\alpha^\lambda(t) \le \frac{\widetilde{M}}{\mu}\sum_{i=1}^n \big[c_i^1(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + c_i^2\big] + \frac{\widetilde{M}}{\mu}\int_0^t m(s)W((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(s))ds,$$

for every $t \in [0,a]$. By defining $\xi^\lambda(t) = (M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(t)$, we find that

$$\xi^\lambda(t) \le (M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + \frac{\widetilde{M}K_a}{\mu}\sum_{i=1}^n \big[c_i^1(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + c_i^2\big] + \frac{\widetilde{M}K_a}{\mu}\int_0^t m(s)W(\xi^\lambda(s))ds.$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, if follows that

$$\beta_\lambda'(t) \le \frac{\widetilde{M}K_a}{\mu} m(t)W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le \frac{\widetilde{M}K_a}{\mu}\int_0^a m(s)ds < \int_C^\infty \frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I,\mathbb{R})$. This show that the set $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{BPC}$.

To prove that the map Γ is completely continuous, we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where $(\Gamma_i x)_0 = 0, i = 1,2$, and

$$\Gamma_1 x(t) = \int_0^t U(t,s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds, \quad t \in I,$$
$$\Gamma_2 x(t) = \sum_{0<t_i<t} U(t,t_i)I_i(\bar{x}_{t_i}), \quad t \in I.$$

Proceeding as in the proof of Theorem 2.6 we can prove that Γ_1 is completely continuous. The continuity of Γ_2 can be proven using the phase space axioms.

To prove that Γ_2 is also completely continuous, we use Lemma 1.2. For $r > 0$, $t \in [t_i, t_{i+1}] \cap (0, a]$, $i \geq 1$, and $u \in B_r = B_r(0, \mathcal{BPC})$ we find that

$$\widetilde{\Gamma_2 u}(t) \in \begin{cases} \sum_{j=1}^{i} U(t, t_j) I_j(B_{r^*}(0, X)), & t \in (t_i, t_{i+1}), \\ \sum_{j=0}^{i} U(t_{i+1}, t_j) I_j(B_{r^*}(0, X)), & t = t_{i+1}, \\ \sum_{j=1}^{i-1} U(t_i, t_j) I_j(B_{r^*}(0, X)) + I_i(B_{r^*}(0; X)), & t = t_i, \end{cases}$$

where $r^* := (M_a + \widetilde{M} H K_a)\|\varphi\|_{\mathcal{B}} + K_a r$, which proves that $[\widetilde{\Gamma_2(B_r)}]_i(t)$ is relatively compact in X for every $t \in [t_i, t_{i+1}]$, since the maps I_j are completely continuous. Moreover, using the compactness of the operators I_i and properties of the evolution family $U(\cdot)$, we can prove that $[\widetilde{\Gamma_2(B_r)}]_i(t)$ is equicontinuous at t, for every $t \in [t_i, t_{i+1}]$ and each $i = 1, 2, \dots, n$, which complete the proof that Γ_2 is completely continuous.

The existence of a mild solution is now a consequence of Theorem 1.6. The proof is complete. □

3. Applications

In this section we consider an application of our abstract results. Consider the partial differential equation

$$\begin{aligned} \frac{\partial u(t,\xi)}{\partial t} &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi) u(t,\xi) \\ &\quad + \int_{-\infty}^{t} a_1(s-t) u(s - \rho_1(t)\rho_2(\int_0^{\pi} a_2(\theta)|u(t,\theta)|^2 d\theta), \xi) ds \end{aligned} \tag{3.1}$$

for $t \in I = [0, a]$, $\xi \in [0, \pi]$. The above equation is subject to the conditions

$$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \tag{3.2}$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \leq 0,\ 0 \leq \xi \leq \pi. \tag{3.3}$$

$$\Delta u(t_j, \xi) = \int_{-\infty}^{t_j} \gamma_j(s - t_j) u(s, \xi) ds, \quad j = 1, 2, \dots, n. \tag{3.4}$$

To study this system, we consider the space $X = L^2([0, \pi])$ and the operator $A : D(A) \subset X \to X$ given by $Ax = x''$ with $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = (\frac{2}{\pi})^{1/2} \sin(n\xi)$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, z_n \rangle z_n$ for $x \in X$ and $t \geq 0$. It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

On the domain $D(A)$, we define the operators $A(t) : D(A) \subset X \to X$ by $A(t)x(\xi) = Ax(\xi) + a_0(t, \xi)x(\xi)$. By assuming that $a_0(\cdot)$ is continuous and that $a_0(t, \xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}, \xi \in [0, \pi]$, it follows that the system

$$\begin{gathered} u'(t) = A(t)u(t) \quad t \geq s, \\ u(s) = x \in X, \end{gathered}$$

has an associated evolution family given by $U(t,s)x(\xi) = [T(t-s)e^{\int_s^t a_0(\tau,\xi)d\tau}x](\xi)$. From this expression, it follows that $U(t,s)$ is a compact linear operator and that $\|U(t,s)\| \le e^{-(1+\delta_0)(t-s)}$ for every $t, s \in I$ with $t > s$.

Proposition 3.1. *Let $\mathcal{B} = \mathcal{PC}_0 \times L^2(g,X)$ and $\varphi \in \mathcal{B}$. Assume that condition* (H0) *holds, $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous and that the following conditions are verified.*

- (a) *The functions $a_i : \mathbb{R} \to \mathbb{R}$ are continuous and $L_f = (\int_{-\infty}^0 \frac{(a_1(s))^2}{g(s)} ds)^{1/2}$ is finite.*
- (b) *The functions $\gamma_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,n$, are continuous, bounded and $L_i := \Big(\int_{-\infty}^0 \frac{(\gamma_i(s))^2}{g(s)} ds\Big)^{1/2} < \infty$ for every $i = 1,2,\dots,n$.*

Then there exists a mild solution of (3.1)–(3.3).

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^0 a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\Big(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\Big),$$

$$I_i(\psi)(\xi) = \int_{-\infty}^0 \gamma_i(s)\psi(s,\xi)ds, \quad i = 1,2,\dots,n,$$

are well defined functions, which permit to transform system (3.1)–(3.3) into the abstract system (1.1)–(1.2). Moreover, the functions f, I_i are bounded linear operator, $\|f\| \le L_1$ and $\|I_i\| \le L_i$ for every $i = 1,2,\dots n$. Now, the existence of a mild solutions can be deduced from a direct application of Theorem 2.7. The proof is complete. □

From Remark 2.1 we have the following result..

Corollary 3.2. *Let $\varphi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (3.1)–(3.3) *on I.*

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