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EXISTENCE AND SMOOTHNESS OF SOLUTIONS TO SECOND INITIAL BOUNDARY VALUE PROBLEMS FOR SCHRÖDINGER SYSTEMS IN CYLINDERS WITH NON-SMOOTH BASES

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Abstract. In this paper, we consider the second initial boundary value problem for strongly general Schrödinger systems in both the finite and the infinite cylinders $Q_T, 0 < T \leq +\infty$, with non-smooth base Ω. Some results on the existence, uniqueness and smoothness with respect to time variable of generalized solution of this problem are given.

1. Introduction

Boundary value problems for Schrödinger equations have been considered in the books by Lions and Magenes [7] in finite cylinders $Q_T = \Omega \times (0,T), (T < +\infty)$, with base Ω, where $\partial\Omega$ is smooth. Their results are restricted to Schrödinger type equations, where coefficients a_{pq} of equations are functions independent of t (except a_{00}). The first initial boundary value problem for general Schrödinger systems, where the coefficients $a_{pq}(x,t)$ are matrices of functions of two variables x and t for all p, q, was considered in [2, 3], in cases the finite cylinder Q_T, $T < +\infty$ or in cases the infinite cylinder $Q_\infty = \Omega \times (0, +\infty)$ as in [3, 4]. In this paper, we consider the second initial boundary value problem for these systems in both the finite and the infinite cylinder $Q_T = \Omega \times (0,T)$, where $0 < T \leq +\infty$ and Ω is a domain with non-smooth boundary. Our main purpose is to study the existence, uniqueness and smoothness with respect to time variable of generalized solution of the mentioned problem. Such results are investigated in a scale of weighted spaces $H^{m,0}_\gamma(Q_T)$ for some $\gamma > 0$.

As we have known, in the first problem, the qualitative properties of solution were indicated by basing on the properties of functions $u \in \overset{o}{H}{}^{m,0}_{\gamma}(Q_T)$, which let us to the Garding inequality (see [2, 3, 4, 6]). But in the second problem, when the solution space is $H^{m,0}_{\gamma}(Q_T)$ and the second boundary condition is hidden in the integral equality in the definition of generalized solution, the Garding inequality is not valid, so it becomes more complicated to establish the unique solvability of the problem. This difficulty is solved in this paper in section 2, Lemma 2.1. Then based on it, we receive our results on the existence and uniqueness of generalized solution

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in section 3 and the smoothness with respect to time variable of solutions in the last section. Moreover, the problem becomes more complicated in technics when we consider with non homogeneously initial condition $u(x,0) = \varphi(x)$ in section 3, and the results that we received are more general than those in [2, 3, 4, 6], in which the authors just considered the problem with homogeneously initial condition $u(x,0) = 0$.

2. Preliminaries

Suppose that Ω is a bounded domain in $\mathbb{R}^n$, and $\overline{\Omega}$, $\partial\Omega$ denote the closure and the boundary of Ω in $\mathbb{R}^n$. We suppose that $\Gamma = \partial\Omega\backslash\{0\}$ is a smooth manifold and Ω coincides with the cone $K = \{x : \frac{x}{|x|} \in G\}$ in a neighborhood of the origin point 0, where G is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. We begin by introducing some notations and functional spaces which are used fluently in the rest.

Denote $Q_T = \Omega \times (0,T)$, $S_T = \Gamma \times (0,T)$, for some $0 < T \le +\infty$; $x = (x_1,\dots,x_n) \in \Omega$, $u(x,t) = (u_1(x,t),\dots,u_s(x,t))$ is a vector complex function; $\alpha = (\alpha_1,\dots,\alpha_n)$ ($\alpha_i \in \mathbb{N}$, $i = 1,\dots,n$) is a multi-index; $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum_{i=1}^s |D^\alpha u_i|^2$, $u_{t^j} = (\partial^j u_1/\partial t^j,\dots,\partial^j u_s/\partial t^j)$, $C_k^s = \frac{k!}{s!(k-s)!}$ $(0 \le s \le k)$.

In this paper we use the usual functional spaces: $C^\infty(\overline{\Omega})$, $L_2(\Omega)$, $H^m(\Omega)$, $L_2(Q_T)$, $H^{l,k}(Q_T)$ when $T < +\infty$ and $m, l, k \in \mathbb{N}$ (see [3, 4] for the precise definitions).

Moreover, when $0 < T \le +\infty$ we define $H_\gamma^{m,0}(Q_T)$ ($\gamma > 0$) as the space of all measurable complex functions $u(x,t)$ that have generalized derivatives up to order m with respect to x with the norm

$$\|u\|_{H_\gamma^{m,0}(Q_T)} = \Big(\sum_{|\alpha|\le m}\int_{Q_T} |D^\alpha u|^2 e^{-2\gamma t}\,dx\,dt\Big)^{\frac12}.$$

The space $L^\infty(0,T;L_2(\Omega))$ consists of all measurable functions $u : (0,T) \to L_2(\Omega)$, $t \mapsto u(t)$ with the norm $\|u\|_\infty = \operatorname{ess\,sup}_{0<t<T} \|u(t)\|_{L_2(\Omega)} < +\infty$.

For convenience, in the rest of this paper we say that $u(x,t)$ belongs to some spaces if all of its components belong to that space. We now introduce a $2m$th-order differential operator

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p(a_{pq}(x,t)\,D^q), \tag{2.1}$$

where a_{pq} are $s \times s$ matrices of bounded measurable complex functions defined on Q_T, $a_{pq} = (-1)^{|p|+|q|}a_{qp}^*$ (a_{qp}^* denotes the transposed conjugate matric of a_{qp}). Moreover, we assume that the operator L satisfies a hypothesis that given as follows (see for example [5, 8, 9]).

For all $(x,t) \in Q_T$ and $(\eta_p)_{|p|=m} \in C^{s.m_1^*} \setminus \{0\}$, we have

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\eta_q\overline{\eta_p} \ge C_0 \sum_{|p|=m} |\eta_p|^2, \tag{2.2}$$

where C_0 is a positive real number, independent of $(\eta_p)_{|p|=m}$; $m_1^* = \sum_{|p|=m} 1$.

Setting $\eta_p = \xi^p\eta$ with $\xi \in \mathbb{R}^n \setminus \{0\}$, $\xi^p = \xi_1^{p_1}\dots\xi_n^{p_n}$ and $\eta \in \mathbb{C}^s \setminus \{0\}$, it follows from condition (2.2) that $\sum_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \ge C_0|\xi|^{2m}|\eta|^2$, for all $(x,t) \in Q_T$, which is equivalent to the strong ellipticity of the operator L. However, one can see easily that the condition of strong ellipticity of the operator L does not imply the condition (2.2).

In the cylinder Q_T we consider *the second initial boundary problem for the Schrödinger system*

$$i(-1)^{m-1}L(x,t,D)u - u_t = f(x,t), \quad (x,t) \in Q_T, \tag{2.3}$$

with initial condition

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2.4}$$

where $L(x,t,D)$ is the operator in (2.1), satisfies the condition (2.2) and u, f, φ are vector functions.

The function $u(x,t)$ is called *generalized solution* in the space $H^{m,0}_\gamma(Q_T)$ of the second initial boundary problem for the Schrödinger system (2.3) and initial condition (2.4) if and only if $u(x,t) \in H^{m,0}_\gamma(Q_T)$, satisfying

$$\begin{aligned} &(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_\tau} a_{pq}(x,t) D^q u(x,t) \overline{D^p \eta}(x,t)\, dx\, dt \\ &+ \int_{Q_\tau} u(x,t) \overline{\eta_t}(x,t)\, dx\, dt \\ &= -\int_\Omega \varphi(x)\eta(x,0) dx + \int_{Q_\tau} f(x,t)\overline{\eta}(x,t)\, dx\, dt \end{aligned} \tag{2.5}$$

for each $0 < \tau < T$ and all test functions $\eta(x,t) \in H^{m,1}(Q_\tau)$, $\eta(x,\tau) = 0$. Set

$$B[u,v](t) = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq}(x,t) D^q u(x,t) \overline{D^p v(x,t)} dx.$$

To consider the problem we need to prove the important following lemma.

Lemma 2.1. *There exist two constants $\mu_0 > 0$ and λ_0 such that the inequality*

$$(-1)^m B[u,u](t) \geq \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)}$$

is valid for all $u \in H^{m,0}_\gamma(Q_T)$, $\gamma > 0$ and almost $t \in (0,T)$.

Proof. It follows from (2.2) that

$$\sum_{|p|=|q|=m} \int_\Omega a_{pq}(x,t)\, D^q u \overline{D^p u} dx \geq C_0 \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)}$$

for all $u(x,t) \in H^{m,0}_\gamma(Q_T)$ and almost $t \in (0,T)$, where C_0 is a positive number, independent of u. Since a_{pq} are bounded, using Cauchy's inequality one has

$$\begin{aligned} &C_0 \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)} \\ &\leq \sum_{|p|=|q|=m} \int_\Omega a_{pq}\, D^q u \overline{D^p u} dx \\ &= (-1)^m B[u,u](t) - (-1)^m \sum_{|p|+|q|<2m,\, |p|,|q|\leq m} (-1)^{|p|} \int_\Omega a_{pq}\, D^q u \overline{D^p u} dx \\ &\leq (-1)^m B[u,u](t) + C(\varepsilon)\|u\|^2_{H^{m-1}(\Omega)} + \varepsilon \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)}, \end{aligned}$$

where $0 < \varepsilon < C_0$, $C(\varepsilon) > 0$. This implies

$$\sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)} \le C_1(-1)^m B[u,u](t) + C_2\|u\|^2_{H^{m-1}(\Omega)}, \tag{2.6}$$

where $C_1 = \frac{1}{C_0-\varepsilon}$, $C_2 = \frac{C(\varepsilon)}{C_0-\varepsilon} > 0$. Following [1, Theorem 4.15], we have for all $\varepsilon > 0$, there exists a constant $C_3(\varepsilon)$ such that the inequality

$$\sum_{|p|=k} \|D^p u\|^2_{L_2(\Omega)} \le \varepsilon \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)} + C_3(\varepsilon)\|u\|^2_{L_2(\Omega)} \tag{2.7}$$

holds for all $k = 1, 2, \dots, m-1$, and for all $u \in H^m(\Omega)$. Note that for all $0 < T \le +\infty$, if $u \in H^{m,0}_\gamma(Q_T)$, $\gamma > 0$, then for almost fixed point $t_1 \in (0,T)$ we have $u(x,t_1) \in H^m(\Omega)$ and (2.7) is valid for $u(x,t_1)$. Because $\varepsilon, C_3(\varepsilon)$ are independent of $t_1 \in (0,T)$, so one gets

$$\sum_{|p|=k} \|D^p u(x,t)\|^2_{L_2(\Omega)} \le \varepsilon \sum_{|p|=m} \|D^p u(x,t)\|^2_{L_2(\Omega)} + C(\varepsilon)\|u(x,t)\|^2_{L_2(\Omega)} \tag{2.8}$$

for all $k = 1, 2, \dots, m-1$, for all $u \in H^{m,0}_\gamma(Q_T)$ and almost $t \in (0,T)$. This follows that for all $0 < \varepsilon < 1$, there exists $C_4 = C_4(\varepsilon)$ such that the following inequality holds

$$\|u\|^2_{H^{m-1}(\Omega)} \le \varepsilon\|u\|^2_{H^m(\Omega)} + C_4\|u\|_{L_2(\Omega)} \tag{2.9}$$

for all $u \in H^{m,0}_\gamma(Q_T)$, almost $t \in (0,T)$.

Hence, from (2.6) and (2.9) we have

$$\begin{aligned}\|u\|^2_{H^m(\Omega)} &\le C_1(-1)^m B[u,u](t) + (C_2+1)\|u\|^2_{H^{m-1}(\Omega)}\\ &\le C_1(-1)^m B[u,u](t) + (C_2+1)[\varepsilon\|u\|^2_{H^m(\Omega)} + C_4\|u\|^2_{L_2(\Omega)}]\end{aligned}$$

for all $0 < \varepsilon < \min\{1, C_0, \frac{1}{C_2+1}\}$. So we obtain

$$(-1)^m B[u,u](t) \ge \mu_0\|u\|^2_{H^m(\Omega)} - \lambda_0\|u\|^2_{L_2(\Omega)},$$

where $\mu_0 = [1-(C_2+1)\varepsilon](C_0-\varepsilon) > 0$, $\lambda_0 = C_4(C_2+1)(C_0-\varepsilon)$. This proves the lemma. $\square$

From lemma 2.1, using the transformation $u(x,t) = e^{i\lambda_0 t}v(x,t)$ if necessary, we can assume that the operator $L(x,t,D)$ satisfies

$$(-1)^m B[u,u](t) \ge \mu_0\|u\|^2_{H^m(\Omega)} \tag{2.10}$$

for all $u \in H^{m,0}_\gamma(Q_T)$, almost $t \in (0,T)$.

3. The uniqueness and existence theorems

In this section we investigate the unique solvability of the second initial boundary value problem for the system (2.3) with non homogeneously initial condition (2.4) in the space $H^{m,0}_\gamma(Q_T)$, $\gamma > 0$, where $0 < T \le +\infty$.

Denote $m^* = \sum_{|p|=1}^m 1$, we begin by studying the uniqueness theorem.

Theorem 3.1. *Assume that for a positive constant μ,*

$$\sup\{|\frac{\partial a_{pq}}{\partial t}| : (x,t) \in Q_T, 0 \le |p|, |q| \le m\} \le \mu\,.$$

Then the second initial boundary value problem for (2.3) *with non homogeneously initial condition* (2.4) *has at most one generalized solution in* $H_{\gamma}^{m,0}(Q_T)$ *for all* $\gamma > 0$ *arbitrary.*

Proof. Suppose that the problem has two solutions u_1, u_2 in $H_{\gamma}^{m,0}(Q_T)$. Put $u = u_1 - u_2$. For all $0 < \tau < T$ and $b \in (0, \tau)$ we set

$$\eta(x,t) = \begin{cases} \int_b^t u(x,s)ds, & 0 \le t \le b, \\ 0, & b < t \le \tau. \end{cases}$$

It is easy to check that $\eta(x,t) \in H^{m,1}(Q_\tau), \eta(x,\tau) = 0$ and $\eta_t(x,t) = u(x,t)$ for all $(x,t) \in Q_b$. It follows from (2.5) that

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_b} a_{pq} D^q \eta_t \overline{D^p \eta} \, dx \, dt + i \int_{Q_b} |\eta_t|^2 \, dx \, dt = 0. \tag{3.1}$$

Adding this equation with its complex conjugate, using $a_{pq} = (-1)^{|p|+|q|} a_{qp}^*$ and integrating by parts with respect to t, we get

$$B[\eta,\eta](0) = -\sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_b} \frac{\partial a_{pq}}{\partial t} D^q \eta \overline{D^p \eta} \, dx \, dt.$$

Since $|\frac{\partial a_{pq}}{\partial t}|$ are bounded, using the Cauchy inequality and (2.10), we obtain

$$\|\eta(x,0)\|^2_{H^m(\Omega)} \le C \|\eta(x,t)\|^2_{H^{m,0}(Q_b)}, \quad (C = \mu m^*/\mu_0 > 0). \tag{3.2}$$

Putting $v_p(x,t) = \int_t^0 D^p u(x,s)ds$, $0 < t < b$, $0 \le |p| \le m$, so we have $D^p \eta(x,t) = \int_b^t D^p u(x,s)ds = v_p(x,b) - v_p(x,t)$, $D^p \eta(x,0) = v_p(x,b)$. Substituting those into (3.2), one gets

$$\sum_{|p|=0}^{m} \int_{\Omega} |v_p(x,b)|^2 dx \le 2Cb \sum_{|p|=0}^{m} \int_{\Omega} |v_p(x,b)|^2 dx + 2C \sum_{|p|=0}^{m} \int_{Q_b} |v_p(x,t)|^2 \, dx \, dt.$$

Setting $J(t) = \sum_{|p|=0}^{m} \int_{\Omega} |v_p(x,t)|^2 dx$, we have $(1 - 2Cb) J(b) \le 2C \int_0^b J(t) dt$, or $J(b) \le 4C \int_0^b J(t) dt$, for all $b \in [0, \frac{1}{4C}]$. This implies that $J(t) \equiv 0$ on $[0, \frac{1}{4C}]$ by Gronwall-Bellman's inequality. It follows $u_1 \equiv u_2$ on $[0, \frac{1}{4C}]$, where C does not depend on τ. Using similar arguments for two functions u_1, u_2 on $[\frac{1}{4C}, \tau]$, we can show that after finite steps we get $u_1 \equiv u_2$ on $[0, \tau]$. Since $0 < \tau < T$ is arbitrary, so $u_1 \equiv u_2$ on $(0, T)$. The theorem is proved. □

Now, we establish the existence of generalized solution of the mentioned problem by Galerkin's approximate method.

Theorem 3.2. *Assume that:*

(i) *For a positive constant* μ, $|a_{pq}(x,0)|$; $\left|\frac{\partial a_{pq}}{\partial t}\right| \le \mu$, *for all* $0 \le |p|, |q| \le m$; *and all* $(x,t) \in Q_T$;
(ii) $f, f_t \in L^\infty(0,T; L_2(\Omega))$, $f(.,0) \in L_2(\Omega)$;
(iii) $\varphi \in H^m(\Omega)$.

Then for every $\gamma > \gamma_0 = \frac{m^*\mu}{2\mu_0}$ *the second initial boundary value problem for* (2.3)–(2.4) *has a generalized solution* $u(x,t)$ *in the space* $H^{m,0}_\gamma(Q_T)$ *and the following estimate holds*

$$\|u\|^2_{H^{m,0}_\gamma(Q_T)} \le C\left[\|\varphi\|^2_{H^m(\Omega)} + \|f(.,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \|f_t\|^2_\infty\right],$$

where the constant C *only depends on* μ, μ_0.

Proof. Let $\{\varphi_k(x)\}_{k=1}^\infty$ be a basis of $H^m(\Omega)$, which is orthonormal in $L_2(\Omega)$. We find an approximate solution $u^N(x,t)$ in the form $u^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x)$, where $\{C_k^N(t)\}_{k=1}^N$ satisfies

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q u^N \overline{D^p\varphi_l}dx - \int_\Omega u_t^N \overline{\varphi_l}dx = \int_\Omega f\overline{\varphi_l}dx, \quad (3.3)$$

$$C_l^N(0) = \int_\Omega \varphi(x)\varphi_l(x)dx, \quad l = 1,\dots,N. \quad (3.4)$$

From (i), (ii) and (3.3)–(3.4) it follows that coefficients $C_k^N(t)$ are defined uniquely and $\|u^N(x,0)\|^2_{H^m(\Omega)} \le \|\varphi(x)\|^2_{H^m(\Omega)}$ for all $N = 1,2,\dots$.

After multiplying (3.3) by $\frac{d\overline{C_l^N(t)}}{dt}$, taking sum with respect to l from 1 to N, we get

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q u^N \overline{D^p u_t^N}dx - i\int_\Omega u_t^N \overline{u_t^N}dx = i\int_\Omega f\overline{u_t^N}dx. \quad (3.5)$$

Adding this equality and its complex conjugate, we have

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq}\frac{\partial}{\partial t}(D^q u^N \overline{D^p u^N})dx = -2\,\mathrm{Im}\int_\Omega f\overline{u_t^N}dx.$$

So for all $0 < \tau < T$, by integrating with respect to t from 0 to τ, and integrating by parts, we obtain

$$\begin{aligned}(-1)^m B[u^N,u^N](\tau) &= (-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_\tau} \frac{\partial a_{pq}}{\partial t} D^q u^N \overline{D^p u^N}\,dx\,dt \\ &\quad + (-1)^m B[u^N,u^N](0) - 2\,\mathrm{Im}\int_\Omega f(x,0)\overline{u^N}(x,0)dx \\ &\quad + 2\,\mathrm{Im}\Big[\int_\Omega f(x,\tau)\overline{u^N}(x,\tau)dx - \int_{Q_\tau} f_t\overline{u^N}\,dx\,dt\Big].\end{aligned}$$

This implies by Cauchy's inequality and (2.10) that for all $0 < \varepsilon < \mu_0$,

$$\begin{aligned}\|u^N(x,\tau)\|^2_{H^m(\Omega)} &\le \frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}\int_0^\tau \|u^N(x,t)\|^2_{H^m(\Omega)}dt + \frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}\|u^N(x,0)\|^2_{H^m(\Omega)} \\ &\quad + \frac{1}{\varepsilon(\mu_0-\varepsilon)}\left[\|f(x,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \tau\|f_t\|^2_\infty\right].\end{aligned}$$

Applying Gronwall-Bellman's inequality, one gets

$$\begin{aligned}&\|u^N(x,\tau)\|^2_{H^m(\Omega)} \\ &\le C_1\Big[\|u^N(x,0)\|^2_{H^m(\Omega)} + \|f(x,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \|f_t\|^2_\infty\Big]e^{\frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}\tau},\end{aligned}$$

where $C_1 = \max\{\frac{m^*\mu}{\mu_0-\varepsilon},\ \frac{1}{\varepsilon(\mu_0-\varepsilon)}\} > 0$. Therefore,

$$\|u^N(x,\tau)\|^2_{H^m(\Omega)} \le C_1\Big[\|\varphi\|^2_{H^m(\Omega)} + \|f(x,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \|f_t\|^2_\infty\Big] e^{\frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}\tau}. \quad (3.6)$$

For each $\gamma > \gamma_0 = \frac{\mu m^*}{2\mu_0} = \inf_{(0,\mu_0)} \frac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}$ we can choose $\varepsilon \in (0,\mu_0)$ such that $\gamma > \frac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}$; i.e., $-2\gamma + \frac{m^*\mu+\varepsilon}{(\mu_0-\varepsilon)} < 0$. Multiplying (3.6) with $e^{-2\gamma\tau}$, then integrating with respect to τ from 0 to T, we obtain

$$\|u^N\|^2_{H^{m,0}_\gamma(Q_T)} \le C\big[\|\varphi\|^2_{H^m(\Omega)} + \|f(.,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \|f_t\|^2_\infty\big], \quad (3.7)$$

where $C > 0$ independent of N. Since the sequence $\{u^N\}$ is uniformly bounded in $H^{m,0}_\gamma(Q_T)$, we can take a subsequence, denoted also by $\{u^N\}$ for convenience, which converges weakly to a vector function $u(x,t)$ in $H^{m,0}_\gamma(Q_T)$.

We will prove that $u(x,t)$ is a generalized solution of the problem. Since

$$M = \bigcup_{N=1}^{\infty}\{\sum_{l=1}^{N} d_l(t)\varphi_l(x), d_l(t) \in H^1(0,\tau),\ d_l(\tau) = 0,\ \forall l = 1,2,\dots,N\}$$

is dense in the space of test functions $\hat{H}^{m,1}(Q_\tau) = \{\eta(x,t) \in H^{m,1}(Q_\tau), \eta(x,\tau) = 0\}$ for all $0 < \tau < T$ so it suffices to show that $u(x,t)$ satisfies (2.5) for all $\eta(x,t) \in M$. Note that the denseness of the set M in the space $\hat{H}^{m,1}(Q_\tau)$ can be proved easily by using lemma 2.1 and arguments analogous as that used in the first problem (see in [2, 3]).

Taking $\eta(x,t) \in M$ arbitrarily, there exists N_0 such that η can be written in the form $\eta(x,t) = \sum_{l=1}^{N_0} d_l(t)\varphi_l(x), d_l(t) \in H^1(0,\tau),\ d_l(\tau) = 0,\ \forall l = 1,\dots,N_0$. Multiplying (3.3) (with $N \ge N_0$) by $d_l(t)$, taking sum with respect to l from 1 to N, then integrating with respect to t from 0 to τ, we obtain

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_\tau} a_{pq} D^q u^N \overline{D^p\eta}\,dx\,dt - \int_{Q_\tau} u^N_t \overline{\eta}\,dx\,dt = \int_{Q_\tau} f\overline{\eta}\,dx\,dt.$$

It is easy to check that $\int_{Q_\tau} u^N_t\overline{\eta}\,dx\,dt = -\int_\Omega \varphi(x)\overline{\eta(x,0)}dx - \int_{Q_\tau} u^N\overline{\eta_t}\,dx\,dt$, so one has

$$\begin{aligned}&(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_\tau} a_{pq} D^q u^N \overline{D^p\eta}\,dx\,dt + \int_{Q_\tau} u^N\overline{\eta}_t\,dx\,dt\\ &= -\int_\Omega \varphi(x)\overline{\eta(x,0)}dx + \int_{Q_\tau} f\overline{\eta}\,dx\,dt.\end{aligned}$$

Passing to the limit for the weakly convergent subsequence, we get

$$\begin{aligned}&(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_\tau} a_{pq} D^q u\overline{D^p\eta}\,dx\,dt + \int_{Q_\tau} u\overline{\eta}_t\,dx\,dt\\ &= -\int_\Omega \varphi(x)\overline{\eta(x,0)}dx + \int_{Q_\tau} f\overline{\eta}\,dx\,dt.\end{aligned}$$

Hence $u(x,t)$ is a generalized solution of the second initial boundary value problem for the system (2.3)–(2.4). Moreover, the weak convergence of the subsequence of $\{u^N(x,t)\}$ and (3.7) imply that this solution satisfies the inequality

$$\|u\|^2_{H^{m,0}_\gamma(Q_T)} \le \liminf_{N\to\infty} \|u^N\|^2_{H^{m,0}_\gamma(Q_T)}$$

$$\le C\big[\|\varphi\|^2_{H^m(\Omega)} + \|f(.,0)\|^2_{L_2(\Omega)} + \|f\|^2_\infty + \|f_t\|^2_\infty\big],$$

where C only depends on μ, μ_0. This completes the proof. □

4. Smoothness of generalized solutions with respect to time

In this section, we consider the second initial boundary value problem for the system

$$i(-1)^{m-1}L(x,t,D)u - u_t = f(x,t), \quad (x,t) \in Q_T, \tag{4.1}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{4.2}$$

We will prove that the smoothness with respect to time variable of generalized solution of the second initial boundary value problem for the Schrödinger system (4.1)–(4.2) depends on only the smoothness with respect to time variable of the coefficients and the right side of the systems. Indeed, we have the following theorem.

Theorem 4.1. *Suppose that*

(i) *for some positive constant μ, $\{|a_{pq}(x,0)|,\ |\frac{\partial^k a_{pq}}{\partial t^k}(x,t)|\} \le \mu$, for all $0 \le |p|, |q| \le m$, all $(x,t) \in Q_T$, all $1 \le k \le h+1$;*
(ii) *$f_{t^k} \in L^\infty(0,T;L_2(\Omega))$, for all $0 \le k \le h+1$, $f(x,0) = 0$, if $h \ge 2$ then we assume that $f_{t^k}(x,0) = 0$, for all $1 \le k \le h-1$, all $x \in \Omega$.*

Then for every $\gamma > \gamma_0 = \frac{m^*\mu}{2\mu_0}$ the generalized solution $u(x,t)$ of the second problem for (4.1)–(4.2) has the generalized derivatives with respect to t up to order h in the space $H^{m,0}_{(2h+1)\gamma}(Q_T)$ and the following estimate holds

$$\|u_{t^h}\|^2_{H^{m,0}_{(2h+1)\gamma}(Q_T)} \le C\sum_{k=0}^{h+1}\|f_{t^k}\|^2_\infty, \tag{4.3}$$

where the constant C does not depend on u and f.

Proof. Let $\{\varphi_k(x)\}_{k=1}^\infty$ be a basis of $H^m(\Omega)$, which is orthonormal in $L_2(\Omega)$. For each natural number N, we set $u^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x)$, where $\{C_k^N(t)\}_{k=1}^N$ is the solution of the ordinary differential system

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_\Omega a_{pq}D^q u^N\overline{D^p\varphi_l}dx - \int_\Omega u_t^N\overline{\varphi_l}dx = \int_\Omega f\overline{\varphi_l}dx, \tag{4.4}$$

with $C_l^N(0) = 0$, $l = 1,\dots,N$.

From (i), (ii), it follows that coefficients $C_k^N(t)$, defined uniquely by (4.4), have derivatives up to order $h+1$ and $u^N(x,0) = 0$.

We will prove that

$$D^p u^N_{t^k}(x,0) = 0, \quad \forall 0 \le k \le h,\ 0 \le |p| \le m, \forall x \in \Omega. \tag{4.5}$$

Indeed, it is clear that (4.5) holds for $k=0$. Differentiating (4.4) $(k-1)$ times with respect to t, multiplying by $\frac{d^k}{dt^k}\big(\overline{C_l^N(t)}\big)$, then taking sum with respect to l from 1 to N, we obtain

$$
\begin{aligned}
&- i\int_\Omega \left|u^N_{t^k}\right|^2 dx + (-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_\Omega a_{pq}D^q u^N_{t^{k-1}}\overline{D^p u^N_{t^k}}dx \\
&= (-1)^{m-1}\sum_{|p|,|q|=0}^m (-1)^{|p|}\sum_{s=0}^{k-2} C^s_{k-1}\int_\Omega \frac{\partial^{k-s-1}a_{pq}}{\partial t^{k-s-1}} D^q u^N_{t^s}\overline{D^p u^N_{t^k}}dx \\
&\quad + i\int_\Omega f_{t^{k-1}}\overline{u^N_{t^k}}dx.
\end{aligned}
\tag{4.6}
$$

By using (ii) and induction on k, we obtain (4.5) holds for all $0 \le k \le h$.

In the following part, we shall prove the inequalities

$$
\left\|u^N_{t^h}(x,\tau)\right\|^2_{H^m(\Omega)} \le Ce^{\lambda_h\tau}\sum_{k=0}^{h+1}\|f_{t^k}\|^2_\infty, \quad \forall 0<\tau<T,\ \forall N=1,2,\dots, \tag{4.7}
$$

$$
\left\|u^N_{t^h}\right\|^2_{H^{m,0}_{(2h+1)\gamma}(Q_T)} \le C\sum_{k=0}^{h+1}\left\|f_{t^k}\right\|^2_\infty \tag{4.8}
$$

are valid with $0<\varepsilon<\mu_0$, $\lambda_h = \frac{(2h+1)m^*\mu+\varepsilon}{\mu_0-\varepsilon}$; C does not depend on N, f.

From the inequalities (3.6)–(3.7) (with $\varphi(x)=0, f(x,0)=0$), we can see easily that (4.7)–(4.8) hold for $h=0$, and $\{u^N\}$ convergent weakly to the solution u of the problem in $H^{m,0}_\gamma(Q_T)$.

Now let (4.7)–(4.8) be true for $h-1$. We will prove that these also hold for h. Integrating (4.6), for $k=h+1$, with respect to t from 0 to τ, we get

$$
\begin{aligned}
&- i\int_{Q_\tau} \left|u^N_{t^{h+1}}\right|^2 dx\,dt + (-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_{Q_\tau} a_{pq}D^q u^N_{t^h}\overline{D^p u^N_{t^{h+1}}}\,dx\,dt \\
&= (-1)^{m-1}\sum_{|p|,|q|=0}^m (-1)^{|p|}\sum_{s=0}^{h-1} C^s_h\int_{Q_\tau} \frac{\partial^{h-s}a_{pq}}{\partial t^{h-s}} D^q u^N_{t^s}\overline{D^p u^N_{t^{h+1}}}\,dx\,dt \\
&\quad + i\int_{Q_\tau} f_{t^h}\overline{u^N_{t^{h+1}}}\,dx\,dt.
\end{aligned}
$$

Adding this equation with its complex conjugate then integrating by parts with respect to t, using (ii), (4.5), we obtain

$$
\begin{aligned}
&(-1)^m B[u^N_{t^h}(x,\tau), u^N_{t^h}(x,\tau)] \\
&= (-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_{Q_\tau} \frac{\partial a_{pq}}{\partial t} D^q u^N_{t^h}\overline{D^p u^N_{t^h}}\,dx\,dt \\
&\quad + (-1)^m 2\operatorname{Re}\sum_{|p|,|q|=0}^m (-1)^{|p|}.h.\int_{Q_\tau} \frac{\partial a_{pq}}{\partial t} D^q u^N_{t^h}\overline{D^p u^N_{t^h}}\,dx\,dt \\
&\quad + (-1)^m 2\operatorname{Re}\sum_{|p|,|q|=0}^m (-1)^{|p|}\sum_{s=0}^{h-1} C^s_h\int_{Q_\tau} \frac{\partial^{h-s+1}a_{pq}}{\partial t^{h-s+1}} D^q u^N_{t^s}\overline{D^p u^N_{t^h}}\,dx\,dt \\
&\quad + (-1)^m 2\operatorname{Re}\sum_{|p|,|q|=0}^m (-1)^{|p|}\sum_{s=0}^{h-2} C^s_h\int_{Q_\tau} \frac{\partial^{h-s}a_{pq}}{\partial t^{h-s}} D^q u^N_{t^{s+1}}\overline{D^p u^N_{t^h}}\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- (-1)^m\, 2\operatorname{Re}\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\sum_{s=0}^{h-1}C_h^s\int_\Omega \frac{\partial^{h-s}a_{pq}}{\partial t^{h-s}}(x,\tau)D^q u^N_{t^s}(x,\tau)\overline{D^p u^N_{t^h}(x,\tau)}dx\\
&- 2\operatorname{Im}\int_\Omega f_{t^h}(x,\tau)\overline{u^N_{t^h}(x,\tau)}dx \ - 2\operatorname{Im}\int_{Q_\tau} f_{t^{h+1}}\overline{u^N_{t^h}}\,dx\,dt.
\end{aligned}
$$

For all $\varepsilon_1>0$, using Cauchy's inequality and (2.10), we have

$$
\begin{aligned}
&[\mu_0-(\mu m^*(2^h-1)+1)\varepsilon_1]\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)}\\
&\le [(2h+1)m^*\mu+((2^{h+1}-2-h)\mu m^*+1)\varepsilon_1]\int_0^\tau \big\|u^N_{t^h}(x,t)\big\|^2_{H^m(\Omega)}dt\\
&+C\Big[\sum_{k=0}^{h-1}\big\|u^N_{t^k}\big\|^2_{H^{m,0}(Q_\tau)}+\sum_{k=0}^{h-1}\big\|u^N_{t^k}(x,\tau)\big\|^2_{H^m(\Omega)}+\|f_{t^h}\|^2_\infty+\tau\big\|f_{t^{h+1}}\big\|^2_\infty\Big],
\end{aligned}
$$

where $C=\max\{\frac{2\mu m^* M}{\varepsilon_1},\frac{1}{\varepsilon_1}\}$, $M=\max_{s=\overline{0,h-1}}C_h^s$.

Set $\varepsilon=((2^{h+1}-2-h)\mu m^*+1)\varepsilon_1\ge((2^h-1)\mu m^*+1)\varepsilon_1>0$ for $h>0$. This implies that for all $0<\varepsilon<\mu_0$,

$$
\begin{aligned}
&\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)}\\
&\le \frac{(2h+1)m^*\mu+\varepsilon}{\mu_0-\varepsilon}\int_0^\tau \big\|u^N_{t^h}(x,t)\big\|^2_{H^m(\Omega)}dt\\
&+C_1\Big[\sum_{k=0}^{h-1}\big\|u^N_{t^k}\big\|^2_{H^{m,0}(Q_\tau)}+\sum_{k=0}^{h-1}\big\|u^N_{t^k}(x,\tau)\big\|^2_{H^m(\Omega)}+\|f_{t^h}\|^2_\infty+\tau\big\|f_{t^{h+1}}\big\|^2_\infty\Big],
\end{aligned}
$$

where C_1 is a positive constant.

Using the induction assumption, one has

$$
\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)}\le\lambda_h\int_0^\tau \big\|u^N_{t^h}(x,t)\big\|^2_{H^m(\Omega)}dt+C_2e^{\lambda_{h-1}\tau}(1+\tau)\sum_{k=0}^{h+1}\big\|f_{t^k}\big\|^2_\infty, \tag{4.9}
$$

where $C_2=\text{const}>0$. Applying Gronwall-Bellman's inequality, we obtain

$$
\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)}\le C_3\,e^{\lambda_h\tau}\sum_{k=0}^{h+1}\big\|f_{t^k}\big\|^2_\infty, \tag{4.10}
$$

where C_3 is a positive constant. We can choose $0<\varepsilon<\mu_0$ such that $(2h+1)\gamma>\frac{\lambda_h}{2}$ for all $\gamma>\gamma_0=\frac{\mu m^*}{2\mu_0}$, because

$$
\inf_{0<\varepsilon<\mu_0}\frac{(2h+1)m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}=\frac{(2h+1)m^*\mu}{2\mu_0}<(2h+1)\gamma\,.
$$

After multiplying (4.10) with $e^{-2(2h+1)\,\gamma\,\tau}$, then integrating with respect to τ from 0 to T, we have the inequality

$$
\big\|u^N_{t^h}\big\|^2_{H^{m,0}_{(2h+1)\gamma}(Q_T)}\le C\sum_{k=0}^{h+1}\big\|f_{t^k}\big\|^2_\infty, \tag{4.11}
$$

where C is a positive number, independent of N, f. Hence (4.7)–(4.8) hold for h. Since $\{u^N_{t^h}\}$ is bounded in $H^{m,0}_{(2h+1)\gamma}(Q_T)$ for all $\gamma>\gamma_0$, we can choose a subsequence which converges weakly to a vector function $u^{(h)}$ in $H^{m,0}_{(2h+1)\gamma}(Q_T)$. On the other hand, one has

$$\int_{Q_T} u^N_{t^h} v\,dx\,dt = -(1)^h \int_{Q_T} u^N v_{t^h}\,dx\,dt, \quad \forall v \in C_0^\infty(Q_T).$$

Passing $N \to \infty$, it follows that $\int_{Q_T} u^{(h)} v\,dx\,dt = -(1)^h \int_{Q_T} u v_{t^h}\,dx\,dt$, for all $v \in C_0^\infty(Q_T)$; i.e., u has generalized derivatives up to order h with respect to t and $u_{t^h} = u^{(h)}$. Furthermore, by passing (4.11) to the limit for the weakly convergent subsequence, we obtain

$$\|u_{t^h}\|^2_{H^{m,0}_{(2h+1)\gamma}(Q_T)} \le C \sum_{k=0}^{h+1} \|f_{t^k}\|^2_\infty. \tag{4.12}$$

The theorem is proved. □

Remark 4.2. We also have the same results of the smoothness with respect to time variable of the solution of the system (2.3)–(2.4) if the initial function $\varphi(x)$ is required to be in $H^m(\Omega)$ space and the coefficients a_{pq} and the right side f are required to satisfy some suitable conditions.

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References

[1] Adams, R. A.; *Sobolev Spaces*, Academic Press, 1975.

[2] Hung, N. M.; *The first initial boundary value problem for Schrödinger systems in non-smooth domains*, Diff. Urav., 34 (1998), pp.1546-1556 (in Russian).

[3] Hung, N. M.; Anh, C.T.; *On the solvability of the first initial boundary value problem for Schrödinger systems in infinite cylinders*, Vietnam J. Math., 32(1) (2004), pp. 41- 48.

[4] Hung, N. M.; Anh, C. T.; *On the smoothness of solutions of the first initial boundary value problem for Schrödinger systems in domains with conical points*, Vietnam J. Math.,32(2) (2005), pp. 135-147.

[5] Kokotov, A.; Plamenevskii, B.; *On the asymptotic on solutions to the Neumann problem for hyperbolic systems in domains with conical points*, Algebra i analiz 16 (2004)(in Russian), English transl., St. Peterburg Math.J.16(2005), N° 3, pp. 477-506.

[6] Ladyzhenskaya, O. A.; *On the non-stationary operator equations and its application to linear problems of Mathematical Physics*, Mat. Sbornik. **45** (87) (1958) 123-158 (in Russian).

[7] Lions, J. L. and Magenes, E.; *Non-homogeneous boundary value problems and applications*,Vol 1,2, Springer,1972.

[8] Maz'ya, V. G.; Kozlov, V. A.; Rossmann, J.; *Elliptic boundary value problems in domains with point singularities,* Mathematical Surveys and Monographs **52**, Amer. Math.Soc., Providence, Rhode Island, 1997.

[9] Maz'ya, V. G.; Kozlov, V. A.; *Points estimates for Green'matrix to boundary value problems for second order elliptic systems in a polyhedral cone,* Z. angew. Math.Mech. 00(2004), No, 0, pp. 1-30.

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