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ASYMPTOTIC BEHAVIOR FOR A DISSIPATIVE PLATE EQUATION IN $\mathbb{R}^N$ WITH PERIODIC COEFFICIENTS

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Abstract. In this work we study the asymptotic behavior of solutions of a dissipative plate equation in $\mathbb{R}^N$ with periodic coefficients. We use the Bloch waves decomposition and a convenient Lyapunov function to derive a complete asymptotic expansion of solutions as $t \to \infty$. In a first approximation, we prove that the solutions for the linear model behave as the homogenized heat kernel.

1. Introduction

The aim of this paper is to study the asymptotic behavior, for large time, of solutions of the following Cauchy problem associated with vibrations of thin plates and beams

$$\begin{gathered} u_{tt} + A^2 u + aAu_{tt} + bAu_t = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = \varphi_0(x)\,, \quad u_t(x,0) = \varphi_1(x). \end{gathered} \tag{1.1}$$

Here, a and b are positive constants and A is the divergence operator

$$A \equiv -\frac{\partial}{\partial x_k}\Big(a_{k\ell}(x)\frac{\partial}{\partial x_\ell}\Big) \tag{1.2}$$

where the coefficients $\{a_{k\ell}(x)\}_{k,\ell=1}^N$ are Y-periodic, with $Y =]0,2\pi[^N$ and

$$a_{k\ell} \in L^\infty_\#(Y) = \{\phi \in L^\infty(\mathbb{R}^N)\,;\, \phi(x+2\pi p) = \phi(x)\,, \forall\, x \in \mathbb{R}^N\,, \forall\, p \in \mathbb{Z}^N\}. \tag{1.3}$$

We also assume that the operator A is elliptic and symmetric, that is

$$\begin{gathered} \exists\, \alpha > 0 \,\text{such that}\, a_{k\ell}(x)\eta_k\eta_\ell \geq \alpha|\eta|^2,\, \forall\, \eta \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N; \\ a_{k\ell} = a_{\ell k}\, \forall\, k,\ell = 1,2,\dots,N. \end{gathered} \tag{1.4}$$

The energy associate with the problem (1.1) is given by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^N} \Big[|u_t|^2 + |Au|^2 + a\, a_{k\ell}(x)\frac{\partial u_t}{\partial x_k}\ \frac{\partial u_t}{\partial x_\ell}\Big]dx \tag{1.5}$$

and satisfies the dissipation law

$$\frac{dE}{dt} = -b\int_{\mathbb{R}^N} |A^{1/2}u_t|^2 dt = -b\int_{\mathbb{R}^N} a_{k\ell}(x)\frac{\partial u_t}{\partial x_k}\ \frac{\partial u_t}{\partial x_\ell}\ dx. \tag{1.6}$$

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This indicates that the term bAu_t in the equation in (1.1) plays the role of a feedback damping mechanism. Consequently, $E(t)$ is a nonincreasing function and the following basic question arises: Does $E(t) \to 0$ as $t \to \infty$ and, if yes, is it possible to find the decay rate of $E(t)$?

An important model associated to (1.1) is the nonlinear plate equation with periodic coefficients

$$\begin{gathered} u_{tt} + A^2 u + aAu_{tt} - M\Big(\int_{\mathbb{R}^N} |A^{\frac{1}{2}} u|^2 dx\Big) Au + bAu_t = 0, \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x) \end{gathered} \tag{1.7}$$

where $M = M(s) \geq 0$ for all s. When $n = 1$ such a model is a general mathematical formulation of a problem arising in the dynamic buckling of a hinged extensible beam with infinite measure under an axial force. If $n = 2$, equations in (1.1) represent the "Berger approximation" of the full dynamic von Kármán system modelling the nonlinear vibrations of a plate. A rigorous mathematical justification for this fact was given in [15] where it was shown that the limit is a linear plate model, i.e., $M \equiv 0$.

Roughly speaking, when the medium is homogeneous, i.e., the coefficients are constant, the plane waves $e^{i\xi\cdot x}$ serve as an effective tool for transforming the differential equation into a set of algebraic equations. If the medium is periodic (which is true in the present case) there exists an exact theory, by which the response of the medium can be obtained, that serves the same purpose. This method is based on Floquet theory in ordinary differential equations, and known in the waves literature as Bloch waves decomposition. Simply put, Bloch waves decomposition gives a representation for the solution of the problem in terms of an eigenvalue problem. These waves were originally introduced by Bloch (see [3]) in solid state physics in the context of propagation of electrons in a crystal. Since that time, several questions and properties of periodic media were translated in terms of Bloch waves. We refer to [6] for a wide variety of applications in the vibrations of fluid-solid structures and to [1] and [20] for additional references on Bloch waves.

Equations of fourth-order appear in problems of solid mechanics, in particular, in the theory of thin plates and beams. Also, elliptic equations of fourth-order appear in problems related with the Navier-Stokes equations (see Mozolevski-Süli-Bösing[14]). The model we are considering here is an optional one since, in some cases, the vibrations of thin plates are given by the full von Kármán system which have been studied by several authors (see, for instance, Lasiecka [11], [12], [13], Koch-Lasiecka [9], Puel-Tucsnak [18]).

In this work we are interested in using Bloch waves decomposition to study the asymptotic behavior of the solutions of the linear model (1.1). For (1.1) we are going to prove that, in a first approximation, the solutions behave as a linear combination, at some order k, depending on the initial data, of derivatives of the fundamental solution of the homogenized heat equation modulated by periodic functions. By homogenized heat equation we mean the underlying parabolic homogenized system

$$\begin{gathered} u_t - q_{k\ell} \frac{\partial^2 u}{\partial x_k \partial x_\ell} = 0, \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \delta_0(x), \end{gathered} \tag{1.8}$$

where $\delta_0(x)$ is the Dirac delta distribution at the origin and $\{q_{k\ell}\}_{k,\ell=1}^N$ are the homogenized coefficients associated to the periodic matrix with coefficients (1.3)-(1.4). We remark that the homogenized coefficients $q_{k\ell}$ associated to the periodic matrix a are given by (see [1] and [19])

$$q_{k\ell} = \frac{1}{|Y|}\int_Y a_{k\ell} dy + \frac{1}{|Y|}\int_Y a_{km}\,\frac{\partial \chi_\ell}{\partial y_m}\,dy\,, \quad 1 \le k,\ \ell \le N \tag{1.9}$$

where χ_j is the solution of the Y-periodic elliptic problem

$$-\frac{\partial}{\partial x_k}\Big(a_{k\ell}\,\frac{\partial \chi_i}{\partial x_\ell}\Big) = \frac{\partial a_{j\ell}}{\partial y_\ell} \tag{1.10}$$

where χ_j is Y-periodic, and $1 \le j \le N$. The solution χ_j of this equation is uniquely determined up to an additive constant. Moreover, the homogenized matrix $\{q_{k\ell}\}_{k,\ell=1}^N$ given in (1.9) is symmetric, that is,

$$q_{k\ell} = q_{\ell k} \tag{1.11}$$

and elliptic with the same constant α of ellipticity for the matrix $\{a_{k\ell}(x)\}_{k,\ell=1}^N$ in (1.2), that is

$$q_{k\ell}\xi_k\xi_\ell \ge \alpha|\xi|^2\,, \quad \xi \in \mathbb{R}^N\,. \tag{1.12}$$

We refer the reader to [4] and [5] for more details on homogenization.

A similar analysis was done in [16] by J.H. Ortega and E. Zuazua. In this article, the authors obtain a complete asymptotic expansion (for large time) of the solution of linear parabolic equations with periodic coefficients and $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ data. Such problem is somewhat surprisingly related to the problem of homogenization of parabolic equations with periodically oscillating coefficients. This is one of the effects of the scaling laws present in such equations. Furthermore, these scaling laws transform the original initial data to another one approximating Dirac mass. Thus, the large-time behavior of the solution is governed by that of the fundamental solution of the homogenized equation, a fact which is already known from the work of G. Duro and E. Zuazua [8] (see also [7]). Exploiting this idea we address the same issue and we prove the main result of this paper, i.e., we conclude that the solutions of (1.1) behave as the homogenized heat kernel, as $t \to \infty$. In fact, equation in (1.1) can be viewed as a "perturbed" heat equation

$$u_t + Au = -A^{-1}(I + A)u_{tt}$$

and, according to our analysis, the behavior of solutions as, $t \to \infty$, does not change in a first approximation. It is also possible to see the influence of the first eigenvalue due to the presence of the operator A^{-1} on the right hand side of the above equation. Thus, our result is not so similar to the asymptotic expansion obtained in [17] for dissipative wave equation with periodic coefficients and in [2] where the Benjamin-Bona-Mahony equation with periodic coefficients was considered. Moreover, the asymptotic expansion for the plate equation depends on two waves given by two Kernels that will be defined latter. In addition, from the decomposition described above it is possible to see that the total mass of the solution is given by the first term in the asymptotic expansion.

We observe that the Bloch waves decomposition provides an orthogonal decomposition of $L^2(\mathbb{R}^N)$. Therefore, our results is established in the L^2-setting with $L^2 \cap L^1(\mathbb{R}^N) \times H^{-1} \cap L^1(\mathbb{R}^N)$ initial data.

2. Main result

The well-posedness of (1.1) under the conditions (1.3) and (1.4) can be obtained writing (1.1) as an abstract evolution equation in the space of finite energy

$$\mathcal{H} = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$$

with the inner product

$$((u,v),(\tilde{u},\tilde{v}))_{\mathcal{H}} = \int_{\mathbb{R}^N} u\tilde{u}dx + \int_{\mathbb{R}^N} AuA\tilde{u}dx + \int_{\mathbb{R}^N} v\tilde{v}dx + a\int_{\mathbb{R}^N} a_{k\ell}(x)\frac{\partial v}{\partial x_k}\ \frac{\partial \tilde{v}}{\partial x_\ell}dx,$$

with $\{a_{k\ell}(x)\}_{k,l=1}^N$ as in (1.3) and (1.4), whenever $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{H}$. Under these conditions the operator associated to (1.1) is maximal and dissipative on $\mathcal{H}$. Then, Lummer-Phillip's theorem guarantees that the operator associated to (1.1) is the infinitesimal generator of a continuous semigroup. Thus, we deduce that for any initial data $(\varphi_0,\varphi_1) \in L^2(\mathbb{R}^N)\times H^{-1}(\mathbb{R}^N)$, problem (1.1) has a unique weak solution $u = u(x,t)$ such that

$$u \in \mathcal{C}(\mathbb{R}^+, L^2(\mathbb{R}^N)) \cap \mathcal{C}^1(\mathbb{R}^+, H^{-1}(\mathbb{R}^N)).$$

Let us now state the main result of this work.

Theorem 2.1 (Asymptotic expansion). *Let the initial data $\varphi^0 \in L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ and $\varphi^1 \in H^{-1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ with $|x|^k\varphi^0(x)$, $|x|^k\varphi^1(x) \in L^1(\mathbb{R}^N)$ for some fixed integer $k \geq 1$. Let $u = u(x,t)$ be the solution of (1.1) with $b^2 \neq 4$. Then, there exist periodic functions $c^i_\alpha(\cdot) \in L^\infty_\#(Y)$, $|\alpha| \leq k$, $i = 1,2$ and constants $d^i_{\beta,n}$, $i = 1,2$, depending on initial data and the coefficients $a_{k\ell}$ such that*

$$\begin{aligned}
&\Big\| u(\cdot,t) - \sum_{|\alpha|\leq k} \Big\{ c^1_\alpha(\cdot)\Big[G^-_\alpha(\cdot,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^1_{\beta,n} G^-_{\alpha+\beta}(\cdot,t)\Big] \\
&+ c^2_\alpha(\cdot)\Big[G^+_\alpha(\cdot,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^2_{\beta,n} G^+_{\alpha,\beta}(\cdot,t)\Big]\Big\}\Big\|_{L^2(\mathbb{R}^N)} \\
&\leq C_k t^{-\frac{2k+N-2}{4}}
\end{aligned}$$

as $t \to \infty$, where C_k is a positive constant depending on k, the initial data and the coefficients $a_{k\ell}$. The integers $p = p(\alpha)$ and $p_1 = p_1(\alpha,n)$ are given by $p(\alpha) = [\frac{k-|\alpha|}{2}]$ and $p_1 = p(\alpha) - n$. The functions $G^\pm(x,t)$ are defined by

$$G^\pm_\alpha(x,t) = \int_{\mathbb{R}^N} \xi^\alpha |\xi|^{-2} e^{-\frac{b\pm\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t} e^{ix\cdot\xi} d\xi\,, \quad |\alpha| \leq k.$$

Here, $q_{k\ell}$ are the homogenized constant coefficients associated with the matrix $a = \{a_{k\ell}(\frac{x}{\varepsilon})\}_{k,\ell=1}^N$, as $\varepsilon \to 0$.

Remark 2.2. When $b^2 = 4$ the asymptotic expansion is the same, except for the fact that the decay rate is $t^{-\frac{2k+N-4}{4}}$. This can be seen in the proof of Lemma 4.8.

It is important to observe that the convergence result given by Theorem 1.1 indicates that the solution u of (1.1) can be, roughly, approximated at any order by a linear combination of the derivatives of the fundamental solution of the heat equation, modulated by the periodic functions.

Here, as in the previous works, to obtain the main result we use the Bloch waves decomposition. This is done following closely the work of Conca and Vanninathan [5] which shows how classical homogenization results may be recovered using Bloch waves decomposition for elliptic equations. As we shall see, when deriving higher order asymptotic results for (1.1), two types of terms appear: first, we get those terms that are provided by the moments of the initial data and then, those that are generated by the microstructure. This second contribution may be obtained by a careful analysis of the first Bloch mode. The contribution of the higher Bloch modes may be neglected since they provide terms that decay exponentially as $t \to \infty$, which is in agreement with the elliptic results of [5].

The rest of this work is organized as follows: The next section contains the basic results on Bloch waves. In Section 3 we present some technical lemmas that we use in the Section 4. Section 4 is devoted to the asymptotic expansion. In Section 5 we prove the main result of this work, i.e., Theorem 2.1.

3. Bloch Waves Decomposition

In this section we recall same basic results on Bloch wave decomposition. We refer to [4], [5] and [20] for the notations and the proofs.

Let us consider the following spectral problem depending on a parameter $\xi \in \mathbb{R}^N$: to find $\lambda = \lambda(\xi) \in \mathbb{R}^N$ and a function $\psi = \psi(x,\xi)$ not identically zero, such that

$$\begin{gathered} A\psi(\cdot,\xi) = \lambda(\xi)\psi(\cdot,\xi) \quad \text{in } \mathbb{R}^N \\ \psi(\cdot,\xi) \text{is } (\xi,Y)\text{-periodic; i.e.,} \\ \psi(x+2\pi m,\xi) = e^{2\pi i m\cdot\xi}\psi(x,\xi) \quad \forall\, m \in \mathbb{Z}^N,\ x \in \mathbb{R}^N, \end{gathered} \tag{3.1}$$

where $i = \sqrt{-1}$ and A is the elliptic operator in divergence form given in (1.2).

We can write $\psi(x,\xi) = e^{ix.\xi}\phi(x,\xi)$, ϕ being Y-periodic in the variable x. From (3.1) we can observe that the (ξ,Y)-periodicity is unaltered if we replace ξ by $(\delta+n)$, with $n \in \mathbb{Z}^N$. Therefore, ξ can be confined to the dual cell $Y' = [-1/2, 1/2[^N$.

From the ellipticity and symmetry assumption on the matrix $\{a_{k,\ell}(x)\}_{k,\ell=1}^N$ it is possible to prove (see for instance [1]) that for each $\xi \in Y'$ the spectral problem (3.1) admits a sequence of eigenvalues $\{\lambda_m(\xi)\}_{m\in\mathbb{N}}$ with the following properties:

$$\begin{gathered} 0 \le \lambda_1(\xi) \le \dots \le \lambda_m(\xi) \le \dots \to +\infty, \\ \lambda_m(\xi) \text{ is a Lipschitz function of } \xi \in Y',\ \forall\, m \ge 1\,. \end{gathered} \tag{3.2}$$

Besides, the sequences $\{\psi_m(x,\xi)\}_{m\in\mathbb{N}}$ and $\{\phi_m(x,\xi)\}_{m\in\mathbb{N}}$ of the corresponding eigenfunctions constitute orthonormal basis in the subspaces of $L^2_{loc}(\mathbb{R}^N)$ which are (ξ,Y)-periodic and Y-periodic, respectively. Moreover, as a consequence of min-max principle we have that

$$\dots \ge \lambda_m(\xi) \ge \dots \ge \lambda_2(\xi) \ge \lambda_2^N > 0\,, \quad \forall\, \xi \in Y' \tag{3.3}$$

where λ_2^N is the second eigenvalue of the operator A, given in (3.2) in the cell Y with Neumann boundary conditions on ∂Y.

The Bloch waves introduced above enable us to describe the spectral resolution of the unbounded self-adjoint operator A in $L^2(\mathbb{R}^N)$, in the orthogonal basis of Bloch waves

$$\{\psi_m(x,\xi) = e^{ix\cdot\xi}\phi_m(x,\xi) : m \ge 1,\, \xi \in Y'\}.$$

The result related to this subject is as follows.

Proposition 3.1. *Let $g \in L^2(\mathbb{R}^N)$. The m-th Bloch coefficients of g is defined as follows:*

$$\hat{g}_m(\xi) = \int_{\mathbb{R}^N} g(x) e^{-ix\cdot\xi}\, \overline{\phi_m(x,\xi)}\, dx, \ \forall\ m \geq 1,\ \xi \in Y'.$$

Then, the inverse formula

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \hat{g}_m(\xi) e^{ix\cdot\xi} \phi_m(x,\xi) d\xi$$

and the Parseval's identity

$$\int_{\mathbb{R}^N} |g(x)|^2 dx = \int_{Y'} \sum_{m=1}^{\infty} |\hat{g}_m(\xi)|^2 d\xi \tag{3.4}$$

hold. Furthermore, for all g in the domain of A, we have

$$Ag(x) = \int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi) \hat{g}_m(\xi) e^{ix\cdot\xi} \phi_m(x,\xi) d\xi$$

and, consequently, the equivalence of norms in $H^1(\mathbb{R}^N)$ and $H^{-1}(\mathbb{R}^N)$ given by

$$\|g\|^2_{H^s(\mathbb{R}^N)} = \int_{Y'} \sum_{m=1}^{\infty} \left(1 + \lambda_m(\xi)\right)^s |\hat{g}_m(\xi)|^2 d\xi, \quad s = 1, -1.$$

Remark 3.2. Observe that Parseval's identity guarantees that $L^2(\mathbb{R}^N)$ may be identified with $L^2(Y', \ell^2(\mathbb{N}))$.

The following result on the behavior of $\lambda_1(\xi)$ and $\phi_1(x,\xi)$, near $\xi = 0$, will also be necessary in this work (see [4] and [5]).

Proposition 3.3. *We assume that $\{a_{k\ell}(x)\}_{k,\ell=1}^N$ satisfy the conditions (1.4). Then there exists $\delta_1 > 0$ such that the first eigenvalue $\lambda_1(\xi)$ is an analytic function on $B_{\delta_1} := \{\xi \in Y', |\xi| < \delta_1\}$, and there is a choice of the first eigenvector $\phi_1(\cdot,\xi)$ such that*

$$\xi \to \phi_1(\cdot,\xi) \in L^\infty_\#(Y) \cap H^1_\#(Y)$$

is analytic on B_{δ_1} and

$$\phi_1(x,0) = |Y|^{-1/2} = \frac{1}{(2\pi)^{N/2}}\,, \quad x \in \mathbb{R}^N\,.$$

Moreover,

$$\begin{gathered}
\lambda_1(0) = 0\,, \quad \partial_k \lambda_1(0) = 0\,, \quad 1 \leq k \leq N\,, \\
\frac{1}{2}\partial^2_{k\ell}\lambda_1(0) = q_{k\ell}\,, \quad 1 \leq k,\ell \leq N, \\
\partial^\alpha \lambda_1(0) = 0, \quad \forall\, \alpha \text{ such that } |\alpha| \text{ is odd}
\end{gathered}$$

and

$$c_1|\xi|^2 \leq \lambda_1(\xi) \leq c_2|\xi|^2\,, \quad \xi \in B_{\delta_1}, \tag{3.5}$$

where c_1 and c_2 are positive constants.

4. Asymptotic Expansion

We begin this section with a basic lemma on asymptotic analysis and some technical results which will be useful in the proof of the asymptotic expansion of solutions of (1.1). For the proofs, we refer to [16], [17] and [4], respectively.

Lemma 4.1. *Let $c > 0$. Then*

$$\int_{B_\gamma} e^{-c|\xi|^2 t} |\xi|^k d\xi \sim C_k t^{-\frac{k+N}{2}}, \quad \forall\, k \in \mathbb{N}, \tag{4.1}$$

as $t \to +\infty$, where C_k is a positive constant which may be computed explicitly.

Lemma 4.2. *Let $\varphi \in L^1(\mathbb{R}^N)$ be a function such that $|x|^k\varphi \in L^1(\mathbb{R}^N)$. Then its first Bloch coefficient $\hat{\varphi}_1(\xi)$ belongs to $C^k(B_\delta)$, with B_δ a neighborhood of $\xi = 0$ where there first Bloch wave $\phi_1(\cdot,\xi)$ is analytic.*

Lemma 4.3. *Consider the function*

$$G(x) = \int_{Y'} g(\xi) e^{ix\cdot\xi} w(x,\xi) d\xi\,,\ x \in \mathbb{R}^N \tag{4.2}$$

where $g \in L^2(Y')$ and $w \in L^\infty\Big(Y', L^2_\#(Y)\Big)$. Then $G \in L^2(\mathbb{R}^N)$ and

$$\|G\|^2_{L^2(\mathbb{R}^N)} = \int_{Y'} |g(\xi)|^2 \|w(\cdot,\xi)\|^2_{L^2(Y)} d\xi\,.$$

Next, we want to compute the Bloch coefficients of the solution u of (1.1) and derive a result on the dependence of such coefficients with respect to the parameter ξ.

Lemma 4.4. *Let $u = u(x,t)$ be the solution of (1.1). Then,*

$$u(x,t) = \sum_{m=1}^{\infty} \int_{Y'} \Big[\beta^1_m(\xi) e^{-\alpha^1_m(\xi)t} + \beta^2_m(\xi) e^{-\alpha^2_m(\xi)t}\Big] e^{ix\cdot\xi} \phi_m(x,\xi) d\xi \tag{4.3}$$

with

$$\beta^1_m(\xi) = \frac{(1+a\lambda_m(\xi))\alpha^2_m(\xi)}{\lambda_m(\xi)\sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}\ \hat{\varphi}^0_m + \frac{1+a\lambda_m(\xi)}{\lambda_m(\xi)\sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}\ \hat{\varphi}^1_m \tag{4.4}$$

$$\beta^2_m(\xi) = -\frac{(1+a\lambda_m(\xi))\alpha^1_m(\xi)}{\lambda_m(\xi)\sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}\ \hat{\varphi}^0_m - \frac{1+a\lambda_m(\xi)}{\lambda_m(\xi)\sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}\ \hat{\varphi}^1_m \tag{4.5}$$

where $\hat{\varphi}^0_m$ and $\hat{\varphi}^1_m$ are the Bloch coefficients of the initial data φ_0 and φ_1, respectively. The functions α^1_m and α^2_m are given by

$$\begin{aligned} \alpha^1_m(\xi) &= \frac{b - \sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}{2\,(1+a\lambda_m(\xi))}\ \lambda_m(\xi), \\ \alpha^2_m(\xi) &= \frac{b + \sqrt{b^2 - 4\,(1+a\lambda_m(\xi))}}{2\,(1+a\lambda_m(\xi))}\ \lambda_m(\xi). \end{aligned} \tag{4.6}$$

Proof. Since $u(x,t) \in L^2(\mathbb{R}^N)$ for all $t > 0$, we have

$$u(x,t) = \int_{Y'} \sum_{m=1}^{\infty} \hat{u}_m(\xi,t) e^{ix\cdot\xi} \phi_m(x,\xi) d\xi \tag{4.7}$$

where $\hat{u}_m$ are the Bloch coefficients of $u = u(x,t)$ given by Proposition 3.1. Furthermore, since

$$A(e^{-ix\cdot\xi}\overline{\phi_m(x,\xi)}) = \overline{A(e^{ix\cdot\xi}\phi_m(x,\xi))} = \lambda_m(\xi)e^{-ix\cdot\xi}\overline{\phi_m(x,\xi)}$$

and $\{\phi_m(x,\cdot)\}_{m\in\mathbb{N}}$ is orthonormal, it follows from (1.1) that the functions $\hat{u}_m(\xi,t)$ satisfy the following ordinary differential equation

$$\begin{gathered}(1+a\lambda_m(\xi))\,\partial_t^2\hat{u}_m(\xi,t) + \lambda_m^2(\xi)\hat{u}_m(\xi,t) + b\lambda_m(\xi)\partial_t\hat{u}_m(\xi,t) = 0,\\ \text{in } Y' \times (0,+\infty) \\ \hat{u}_m(\xi,0) = \hat{\varphi}_m^0(\xi), \quad \partial_t\hat{u}_m(\xi,0) = \hat{\varphi}_m^1(\xi), \quad \xi \in Y',\ t>0\end{gathered} \tag{4.8}$$

for each $m \geq 1$. Solving the differential equation above we find

$$\hat{u}_m(\xi,t) = \beta_m^1(\xi)e^{-\alpha_m^1(\xi)t} + \beta_m^2(\xi)e^{-\alpha_m^2(\xi)t} \tag{4.9}$$

where $\{\alpha^i(\xi)\}$, $i=1,2$, are defined by (4.6) and they are the two roots of the characteristic equation

$$(1+a\lambda_m(\xi))\,r^2 + b\lambda_m(\xi)r + \lambda_m^2(\xi) = 0. \tag{4.10}$$

It is easy to see that the terms β_m^1 and β_m^2 given in (4.4) and (4.5), respectively, are obtained in order to satisfy the initial data in (4.8). $\square$

Since $\alpha_1^1(\xi)$ and $\alpha_1^2(\xi)$ are also defined by (4.6) we use Proposition 3.3 to obtain the following result.

Lemma 4.5. *Assume that the $\{a_{k\ell}(x)\}_{k,\ell=1}^N$ satisfy* (1.4)*. Then, there exists $\delta>0$ such that the functions $\alpha_1^i(\xi)$ and $\beta_1^i(\xi)$, i=1, 2, defined in* (4.4)-(4.6) *are analytic functions on $B_\delta := \{\xi \in Y', |\xi| < \delta\}$. Furthermore, $\alpha_1^1(\xi)$ and $\alpha_1^2(\xi)$ satisfy*

$$\begin{aligned} c_5|\xi|^2 \leq |\alpha_1^1(\xi)| \leq c_6|\xi|^2, \quad \forall\,\xi \in B_\delta,\\ c_7|\xi|^2 \leq |\alpha_1^2(\xi)| \leq c_8|\xi|^2, \quad \forall\,\xi \in B_\delta\end{aligned} \tag{4.11}$$

and

$$\begin{gathered}\alpha_1^i(0) = \partial_k\alpha_1^i(0) = 0, \quad k=1,2,\dots,N,\ i=1,2\\ \partial^\beta\alpha_1^i(0) = 0, \quad \forall\,\beta \text{ such that } |\beta| \text{ is odd, } i=1,2\\ \partial_{k\ell}^2\,\alpha_1^1(0) = \Big(b - \sqrt{b^2-4}\Big)\,q_{k\ell}, \quad k,\ell=1,2,\dots,N\\ \partial_{k\ell}^2\,\alpha_1^2(0) = \Big(b + \sqrt{b^2-4}\Big)\,q_{k\ell}, \quad k,\ell=1,2,\dots,N\end{gathered} \tag{4.12}$$

where c_5, c_6, c_7 and c_8 are positive constants.

Proof. Let $0<\delta\leq\delta_1$, with δ_1 given by Proposition 3.3. Then we can consider two cases:
(a) If $b^2>4$ we choose $\delta>0$ satisfying $b^2-4-4ac_2\delta^2>0$. Then, for $|\xi|\leq\delta$, Proposition 3.3 give us that

$$\alpha_1^1(\xi) = \frac{b-\sqrt{b^2-4(1+a\lambda_1(\xi))}}{2\,(1+a\lambda_1(\xi))}\ \lambda_1(\xi) \geq \frac{\big(b-\sqrt{b^2-4}\big)\,c_1|\xi|^2}{2(1+ac_2\delta^2)} = c_3|\xi|^2$$

and

$$\alpha_1^1(\xi) \leq \frac{b\lambda_1(\xi)}{2} \leq \frac{bc_2|\xi|^2}{2} = c_4|\xi|^2\,.$$

(b) If $b^2 \le 4$ we can choose any $\delta \le \delta_1$. Then, since

$$\alpha_1^1(\xi) = \Big[\frac{b}{2\,(1+a\lambda_1(\xi))} - i\,\frac{\sqrt{-b^2+4\,(1+a\lambda_1(\xi))}}{2\,(1+a\lambda_1(\xi))}\Big]\lambda_1(\xi)\,,$$

it is easy to see that there exist positive constants c_5 and c_6 such that

$$c_5|\xi|^2 \le \big|\alpha_1^1(\xi)\big| \le c_6|\xi|^2 \quad \text{for } |\xi| \le \delta.$$

In the same way we can obtain $c_7 > 0$ and $c_8 > 0$ satisfying

$$c_7|\xi|^2 \le |\alpha_1^2(\xi)| \le c_8|\xi|^2 \quad \text{for } |\xi| \le \delta$$

with $\delta > 0$ as given in (a) or (b). The second part of the Lemma is straightforward and follows from Proposition 3.3. □

The next steps are devoted studying the asymptotic behavior of the Bloch coefficients of the solution u computed in Lemma 4.4

4.1. **Bloch components of u with exponential decay.** First, we prove that the terms in (4.3) corresponding to the eigenvalues $\lambda_m(\xi),\ m \ge 2$, decay exponentially to zero as $t \to \infty$. Then, we show that the term corresponding to $\lambda_1(\xi)$ also goes to zero exponentially, whenever $\xi \in Y'\backslash B_\delta = \{\xi \in Y',\ |\xi| > \delta\}$.

Lemma 4.6. *Let $\hat u_m = \hat u_m(\xi,t)$ be the Bloch coefficients of the solution $u = u(x,t)$ of (1.1). Then, there exist positive constants c and ν_0 such that*

$$\int_{Y'} \sum_{m\ge 2} |\hat u_m(\xi,t)|^2 d\xi \le ce^{-\nu_0 t}\Big(\|\varphi^0\|^2_{L^2(\mathbb{R}^N)} + \|\varphi^1\|^2_{H^{-1}(\mathbb{R}^N)}\Big) \tag{4.13}$$

for all $t > 0$.

Proof. We consider the Liapunov function associated to ordinary differential equation in (4.8)

$$L_m(\xi,t) = E_m(\xi,t) + \varepsilon F_m(\xi,t),\ \varepsilon > 0 \tag{4.14}$$

where

$$E_m(\xi,t) = \frac{1}{2}\Big[\,|\partial_t\hat u_m(\xi,t|^2 + \frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}\,|\hat u_m(\xi,t)|^2\,\Big],$$

$$F_m(\xi,t) = \overline{\hat{u_m}(\xi,t)}\ \partial_t\hat u_m(\xi,t) + \frac{b\lambda_m(\xi)}{2\,(1+a\lambda_m(\xi))}\ |\hat u_m(\xi,t)|^2\,.$$

Then, since $\lambda_m(\xi) \ge \lambda_2^N > 0$, $\forall\ m \ge 2$, we have

$$\begin{aligned}
&|L_m(\xi,t) - E_m(\xi,t)| = \varepsilon|F_m(\xi,t)|\\
&\le \varepsilon\Big[|\hat u_m(\xi,t)||\partial_t\hat u_m(\xi,t)| + \frac{b\lambda_m(\xi)}{2\,(1+a\lambda_m(\xi))}|\hat u_m(\xi)|^2\Big]\\
&\le \varepsilon\Big[\frac{\lambda_m^2(\xi)}{2\,(1+a\lambda_m(\xi))}\,|\hat u_m(\xi,t)|^2 + \frac{1+a\lambda_m(\xi)}{2\lambda_m^2(\xi)}|\partial_t\hat u_m(\xi,t)|^2\\
&\quad + \frac{b\lambda_m^2(\xi)}{2\lambda_2^N\,(1+a\lambda_m(\xi))}|\hat u_m(\xi,t)|^2\Big]\\
&= \frac{\varepsilon}{2}\Big[(1+\frac{b}{\lambda_2^N})\frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}|\hat u_m(\xi,t)|^2 + \frac{1+a\lambda_m(\xi)}{\lambda_m^2(\xi)}|\partial_t\hat u_m(\xi,t)|^2\Big].
\end{aligned}$$

Moreover, due to

$$\frac{1+a\lambda_m(\xi)}{\lambda_m^2(\xi)} \le \frac{1}{\left(\lambda_2^N\right)^2} + \frac{a}{\lambda_2^N}, \quad m \ge 2,$$

we obtain

$$\begin{aligned}&|L_m(\xi,t)-E_m(\xi,t)|\\ &\le \frac{\varepsilon}{2}\Big[\big(1+\frac{b}{\lambda_2^N}\big)\ \frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}|\hat{u}_m(\xi,t)|^2+\big(\frac{1}{(\lambda_2^N)^2}+\frac{a}{\lambda_2^N}\big)\Big]|\partial_t\hat{u}_m(\xi,t)|^2\end{aligned}$$

for all $m \ge 2$. Thus,

$$|L_m(\xi,t)-E_m(\xi,t)| \le \varepsilon c_0 E_m(\xi,t), \quad \forall\, m \ge 2 \tag{4.15}$$

with $c_0 = \max\big\{\frac{1}{(\lambda_2^N)^2}+\frac{a}{\lambda_2^N}, 1+\frac{b}{\lambda_2^N}\big\}$. Consequently, for $0<\varepsilon<\frac{1}{c_0}$ we have

$$0 < (1-\varepsilon c_0)E_m(\xi,t) \le L_m(\xi,t) \le (1+\varepsilon c_0)E_m(\xi,t). \tag{4.16}$$

Now, we claim that

$$\partial_t L_m(\xi,t) \le -\,cL_m(\xi,t) \tag{4.17}$$

holds for some positive constant c independent of ξ, whenever $m \ge 2$. To prove this claim we proceed as follows:

Multiplying the equation in (4.8) by $\overline{\hat{u}_m(\xi,t)}$, we have

$$\partial_t F_m(\xi,t) = -\frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}\ |\hat{u}_m(\xi,t)|^2 + |\partial_t\hat{u}_m(\xi,t)|^2. \tag{4.18}$$

Next, we multiply the equation in (4.8) by $\overline{\partial_t\hat{u}_m(\xi,t)}$ to obtain

$$\partial_t E_m(\xi,t) = -\frac{b\lambda_m(\xi)}{1+a\lambda_m(\xi)}\ |\partial_t\hat{u}_m(\xi,t)|^2\,. \tag{4.19}$$

Then, multiplying (4.18) by ε and adding with (4.19) it results

$$\partial_t L_m(\xi,t) = \Big(\varepsilon - \frac{b\lambda_m(\xi)}{1+a\lambda_m(\xi)}\Big)|\partial_t\hat{u}_m(\xi,t)|^2 - \frac{\varepsilon\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}|\hat{u}_m(\xi,t)|^2. \tag{4.20}$$

On the other hand, since $\lambda_m(\xi) \ge \lambda_2^N > 0$, $\forall\, m \ge 2$, we get

$$\frac{\lambda_2^N}{1+a\lambda_2^N} \le \frac{\lambda_m(\xi)}{1+a\lambda_m(\xi)} \le \frac{1}{a}, \quad \forall\ m \ge 2\,.$$

Consequently, choosing $0<\varepsilon\le \min\big\{\frac{1}{2c_0}, \frac{b\lambda_2^N}{2(1+a\lambda_2^N)}\big\}$, where c_0 is given in (4.15), we deduce that

$$\partial_t L_m(\xi,t) \le -c\,E_m(\xi,t) \tag{4.21}$$

for some positive constant $c=c(\varepsilon)$. Now, combining the above inequality and (4.16) the following holds

$$E_m(\xi,t) \le c_9 E_m(\xi,0)e^{-\nu_0 t} \tag{4.22}$$

for some positive constant ν_0 which does not depend on t and ξ and $c_9 = \frac{1+\varepsilon c_0}{1-\varepsilon c_0} > 0$. Recalling the definition of $E_m(\xi,t)$, from (4.22) we deduce that

$$\frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}\ |\hat{u}_m(\xi,t)|^2 \le c_9\Big[|\partial_t\hat{u}_m(\xi,0)|^2 + \frac{\lambda_m^2(\xi)}{1+a\lambda_m(\xi)}\ |\hat{u}_m(\xi,0)|^2\Big]e^{-\nu_0 t}.$$

Due to $\lambda_m(\xi) \geq \lambda_2^N$, for $m \geq 2$, there exists a positive constant c_{10} such that

$$\frac{1 + a\lambda_m(\xi)}{\lambda_m^2(\xi)} \leq \frac{c_{10}}{1 + \lambda_m(\xi)}, \quad \forall\, m \geq 2.$$

Then, we obtain that

$$|\hat{u}_m(\xi,t)|^2 \leq \Big\{\frac{c_9\, c_{10}}{1 + \lambda_m(\xi)} |\hat{\varphi}_m^1(\xi)|^2 + c_9 |\hat{\varphi}_m^0(\xi)|^2\Big\} e^{-\nu_0 t}$$

where $\hat{\varphi}_m^1(\xi)$ and $\hat{\varphi}_m^0(\xi)$ are the Bloch coefficients of the initial data φ_1 and φ_0, respectively. Consequently, according to Proposition 3.1

$$\int_{Y'} \sum_{m \geq 2} |\hat{u}_m(\xi,t)|^2 d\xi \leq c\left[\|\varphi^1\|^2_{H^{-1}(\mathbb{R}^N)} + \|\varphi^0\|^2_{L_2(\mathbb{R}^N)}\right] e^{-\nu_0 t} \tag{4.23}$$

for $t > 0$, where $c > 0$ is a constant which does not depend on ξ and t. □

Lemma 4.7. *Let $\hat{u}_1(\xi,t)$ be the first Bloch coefficients of the solution u of* (1.1) *given in (4.3). Then, there exist positive constants c and ν_1 such that*

$$\int_{Y' \setminus B_\delta} |\hat{u}_1(\xi,t)|^2 d\xi \leq c\left[\, \|\varphi^0\|^2_{L^2(\mathbb{R}^N)} + \|\varphi^1\|^2_{H^{-1}(\mathbb{R}^N)}\right] e^{-\nu_1 t} \tag{4.24}$$

for all $t \geq 0$, where δ is given in Lemma 4.5 and satisfies $0 \leq \delta \leq \delta_1$, with δ_1 as in Proposition 3.3.

Proof. To prove (4.24) we argue as in the previous lemma. But, instead of using the fact that $\lambda_m(\xi) \geq \lambda_2^N > 0$, $\forall\, \xi \in Y'$ and $m \geq 2$, we use the fact that $c_1|\xi|^2 \leq \lambda_1(\xi) \leq c_2|\xi|^2$ for all $\xi \in B_\delta$ (see Proposition 3.3). □

4.2. **Bloch component of u with polynomial decay.** According to the previous analysis, To prove the asymptotic expansion of the solution $u(x,t)$ of (1.1), it is sufficient to analyze

$$I(x,t) = \int_{B_\delta} \left[\beta_1^1(\xi) e^{-\alpha_1^1(\xi)t} + \beta_1^2(\xi) e^{-\alpha_1^2(\xi)t}\right] e^{ix\cdot\xi} \phi_1(x,\xi) d\xi \tag{4.25}$$

with $\delta > 0$ given in Lemma 4.5, since the other components of u, in particular the term $\int_{Y' \setminus B_\delta} |\hat{u}_1(\xi,t)|^2 d\xi$ decay exponentially. The asymptotic expansion of the solution is obtained from the term in (4.25). To analyze $I(x,t)$ defined above we make use of classical asymptotic lemmas and assume that the initial data $\varphi^0 \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $\varphi^1 \in H^{-1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ are such that $|x|^k \varphi^0(x)$, $|x|^k \varphi^1(x) \in L^1(\mathbb{R}^{\mathbb{N}})$ for some $k \geq 1$. Under these conditions the first Bloch coefficients $\hat{\varphi}_1^0(\xi)$ and $\hat{\varphi}_1^1(\xi)$ of the initial data belong to $C^k(B_\delta)$, which is crucial in the proof of the asymptotic expansion. Indeed, a further Taylor's development of the first term in the asymptotics shows a connection with the fundamental solution of the heat equation.

In this way, we begin by considering

$$J(x,t) = \int_{B_\delta} \frac{1}{\lambda_1(\xi)} \sum_{|\alpha| \leq k} \left[d_\alpha^1 e^{-\alpha_1^1(\xi)t} + d_\alpha^2 e^{-\alpha_1^2(\xi)t}\right] \xi^\alpha e^{ix\cdot\xi} \phi_1(x,\xi) d\xi \tag{4.26}$$

where

$$d_\alpha^1 = \frac{1}{\alpha!}\, \partial^\alpha \left(\lambda_1 \beta_1^1\right)(0) \;\text{ and }\; d_\alpha^2 = \frac{1}{\alpha!}\, \partial^\alpha \left(\lambda_1 \beta_1^2\right)(0),\;\; \alpha \in \mathbb{N}^N$$

which are the Taylor's coefficients of $\beta_1^1(\xi)$ and $\beta_1^2(\xi)$ around $\xi = 0$, respectively.

Then, we have the following result.

Lemma 4.8. *Let $\varphi^0 \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\varphi^1 \in H^{-1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ such that $|x|^{k+1}\varphi^0(x)$ and $|x|^{k+1}\varphi^1(x) \in L^1(\mathbb{R}^N)$. Then, there exists $\delta > 0$ and a positive constant c_k such that*

$$\|I(\cdot,t) - J(\cdot,t)\|_{L^2(\mathbb{R}^N)} \le c_k t^{-\frac{2k+N-2}{4}} \tag{4.27}$$

as $t \to \infty$, where $I(x,t)$ and $J(x,t)$ are defined in (4.25) and (4.26), respectively.

Proof. By the assumptions on $\varphi^0(x)$ and $\varphi^1(x)$ if follows that $\hat{\varphi}_1^0(\xi)$ and $\hat{\varphi}_1^1(\xi) \in C^{k+1}(B_\delta)$. According to the proof of Lemma 4.5, $0 \le \delta \le \delta_1$ satisfies

$$b_\delta = b^2 - 4 - 4ac_2\delta^2 > 0 \text{ if } b^2 > 4 \quad \text{or} \quad \delta > 0 \text{ is any value if } b^2 < 4,$$

where c_2 and δ_1 are given in Proposition 3.3. This allows us to conclude that

$$\begin{gathered} b^2 - 4 - 4a\lambda_1(\xi) > b_\delta \quad \text{if } b^2 > 4; \\ b^2 - 4 - 4a\lambda_1(\xi) < b^2 - 4 < 0 \quad \text{if } b^2 < 4 \end{gathered} \tag{4.28}$$

whenever $\xi \in B_\delta$. Here, we observe that

$$\begin{aligned} \beta_1^1(\xi) &= \frac{1}{\lambda_1(\xi)}\Big[\frac{b + \sqrt{b^2 - 4\,(1 + a\lambda_1(\xi))}}{2\sqrt{b^2 - 4\,(1 + a\lambda_1(\xi))}}\ \lambda_1(\xi)\hat{\varphi}_1^0(\xi) \\ &\quad + \frac{1 + a\lambda_1(\xi)}{\sqrt{b^2 - 4\,(1 + a\lambda_1(\xi))}}\hat{\varphi}_1^1(\xi)\Big] \\ &= \frac{1}{\lambda_1(\xi)}\,\tilde{\beta}_1^1(\xi) \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \beta_1^2(\xi) &= \frac{1}{\lambda_1(\xi)}\Big[-\frac{b + \sqrt{b^2 - 4(1 + a\lambda_1(\xi))}}{2\sqrt{b^2 - 4\,(1 + a\lambda_1(\xi))}}\ \lambda_1(\xi)\hat{\varphi}_1^0(\xi) \\ &\quad - \frac{1 + a\lambda_1(\xi)}{\sqrt{b^2 - 4(1 + a\lambda_1(\xi))}}\hat{\varphi}_1^1(\xi)\Big] \\ &= \frac{1}{\lambda_1(\xi)}\tilde{\beta}_1^2(\xi). \end{aligned} \tag{4.30}$$

Since $\lambda_1(\xi)$ is analytic and $\hat{\varphi}_1^0$, $\hat{\varphi}_1^1 \in C^{k+1}(B_\delta)$, we have $\tilde{\beta}_1^1$, $\tilde{\beta}_1^2 \in C^{k+1}(B_\delta)$, for $\delta > 0$ sufficiently small. Then, we can write

$$\tilde{\beta}_1^1(\xi) = \sum_{|\alpha| \le k} d_\alpha^1 \xi^\alpha \quad \text{and} \quad \tilde{\beta}_1^2(\xi) = \sum_{|\alpha| \le k} d_\alpha^2 \xi^\alpha$$

where $d_\alpha^1 = \frac{\partial^\alpha \tilde{\beta}_1^1(0)}{\alpha!}$ and $d_\alpha^2 = \frac{\partial^\alpha \tilde{\beta}_1^2(0)}{\alpha!}$. Thus, from Taylor's expansion we obtain positive constants $\tilde{c}_k^1$ and $\tilde{c}_k^2$, depending on the integer k, such that

$$\begin{gathered} |\tilde{\beta}_1^1(\xi) - \sum_{|\alpha| \le k} d_\alpha^1 \xi^\alpha| \le \tilde{c}_k^1\, |\xi|^{k+1} \\ |\tilde{\beta}_1^2(\xi) - \sum_{|\alpha| \le k} d_\alpha^2 \xi^\alpha| \le \tilde{c}_k^2\, |\xi|^{k+1} \end{gathered} \tag{4.31}$$

for all $\xi \in B_\delta$. Consequently, from Parseval's identity (see Proposition 3.1) we get

$$\begin{aligned} \|I(\cdot,t)-J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le \int_{B_\delta} \frac{1}{\lambda_1^2(\xi)} \Big|\tilde\beta_1^1(\xi) - \sum_{|\alpha|\le k} d^1_\alpha \xi^\alpha\Big|^2 \Big|\, e^{-\alpha_1^1(\xi)t}\Big|^2 d\xi + \\ + \int_{B_\delta} \frac{1}{\lambda_1^2(\xi)} \Big|\tilde\beta_1^2(\xi) - \sum_{|\alpha|\le k} d^2_\alpha \xi^\alpha\Big|^2 \Big|\, e^{-\alpha_1^2(\xi)t}\Big|^2 d\xi. \end{aligned} \tag{4.32}$$

Now, using (4.31) and Proposition 3.3 it results

$$\|I(\cdot,t)-J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le C_k \int_{B_\delta} |\xi|^{2k-2}\big[\,|e^{-\alpha_1^1(\xi)t}|^2 + |e^{-\alpha_1^2(\xi)t}|^2\,\big] d\xi. \tag{4.33}$$

In order to conclude the proof, we consider two cases:
Case $b^2 > 4$: In this case, it follows from (4.28) that $\alpha_1^1(\xi)$, $\alpha_1^2(\xi) \ge 0$. Therefore, combining Lemma 4.5, Lemma 4.1 and (4.33) we get

$$\|I(\cdot,t)-J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le 2C_k \int_{B_\delta} |\xi|^{2k-2}\ e^{-\tilde c\,|\xi|^2 t} d\xi \le \tilde C_k t^{-\frac{2k+N-2}{2}}, \quad \text{as } t\to\infty,$$

where C_k and $\tilde C_k$ are positive constants.
Case $0 < b^2 < 4$: Now, according to (4.28) (see also item (b) in the proof of Lemma 4.5) the functions $\alpha_1^1(\xi)$ and $\alpha_1^2(\xi)$ are complex functions. Therefore, due to Proposition 3.3 we have

$$|e^{-\alpha_1^i(\xi)t}|^2 = e^{-2Re\alpha_1^i(\xi)t} = e^{\frac{-2b\lambda_1(\xi)t}{2(1+a\lambda_1(\xi))}} \le e^{-\tilde C|\xi|^2 t},\ \forall\, \xi \in B_\delta \text{ and } i=1,2. \tag{4.34}$$

where $\tilde C$ is a positive constant. Then, proceeding as in the first case we conclude that

$$\|I(\cdot,t)-J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le 2C_k \int_{B_\delta} |\xi|^{2k-2} e^{-\tilde C|\xi|^2 t} d\xi \le \tilde C_k t^{-\frac{2k+N-2}{2}} \quad \text{as } t\to\infty$$

where $\tilde C_k$ is a positive constant depending on k. The proof is complete. ☐

Remark 4.9. When $b^2 = 4$ the previous analysis shows that the decay rate in (4.27) is $t^{-\frac{2k+N-4}{4}}$ with $J(x,t)$ given in (4.26) modulated by $\lambda^{-\frac{3}{2}}(\xi)$ instead of $\lambda^{-1}(\xi)$.

Now, we compute the Taylor expansion of $\phi_1(x,\xi)$ around $\xi = 0$ and prove that all terms in (4.26), which we denote by

$$J_\alpha(x,t) = \int_{B_\delta} \frac{\xi^\alpha}{\lambda_1(\xi)} \Big[d^1_\alpha e^{-\alpha_1^1(\xi)t} + d^2_\alpha e^{-\alpha_1^2(\xi)t}\Big] e^{ix\cdot\xi}\ \phi_1(x,\xi) d\xi, \tag{4.35}$$

for $\alpha \in (\mathbb{N}\cup\{0\})^N$, $|\alpha| \le k$ and $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$, can be approximated in L^2-setting by a linear combination of the form

$$\sum_{|\gamma|\le k-|\alpha|} d_\gamma(x) \int_{B_\delta} \frac{\xi^\alpha}{\lambda_1(\xi)} \Big[d^1_\alpha e^{-\alpha_1^1(\xi)t} + d^2_\alpha e^{-\alpha_1^2(\xi)t}\Big] e^{ix\cdot\xi}\ \xi^\gamma d\xi, \tag{4.36}$$

where d_γ are periodic functions defined by

$$d_\gamma(x) = \frac{1}{\gamma!}\ \partial_\xi^\gamma \phi_1(x,0)\,, \quad |\gamma| \le k - |\alpha|. \tag{4.37}$$

This can be done because $\phi_1(\cdot,\xi) \in L^2_\#(Y)$ is an analytic function in B_δ. The result reads as follows.

Lemma 4.10. *There exists a constant $C_k > 0$, such that*

$$\|J_\alpha(\cdot,t) - \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t)\|_{L^2(\mathbb{R}^N)} \sim C_k t^{-\frac{2k+N-2}{4}}$$

as $t \to \infty$, where

$$\begin{aligned} I_{\alpha+\gamma}(x,t) &= \int_{B_\delta} \frac{\xi^{\alpha+\gamma}}{\lambda_1(\xi)} \Big[d_\alpha^1 e^{-\alpha_1^1(\xi)t} + d_\alpha^2 e^{-\alpha_1^2(\xi)t} \Big] e^{ix\cdot\xi} d\xi \\ &:= d_\alpha^1\, I_{\alpha+\gamma}^1(x,t) + d_\alpha^2\, I_{\alpha+\gamma}^2(x,t). \end{aligned} \tag{4.38}$$

Proof. Let

$$R_{k,\alpha}(x,\xi) = \phi_1(x,\xi) - \sum_{|\gamma|\le k-|\alpha|} d_\gamma(x)\xi^\gamma$$

where $d_\gamma(\cdot)$ is defined in (4.37) and $\alpha \in (\mathbb{N}\cup\{0\})^N$ with $|\alpha| \le k$. Since $\phi_1(\cdot,\xi)$ is an analytic function with respect to ξ in B_δ and values in $L^2_\#(Y)$, for all $\xi \in B_\delta$ we obtain that

$$\|R_{k,\alpha}(\cdot,\xi)\|_{L^2_\#(Y)} \le C_k |\xi|^{k+1-|\alpha|}\,. \tag{4.39}$$

Thus,

$$R_{k,\alpha} \in L^\infty(B_\delta, L^2_\#(Y))\,. \tag{4.40}$$

Now, we consider the function G given by

$$\begin{aligned} G(x,t) &= J_\alpha(x,t) - \sum_{|\gamma|\le k-|\alpha|} d_\gamma(x) I_{\alpha+\gamma}(x,t) \\ &= \int_{B_\delta} \frac{\xi^\alpha}{\lambda_1(\xi)} \Big[d_\alpha^1 e^{-\alpha_1^1(\xi)t} + d_\alpha^2 e^{-\alpha_1^2(\xi)t} \Big] R_{k,\alpha}(x,\xi) e^{ix\cdot\xi} d\xi\,. \end{aligned} \tag{4.41}$$

Then, from Lemma 4.3, (4.39) and (4.40) we obtain

$$\begin{aligned} \|G(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} &= \int_{B_\delta} \Big| \frac{\xi^\alpha}{\lambda_1(\xi)} \Big(d_\alpha^1 e^{-\alpha_1^1(\xi)t} + d_\alpha^2 e^{-\alpha_1^2(\xi)t} \Big) \Big|^2 \|R_{k,\alpha}(\cdot,\xi)\|^2_{L^2_\#(Y)} d\xi \\ &\le C_k^2 \int_{B_\delta} \frac{|\xi|^{2k+2}}{\lambda_1^2(\xi)} \Big| d_\alpha^1 e^{-\alpha_1^1(\xi)t} + d_\alpha^2 e^{-\alpha_1^2(\xi)t} \Big|^2 d\xi\,. \end{aligned}$$

Consequently, using Proposition 3.3 and Lemma 4.1, we obtain

$$\|G(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le C_k^2 \int_{B_\delta} |\xi|^{2k-2} e^{-c_9|\xi|^2 t} d\xi \sim \tilde{C}_k t^{-\frac{2k+N-2}{2}} \tag{4.42}$$

as $t \to \infty$, where $k \ge 1$ and $c_9 = \max\{c_5, c_7\}$ (see Lemma 4.5). □

Next, we study the integral in (4.38) which appears in the statement of Lemma 4.10. This will be done considering expansions of type $\sum_{|\alpha|\le k} \tilde{d}_\alpha\, \xi^\alpha$ for the functions $\alpha_1^1(\xi)$ and $\alpha_1^2(\xi)$ around $\xi = 0$. We observe that, according to Proposition

3.3, we have

$$\begin{gathered}
\alpha_1^1(0) = \frac{\partial \alpha_1^1(0)}{\partial \xi_k} = 0\,, \quad k = 1,2,\dots,N \\
\frac{\partial^2 \alpha_1^1(0)}{\partial \xi_k \partial \xi_\ell} = (b - \sqrt{b^2-4}) q_{k\ell}\,, \quad k,\ell = 1,2,\dots,N \\
\alpha_1^2(0) = \frac{\partial \alpha_1^2}{\partial \xi_k}(0) = 0\,, \quad k = 1,2,\dots,N \\
\frac{\partial^2 \alpha_1^2(0)}{\partial \xi_k \partial \xi_\ell} = (b + \sqrt{b^2-4}) q_{k\ell}\,, \quad k,\ell = 1,2,\dots,N \\
\partial^\alpha \alpha_1^1(0) = \partial^\alpha \alpha_1^2(0) = 0 \quad \text{if } |\alpha| \text{ is odd.}
\end{gathered} \tag{4.43}$$

In view of the above consideration, if we define

$$r_1(\xi) = \alpha_1^1(\xi) - \frac{b-\sqrt{b^2-4}}{2}\, q_{k\ell}\xi_k\xi_\ell, \quad r_2(\xi) = \alpha_1^2(\xi) - \frac{b+\sqrt{b^2-4}}{2}\, q_{k\ell}\xi_k\xi_\ell \tag{4.44}$$

the maps $\xi \mapsto r_1(\xi)$ and $\xi \mapsto r_2(\xi)$ are analytic in ξ. Moreover,

$$e^{-\alpha_1^1(\xi)t} = e^{-\frac{b-\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t} e^{-r_1(\xi)t} = e^{-\frac{b-\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{\infty} \frac{t^n}{n!}(-r_1(\xi))^n\Big) \tag{4.45}$$

$$e^{-\alpha_1^2(\xi)t} = e^{-\frac{b+\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t} e^{-r_2(\xi)t} = e^{-\frac{b+\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{\infty} \frac{t^n}{n!}(-r_2(\xi))^n\Big). \tag{4.46}$$

Now, for $p \in \mathbb{N}$ fixed, we define the following two functions

$$\tilde{I}_\gamma^1(x,t) = \int_{B_\delta} \frac{\xi^\gamma}{\lambda_1(\xi)}\ e^{-\frac{b-\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{p} \frac{t^n}{n!}(-r_1(\xi))^n\ e^{ix\cdot\xi}\Big) d\xi \tag{4.47}$$

$$\tilde{I}_\gamma^2(x,t) = \int_{B_\delta} \frac{\xi^\gamma}{\lambda_1(\xi)}\ e^{-\frac{b+\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{p} \frac{t^n}{n!}(-r_2(\xi))^n\ e^{ix\cdot\xi}\Big) d\xi. \tag{4.48}$$

Replacing (4.44) into $I_\gamma^1(x,t)$ and $I_\gamma^2(x,t)$ defined in Lemma 4.10, identities (4.45) and (4.46) lead us to consider the asymptotic behavior of the differences

$$\begin{aligned}
&I_\gamma^1(x,t) - \tilde{I}_\gamma^1(x,t) \\
&= \int_{B_\delta} \frac{\xi^\gamma}{\lambda_1(\xi)}\ e^{-\frac{b-\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big[e^{-r_1(\xi)t} - \sum_{n=0}^{p} \frac{t^n}{n!}(-r_1(\xi))^n\Big]\ e^{ix\cdot\xi} d\xi\,,
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
&I_\gamma^2(x,t) - \tilde{I}_\gamma^2(x,t) \\
&= \int_{B_\delta} \frac{\xi^\gamma}{\lambda_1(\xi)} e^{-\frac{b+\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t}\Big[e^{-r_2(\xi)t} - \sum_{n=0}^{p} \frac{t^n}{n!}(-r_2(\xi))^n\Big] e^{ix\cdot\xi} d\xi.
\end{aligned} \tag{4.50}$$

This provides an estimate of $I_{\alpha+\gamma}(x,t)$ in L^2-setting.

Lemma 4.11. *Let $2p \geq k - |\gamma| - 1$. Then, there exists a constant $C_k > 0$, such that*

$$\|I_\gamma^i(\cdot,t) - \tilde{I}_\gamma^i(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k t^{-\frac{2k+N-2}{2}}, \quad \text{as } t \to \infty$$

with $I^i_\gamma(x,t)$ *and* $\tilde I^i_\gamma(x,t)$*, defined in (4.38) and* (4.47)-(4.48)*, respectively, for* $i = 1,2$.

Proof. Parseval's identity and formula (4.49) imply

$$\begin{aligned} &\|I^1_\gamma(\cdot,t)-\tilde I^1_\gamma(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \\ &= \int_{B_\delta} \frac{|\xi|^{2|\gamma|}}{\lambda_1^2(\xi)}\Big|e^{-\frac{b-\sqrt{b^2-4}}{2}\ q_{k\ell}\xi_k\xi_\ell t}\Big[e^{-r_1(\xi)t}-\sum_{n=0}^{p}\frac{t^n}{n!}(-r_1(\xi))^n\Big]\Big|^2 d\xi. \end{aligned} \tag{4.51}$$

Since the function e^z, $z\in\mathbb{R}$, is analytic, we obtain $C_p>0$ satisfying

$$\Big|e^{-r_1(\xi)t}-\sum_{n=0}^{p}\frac{(-t)^n}{n!}(r_1(\xi))^n\Big| \le C_p|r_1(\xi)|^{p+1}\,t^{p+1}, \quad \forall\, t>0,\ \xi\in B_\delta. \tag{4.52}$$

Then, combining (4.43) and (4.44) we may conclude that

$$r_1(\xi)=\sum_{m=0}^{\infty}\sum_{|\alpha|=4+2m}\frac{1}{\alpha!}\,\partial_\xi^\alpha\alpha_1^1(0)\xi^\alpha\,,\ \xi\in B_\delta \tag{4.53}$$

which guarantees the existence of a positive constant satisfying

$$|r_1(\xi)|\le C|\xi|^4, \quad \forall\,\xi\,\in B_\delta. \tag{4.54}$$

Now, returning to (4.51) we can proceed as in the proof of Lemma 4.8 (see for instance (4.33)) and consider two cases: if $b^2>4$ we can combine (4.52), (4.54), Proposition 3.3 and Lemma 4.1 to obtain

$$\begin{aligned} &\|I^1_\gamma(\cdot,t)-\tilde I^1_\gamma(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \\ &\le C_p\int_{B_\delta}\frac{|\xi|^{2|\gamma|}}{\lambda_1^2(\xi)}e^{-\frac{b-\sqrt{b^2-4}}{2}q_{k\ell}\xi_k\xi_\ell t}|r_1(\xi)|^{2p+2}t^{2p+2}d\xi \\ &\le c_1^{-2}C_pC\Big(\int_{B_\delta}e^{-\frac{b-\sqrt{b^2-4}}{2}q_{k\ell}\xi_k\xi_\ell t}|\xi|^{2|\gamma|+8p+4}d\xi\Big)t^{2p+2} \\ &\sim C_{p,k}t^{2p+2-\frac{2|\gamma|+8p+4+N}{2}}, \quad \text{for } t \text{ large}, \end{aligned} \tag{4.55}$$

with $C_{p,k}>0$ and $|\gamma|\le k$. Now, since p is an integer, such that

$$2p\ge k-|\gamma|-1 \tag{4.56}$$

then, (4.55) and (4.56) give us the result for the case $i=1$ and $b^2>4$. When $0<b^2\le 4$ the proof is similar (see (4.33) and the end of the proof of Lemma 4.8). Finally, the case $i=2$ is obtained in the same way. □

Before stating the next result let us recall that the idea is to prove that the solution of (1.1) behave as a linear combination of the derivatives of the fundamental solution of the homogenized heat equation, modulated by the first eigenvalue $\lambda_1(\xi)$ of the operator A. So, taking the last result into account the next step is to study the asymptotic behavior of $r_1(\xi)$ and $r_2(\xi)$ defined in (4.44). However, before doing that, we consider the Taylor's expansion of $r_1(\xi)$ and $r_2(\xi)$ around $\xi=0$ to obtain, for $n \geq 1$,

$$\begin{aligned}(r_1(\xi))^n &= \Big(\sum_{\beta\geq 0}\frac{1}{\beta!}\partial_\xi^\beta r_1(0)\xi^\beta\Big)^n\\ &= \Big(\sum_{m=0}^{\infty}\sum_{|\beta|=4+2m}\frac{1}{\beta!}\partial_\xi^\beta \alpha_1^1(0)\xi^\beta\Big)^n\\ &= \sum_{m=0}^{\infty}\sum_{|\beta|=4n+2m} d_{\beta,n}^1\xi^\beta\end{aligned}\tag{4.57}$$

and

$$\begin{aligned}(r_2(\xi))^n &= \Big(\sum_{\beta\geq 0}\frac{1}{\beta!}\partial_\xi^\beta r_2(0)\xi^\beta\Big)^n\\ &= \Big(\sum_{m=0}^{\infty}\sum_{|\beta|=4+2m}\frac{1}{\beta!}\partial_\xi^\beta \alpha_1^2(0)\xi^\beta\Big)^n\\ &= \sum_{m=0}^{\infty}\sum_{|\beta|=4n+2m} d_{\beta,n}^2,\xi^\beta\end{aligned}\tag{4.58}$$

because $\partial_\xi^\beta\alpha_1^1(0)=\partial_\xi^\beta\alpha_1^2(0)=0$ when $|\beta|=0$ and $|\beta|$ odd (see also 4.43).

Now, we note that

$$\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(r_1(\xi))^n = 1+\sum_{n=1}^{\infty}\frac{(-t)^n}{n!}\sum_{m=0}^{\infty}\sum_{|\beta|=4n+2m} d_{\beta,n}^1\xi^\beta,\tag{4.59}$$

$$\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(r_2(\xi))^n = 1+\sum_{n=1}^{\infty}\frac{(-t)^n}{n!}\sum_{m=0}^{\infty}\sum_{|\beta|=4n+2m} d_{\beta,n}^2\xi^\beta.\tag{4.60}$$

These facts suggest us the following approximation for $\tilde I_\gamma^i(x,t)$, $i=1,2$:

$$\begin{aligned}&I_{\gamma^*}^1(x,t)\\ &= \int_{B_\delta}\frac{\xi^\gamma}{\lambda_1(\xi)}e^{-\frac{b-\sqrt{b^2-4}}{2}q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m} d_{\beta,n}^1\xi^\beta\Big)e^{ix\cdot\xi}d\xi\end{aligned}\tag{4.61}$$

and

$$\begin{aligned}&I_{\gamma^*}^2(x,t)\\ &= \int_{B_\delta}\frac{\xi^\gamma}{\lambda_1(\xi)}e^{-\frac{b+\sqrt{b^2-4}}{2}q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m} d_{\beta,n}^2\xi^\beta\Big)e^{ix\cdot\xi}d\xi\end{aligned}\tag{4.62}$$

where $p_1=p_1(n,\gamma)$ will be chosen later. The constants $d_{\beta,n}^1$ and $d_{\beta,n}^2$ can be calculated explicitly in terms of $\partial_\xi^\beta\alpha_1^1(0)$ and $\partial_\xi^\beta\alpha_1^2(0)$, respectively.

Lemma 4.12. *Let $\tilde I_\gamma^i(x,t)$ and $I_{\gamma^*}^i(x,t)$ defined in* (4.47)-(4.48) *and* (4.61)-(4.62), *respectively. Then, for $|\gamma|\leq k$,*

$$\|\tilde I_\gamma^i(\cdot,t)-I_{\gamma^*}^i(\cdot,t)\|_{L^2(\mathbb{R}^N)}^2\sim C_k t^{-\frac{2k+N-2}{2}},\quad \text{as } t\to\infty,\tag{4.63}$$

for $i=1,2$, where C_k is a positive constant.

Proof. It will be done for $i = 1$. The case $i = 2$ follows the same arguments and is omitted. First we suppose that $b^2 > 4$. Then, from Parseval's theorem we get

$$\begin{aligned} &\|\tilde{I}^1_\gamma(\cdot,t) - I^1_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \\ &= \int_{B_\delta} \Big|\frac{\xi^\gamma}{\lambda_1(\xi)} e^{-\frac{b-\sqrt{b^2-4}}{2} q_{k\ell}\xi_k\xi_\ell t} \\ &\quad\times \Big[\sum_{n=0}^{p} \frac{(-t)^n}{n!}(r_1(\xi))^n - \sum_{n=0}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^1_{\beta,n}\xi^\beta\Big]\Big|^2 d\xi \\ &\le \int_{B_\delta} \frac{|\xi|^{2|\gamma|}}{\lambda_1^2(\xi)} e^{-(b-\sqrt{b^2-4})q_{k\ell}\xi_k\xi_\ell t} \\ &\quad\times \Big|\sum_{n=0}^{p} \frac{t^n}{n!}\Big[(r_1(\xi))^n - \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^1_{\beta,n}\xi^\beta\Big]\Big|^2 d\xi. \end{aligned} \tag{4.64}$$

On the other hand, since $r_1(\xi)$ is analytic, by using the Taylor expansion (4.58), we obtain a positive constant C_n, such that

$$\Big|(r_1(\xi))^n - \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^1_{\beta,n}\xi^\beta\Big| \le C_n|\xi|^{2p_1+4n+1}, \quad \xi \in B_\delta$$

with $p_1 = p_1(n,\gamma)$ to be chosen. Thus, using (4.64) we deduce that

$$\begin{aligned} &\|\tilde{I}^1_\gamma(\cdot,t) - I^1_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R})} \\ &\le \int_{B_\delta} \frac{|\xi|^{2|\gamma|}}{\lambda_1^2(\xi)} e^{-(b-\sqrt{b^2-4})q_{k\ell}\xi_k\xi_\ell t}\Big(\sum_{n=0}^{p} \frac{t^n}{n!} C_n |\xi|^{2p_1+4n+1}\Big)^2 d\xi \\ &\le C_p \sum_{n=0}^{p} t^{2n} \int_{B_\delta} |\xi|^{2|\gamma|+4p_1+8n-2}\ e^{-(b-\sqrt{b^2-4})q_{k\ell}\xi_k\xi_\ell t} d\xi. \end{aligned} \tag{4.65}$$

Now, for $|\gamma| \le k$ we can apply Lemma 4.1 to obtain

$$\|\tilde{I}^1_\gamma(\cdot,t) - I^1_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le C_{p,k}\ \sum_{n=0}^{p} t^{2n}\ t^{-\frac{2|\gamma|+4p_1+8n-2+N}{2}}, \quad \text{as } t \to \infty$$

where $C_{p,k}$ is a positive constant. Thus, choosing $p_1 = p_1(n,\gamma)$ such that

$$2p_1 \ge k - |\gamma| - 2n \tag{4.66}$$

where $|\gamma| \le k$ and $k \ge 1$, we get the following inequality

$$\|\tilde{I}^1_\gamma(\cdot,t) - I^1_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le C_k\ t^{-\frac{2k+N-2}{2}}, \quad \text{as } t \to \infty.$$

This completes the proof for the case $b^2 > 4$. The case $0 < b^2 \le 4$ is similar. □

Remark 4.13. Although the heat Kernel, modulated by $\lambda_1^{-1}(\xi)$, is defined as an integral in $\mathbb{R}^N$ and in our case only in B_δ, we observe that the difference between the two integrals decays exponentially in $L^2(\mathbb{R}^N)$ due to the coercivity of matrix $\{q_{k\ell}\}_{k,\ell=1}^N$ in $\mathbb{R}^N$. Indeed, for $|\gamma| \le k$ and $t > 0$, we have

$$\Big\| \int_{\mathbb{R}^N\setminus B_\delta} \frac{\xi^\gamma}{\lambda_1(\xi)} e^{-\frac{b\pm\sqrt{b^2-4}}{2}\, q_{k\ell}\xi_k\xi_\ell t}\ e^{ix\cdot\xi} d\xi\Big\|_{L^2(\mathbb{R}^N)} \le Ce^{-\sigma\delta^2 t}$$

where $\sigma > 0$ is a constant that depends on the coercivity constant of the matrix and the constant b. The constant $C > 0$ depends on k and the constant c_1 introduced in

Proposition 3.3. Moreover, due to Proposition 3.3 we can see the analogy between $G^{\pm}_{\alpha}$ and the Kernels introduced above. In particular, since $c_1|\xi|^2 \le \lambda_1(\xi) \le c_2|\xi|^2$, $\forall\, \xi \in B_\delta$, they have the same polynomial decay.

5. Proof of Theorem 2.1

Taking Remark 4.13 into account, we define

$$\begin{aligned} H(x,t) = \sum_{|\alpha|\le k} \Big\{ C^1_\alpha(x) \Big[G^-_\alpha(x,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^1_{\beta,n} G^-_{\alpha+\beta}(x,t) \Big] \\ + C^2_\alpha(x) \Big[G^+_\alpha(x,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} d^2_{\beta,n} G^+_{\alpha+\beta}(x,t) \Big] \Big\} \end{aligned} \tag{5.1}$$

where $G^{\pm}_{\alpha}(x,t)$ was defined in Theorem 2.1, $p = p(\alpha)$ satisfies (4.56), $p_1 = p_1(n,\alpha)$ is given in the proof of Lemma 4.12 and

$$C^i_\alpha(x) = \sum_{\gamma\le\alpha} d_\gamma(x) d^i_\alpha\,, \quad i = 1,2$$

with d^i_α and d_γ as in (4.26) and (4.37), respectively. We now fix

$$p = p(\alpha) = \big[\frac{k-|\alpha|}{2}\big] \quad \text{and} \quad p_1 = p_1(n,\alpha) = p(\alpha) - n.$$

Then, according to (4.59), (4.60) and Remark 4.13, we obtain

$$\begin{aligned} &\|u(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\le \|u(\cdot,t) - I(\cdot,t)\|_{L^2(\mathbb{R}^N)} + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\le \Big\| \sum_{m=2}^{\infty} \int_{Y'} \Big[\beta^1_m(\xi) e^{-\alpha^1_m(\xi)t} + \beta^2_m(\xi) e^{-\alpha^2_m(\xi)t}\Big] e^{ix\cdot\xi} \phi_m(x,\xi) d\xi \Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \Big\| \int_{Y'\setminus B_\delta} \Big[\beta^1_1(\xi) e^{-\alpha^1_1(\xi)t} + \beta^2_1(\xi) e^{-\alpha^2_1(\xi)t}\Big] e^{ix\cdot\xi} \phi_1(x,\xi) d\xi \Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)}, \end{aligned}$$

with $I(x,t)$ defined in (4.25). Consequently, from Lemma 4.6, Lemma 4.7 and Parseval's identity we get

$$\|u(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \le Ce^{-\nu t} + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \tag{5.2}$$

where C and ν are positive constants, with C depending on the initial data φ^0 and φ^1.

To estimate the difference $I(x,t) - H(x,t)$ that appears on the right hand side of the above inequality, we use Lemma 4.8:

$$\begin{aligned} \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} &\le \|I(\cdot,t) - J(\cdot,t)\|_{L^2(\mathbb{R}^N)} + \|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\le Ct^{-\frac{2k+N-2}{4}} + \|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{5.3}$$

as $t \to \infty$, where $k \geq 1$. We also have

$$\begin{aligned}\|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} &= \Big\| \sum_{|\alpha|\leq k} J_\alpha(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\ &\leq \sum_{|\alpha|\leq k} \Big\| J_\alpha(\cdot,t) - \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \Big\| \sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Thus, from Lemma 4.10 it results

$$\begin{aligned}&\|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\leq C_k t^{-\frac{2k+N-2}{4}} + \Big\| \sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Here we observe that

$$\sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t) = \sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(x) \left[d_\alpha^1 I_{\alpha+\gamma}^1(x,t) + d_\alpha^2 I_{\alpha+\gamma}^2(x,t)\right] \tag{5.4}$$

where $I_\gamma^1(x,t)$ and $I_\gamma^2(x,t)$ are defined by (4.38). Consequently, we can write

$$\sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma I_{\alpha+\gamma}(x,t) = \sum_{|\alpha|\leq k} [C_\alpha^1(x) I_\alpha^1(x,t) + C_\alpha^2(x) I_\alpha^2(x,t)] \tag{5.5}$$

with C_α^i given above, which allows to conclude that

$$\begin{aligned}&\Big\| \sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\ &\leq \Big\| \sum_{|\alpha\leq k} C_\alpha^1(\cdot)[I_\alpha^1(\cdot,t) - \tilde{I}_\alpha^1(\cdot,t)] + C_\alpha^2(\cdot)[I_{\alpha(\cdot,t)}^2 - \tilde{I}_\alpha^2(\cdot,t)]\Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \Big\| \sum_{|\alpha|\leq k} [C_\alpha^1(\cdot)\tilde{I}_\alpha^1(\cdot,t) + C_\alpha^2(\cdot,t)\tilde{I}_\alpha^2(\cdot,t)] - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)}\end{aligned} \tag{5.6}$$

where $\tilde{I}_\alpha^i$, $i = 1,2$, are given by (4.47) and (4.48), respectively, and $C_\alpha^i(\cdot) \in L_\#^\infty(Y)$, because $d_\gamma(\cdot) \in L_\#^\infty(Y)$.

Now, using Lemma 4.11 and (5.6) we get

$$\begin{aligned}&\Big\| \sum_{|\alpha|\leq k} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot) I_{\alpha+\gamma}(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\ &\leq C_k \sum_{|\alpha|\leq k} \Big[\big\|I_\alpha^1(\cdot,t) - \tilde{I}_\alpha^1(\cdot,t)\big\|_{L^2(\mathbb{R}^N)} + \big\|I_\alpha^2(\cdot,t) - \tilde{I}_\alpha^2(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}\Big] \\ &\quad + \Big\| \sum_{|\alpha|\leq k} \Big[C_\alpha^1(\cdot)\tilde{I}_\alpha^1(\cdot,t) + C_\alpha^2(\cdot)\tilde{I}_\alpha^2(\cdot,t)\Big] - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\ &\leq \tilde{C}_k t^{-\frac{2k+N-2}{4}} + \Big\| \sum_{|\alpha|\leq k} \Big[C_\alpha^1(\cdot)\tilde{I}_\alpha^1(\cdot,t) + C_\alpha^2(\cdot)\tilde{I}_\alpha^2(\cdot,t)\Big] - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)}\end{aligned} \tag{5.7}$$

with the constants C_k, $\tilde{C}_k$ depending on k and $\sup_{|\alpha|\leq k} \|C_\alpha^i(\cdot)\|_{L^\infty(\mathbb{R}^N)}$, $i = 1,2$.

The next step is devoted to estimate the last term in the right hand side of (5.7). Therefore we observe that

$$
\begin{aligned}
&\sum_{|\alpha|\le k}[C^1_\alpha(x)\tilde I^1_\alpha(x,t)+C^2_\alpha(x)\tilde I^2_\alpha(x,t)]-H(x,t)\\
&=\sum_{|\alpha|\le k}\Big\{C^1_\alpha(x)[\tilde I^1_\alpha(x,t)-I^1_{\alpha^*}(x,t)]+C^2_\alpha(x)[\tilde I^2_\alpha(x,t)-I^2_{\alpha^*}(x,t)]\Big\}\\
&\quad+\sum_{|\alpha|\le k}[C^1_\alpha(x)I^1_{\alpha^*}(x,t)+C^2_\alpha(x)I^2_{\alpha^*}(x,t)]-H(x,t)
\end{aligned}
$$

where $I^i_{\alpha^*}$, $i=1,2$, are defined in (4.61) and (4.62). Thus, applying Lemma 4.12 we have

$$
\begin{aligned}
&\sum_{|\alpha|\le k}\big\|[C^1_\alpha(\cdot,t)\tilde I^1_\alpha(\cdot)+C^2_\alpha(\cdot)\tilde I^2_\alpha(\cdot,t)]-H(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}\\
&\le C_k t^{-\frac{2k+N-2}{4}}+\big\|\sum_{|\alpha|\le k}[C^1_\alpha(\cdot)I^1_{\alpha^*}(\cdot,t)+C^2_\alpha(\cdot)I^2_{\alpha^*}(\cdot,t)]-H(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}
\end{aligned}
\tag{5.8}
$$

with C_k a positive constant.

Now, returning to the definition of $H(x,t)$ in (5.1) we have

$$
\begin{aligned}
&\big\|\sum_{|\alpha|\le k}[C^1_\alpha(\cdot)I^1_{\alpha^*}(\cdot,t)+C^2_\alpha(\cdot)I^2_{\alpha^*}(\cdot,t)]-H(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}\\
&\le\big\|\sum_{|\alpha|\le k}C^1_\alpha(\cdot)I^1_{\alpha^*}(\cdot,t)-G^-_\alpha(\cdot,t)-\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m}d^1_{\beta,n}G^-_{\alpha+\beta}(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}\\
&+\big\|\sum_{|\alpha|\le k}C^2_\alpha(\cdot)I^2_{\alpha^*}(\cdot,t)-G^+_\alpha(\cdot,t)-\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m}d^2_{\beta,n}G^+_{\alpha+\beta}(\cdot,t)\big\|_{L^2(\mathbb{R}^N)}\\
&\le e^{-\nu\delta^2t}+\big\|\sum_{|\alpha|\le k}C^1_\alpha(\cdot)\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m}d^1_{\beta,n}\\
&\quad\times\int_{\mathbb{R}^N\setminus B_\delta}\frac{\xi^{\alpha+\beta}}{\lambda_1(\xi)}e^{-\frac{b-\sqrt{b^2-4}}{2}\,q_{k\ell}\xi_k\xi_\ell t}e^{ix\cdot\xi}d\xi\big\|_{L^2(\mathbb{R}^N)}\\
&\quad+\big\|\sum_{|\alpha|\le k}C^2_\alpha(\cdot)\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m}d^2_{\beta,n}\\
&\quad\times\int_{\mathbb{R}^N\setminus B_\delta}\frac{\xi^{\alpha+\beta}}{\lambda_1(\xi)}e^{-\frac{b+\sqrt{b^2-4}}{2}\,q_{k\ell}\xi_k\xi_\ell t}e^{ix\cdot\xi}d\xi\big\|_{L^2(\mathbb{R}^N)}
\end{aligned}
\tag{5.9}
$$

due to Remark 4.13 stated in the previous section.

It remains to estimate the term

$$
\begin{aligned}
F^i(x,t)=&\sum_{|\alpha|\le k}C^i_\alpha(\cdot)\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4n+2m}d^i_{\beta,n}\\
&\times\int_{\mathbb{R}^N\setminus B_\delta}\frac{\xi^{\alpha+\beta}}{\lambda_1(\xi)}e^{-\frac{b\pm\sqrt{b^2-4}}{2}\,q_{k\ell}\xi_k\xi_\ell t}\ e^{ix\cdot\xi}d\xi,
\end{aligned}
\tag{5.10}
$$

in L^2- setting. We observe that the signs $-$ and $+$ in $b \pm \sqrt{b^2-4}$ correspond to $i=1$ and $i=2$, respectively.

Due to Parseval's identity, for the case $b^2 \geq 4$ we have that

$$
\begin{aligned}
&\|F^i(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\
&\leq C \sum_{|\alpha|\leq k} \sum_{n=1}^{p} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} t^p \Big(\int_{\mathbb{R}^N\setminus B_\delta} \frac{|\xi|^{2|\alpha|+2|\beta|}}{\lambda_1^2(\xi)} e^{-(b\pm\sqrt{b^2-4})\, q_{k\ell}\xi_k\xi_\ell\, t} d\xi \Big)^{1/2} \\
&\leq C \sum_{|\alpha|\leq k} \sum_{n=1}^{p} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} t^p \Big(\int_{\mathbb{R}^N\setminus B_\delta} |\xi|^{2|\alpha|+2|\beta|-4} e^{-(b\pm\sqrt{b^2-4})\, q_{k\ell}\xi_k\xi_\ell\, t} d\xi \Big)^{1/2} \\
&\leq C \sum_{|\alpha|\leq k} \sum_{n=1}^{p} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} t^p \Big(t^{-\frac{2|\alpha|+2|\beta|-4+N}{2}} e^{-\nu\delta^2 t} \Big)^{1/2} \\
&\leq C_k e^{-\frac{\nu\delta^2 t}{2}}, \quad \text{as } t\to\infty
\end{aligned}
\tag{5.11}
$$

where C_k is a positive constant and ν is the constant of coercivity of the matrix $\{q_{k\ell}\}_{k,\ell=1}^N$. To obtain this result we have used that, for $m \in \mathbb{N}$,

$$
\int_{\mathbb{R}^N\setminus B_\delta} |\xi|^m e^{-\alpha|\xi|^2 t} d\xi = \int_\delta^\infty r^m e^{-\alpha r^2 t} \Big(\int_{|\xi|=r} dS_\xi \Big) dr \leq C w_N\, t^{-\frac{m+N}{2}} e^{-\frac{\alpha\delta^2 t}{2}},
\tag{5.12}
$$

for all $t>0$, where w_N denotes the measure of the sphere S^{N-1}. Finally, returning to (5.2) and using estimates (5.3) up to (5.11), we conclude that

$$
\|u(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \leq C_k t^{-\frac{2k+N-2}{4}}, \quad \text{as } t\to\infty
$$

where $k \geq 1$ and C_k is a positive constant that depends on the initial data.

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