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EXISTENCE OF SOLUTIONS FOR SYSTEMS OF SELF-REFERRED AND HEREDITARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the existence of solutions of a system of self-referred and hereditary differential equations. The initial data are assumed to be lower semi-continuous. We also formulate some open questions.

1. Introduction

If x is an event, t is the time, and $u(x,t), v(x,t)$ are two reasonings about x at time t, then the term $v(\int_0^t u(x,s)ds, t)$ can be considered as a "criticism" of v over all previous reasonings of u on x, up to time t. The following system of differential equations

$$\begin{gathered}\frac{\partial}{\partial t}u(x,t) = u\Big(v\Big(\int_0^t u(x,s)ds, t\Big), t\Big),\\ \frac{\partial}{\partial t}v(x,t) = v\Big(u\Big(\int_0^t v(x,s)ds, t\Big), t\Big),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x).\end{gathered} \tag{1.1}$$

serves as a mathematical model for the evolution of two reasonings. It relates self-reference and heredity.

Hereditary phenomena with memories depending on past time histories have an extensive literature. Differential equations modelling such phenomena have been considered by several authors [3, 4, 5, 6, 9, 10, 11, 15]. The main idea in finding a mathematical model consists of formalizing mathematically a constitutive law of a given physical phenomenon. In particular, phenomena whose evolution depends on their states have been studied in [1, 2, 12, 13, 14].

Phenomena depending on their past history and with unknown constitutive laws have attracted considerable interest recently. To study these phenomena, Miranda and Pascali [9] introduced a new class of functional differential equations. The mathematical model of these phenomena can be described in the following way. Let $A : X \to \mathbb{R}$ and $B : X \to \mathbb{R}$ be two functionals, where X is a space of functions.

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We consider the equation

$$(Au)(x,t) = u\big(Bu(x,t),t\big), \tag{1.2}$$

where $u(x,t)$ is unknown function satisfying some initial conditions at $t=0$. If Bu is a "hereditary" operator, for example

$$Bu(x,t) = \int_0^t u(x,s)ds,$$

then (1.2) is said to be of a hereditary and self-referred type. In particular, Miranda and Pascali [8] proved a local existence and uniqueness for equations of the type

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\Big(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\Big),$$

where k_i are given real numbers or real valued functions $k_i = k_i(x,t)$.

The article [9] also contains some results on the existence and uniqueness of local solutions for the integral equations:

$$u(x,t) = u_0(x) + \int_0^t u\Big(\int_0^\tau u(x,s)ds, \tau\Big)d\tau, \tag{1.3}$$

$$u(x,t) = u_0(x) + \int_0^t u\Big(\frac{1}{\tau}\int_0^\tau u(x,s)ds, \tau\Big)d\tau, \tag{1.4}$$

$$u(x,t) = u_0(x) + \int_0^t u\Big(\int_0^\tau \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u(\varepsilon,s)d\varepsilon ds, \tau\Big)d\tau, \tag{1.5}$$

for $t \ge 0, x \in \mathbb{R}$. These equations are equivalent to the initial value problems:

$$\begin{gathered}\frac{\partial}{\partial t}u(x,t) = u\Big(\int_0^t u(x,s)ds, t\Big),\\ u(x,0) = u_0(x),\end{gathered} \tag{1.6}$$

$$\begin{gathered}\frac{\partial}{\partial t}u(x,t) = u\Big(\frac{1}{t}\int_0^t u(x,s)ds, t\Big),\\ u(x,0) = u_0(x),\end{gathered} \tag{1.7}$$

$$\begin{gathered}\frac{\partial}{\partial t}u(x,t) = u\Big(\int_0^t \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u(\xi,\tau)d\xi d\tau, t\Big),\\ u(x,0) = u_0(x),\end{gathered} \tag{1.8}$$

for $t \ge 0, x \in \mathbb{R}$, where u_0 is a real valued bounded Lipschitz function and δ is a given function satisfying suitable conditions.

Later, Pascali and Lê [11], obtained an existence result for (1.3) under less restrictive assumptions than in [9]. Also Pascali [10] obtained the results on the existence and uniqueness for (1.1) in the case when u_0, v_0 are bounded and Lipschitz functions. The main goal of this paper is to obtain the existence of solutions of (1.1) under weaker assumptions than in [10]. In our approach we use some ideas from [11].

2. Existence theorems

We consider the system of integral equations

$$\begin{aligned} u(x,t) &= u_0(x) + \int_0^t u\Big(v\Big(\int_0^s u(x,\tau)d\tau, s\Big), s\Big)ds, \\ v(x,t) &= v_0(x) + \int_0^t v\Big(u\Big(\int_0^s v(x,\tau)d\tau, s\Big), s\Big)ds. \end{aligned} \tag{2.1}$$

It is assumed throughout this paper that:

- (A1) u_0, v_0 are non-negative,
- (A2) u_0, v_0 are non-decreasing,
- (A3) u_0, v_0 are bounded,
- (A4) u_0, v_0 are lower semicontinuous

It is worth pointing out that u_0, v_0, in this paper, are only lower semicontinuous, while Lipschitz continuity was required in [10]. The first part of this section will be devoted to the proof of the existence of the solutions for (2.1).

Theorem 2.1. *Let* (A1)–(A4) *hold. Then there exist two functions* $u, v : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *such that*

- (B1) *u, v are non-negative,*
- (B2) *u, v are non-decreasing with respect to each of variables,*
- (B3) *u, v are bounded on the sets* $\mathbb{R} \times [0, +\infty)$,
- (B4) *u, v are lower semicontinuous with respect to x, for all fixed t in* $[0, +\infty)$,
- (B5) *u, v are Lipschitz with respect to* $t \in [0, +\infty)$, *uniformly with respect to* $x \in \mathbb{R}$,

satisfying (2.1).

Proof. The basic idea of the proof is to associate with (2.1) linear recursive schemes as follows

$$\begin{aligned} u_1(x,t) &= u_0(x) + \int_0^t u_0\big(v_0\big(u_0(x)s\big)\big)ds, \\ v_1(x,t) &= v_0(x) + \int_0^t v_0\big(u_0\big(v_0(x)s\big)\big)ds, \\ u_{n+1}(x,t) &= u_0(x) + \int_0^t u_n\Big(v_n\Big(\int_0^s u_n(x,\tau)d\tau, s\Big), s\Big)ds, \\ v_{n+1}(x,t) &= v_0(x) + \int_0^t v_n\Big(u_n\Big(\int_0^s v_n(x,\tau)d\tau, s\Big), s\Big)ds, \end{aligned} \tag{2.2}$$

for all $x \in \mathbb{R}$, $t \in [0, +\infty)$, $n \in \mathbb{N}$. It follows from (A1) that

$$\begin{aligned} u_1(x,t) &\ge u_0(x) \ge 0, \\ v_1(x,t) &\ge v_0(x) \ge 0, \end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}$, $t \in [0, +\infty)$. According to (A2) and $(2.2)_{1,2}$, for $t_2 > t_1, x_2 > x_1$, for all $t_1, t_2 \in [0, +\infty)$, $x_1, x_2 \in \mathbb{R}$, we can conclude that

$$\begin{aligned} u_1(x,t_2) &\ge u_1(x,t_1), \quad \forall x \in \mathbb{R}, \\ u_1(x_2,t) &\ge u_1(x_1,t), \quad \forall t \in [0, +\infty), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} v_1(x,t_2) &\geq v_1(x,t_1), \quad \forall x \in \mathbb{R}, \\ v_1(x_2,t) &\geq v_1(x_1,t), \quad \forall t \in [0,+\infty). \end{aligned} \tag{2.5}$$

Consequently, (2.4) and (2.5) imply that u_1, v_1 are non-decreasing with respect to x and t, separately. Moreover, $(2.2)_{1,2}$ and (A3) yield

$$\begin{aligned} 0 \leq u_1(x,t) \leq (1+t)\|u_0\|_{L^\infty}, \\ 0 \leq v_1(x,t) \leq (1+t)\|v_0\|_{L^\infty}, \end{aligned} \tag{2.6}$$

for all $x \in \mathbb{R}$, $t \in [0,+\infty)$. Hence, u_1, v_1 are uniformly bounded with respect to x, for every $t \in [0,+\infty)$. Furthermore, the Lipschitz property of u_1, v_1 with respect to t (uniformly with respect to x) and their lower semi-continuity with respect to x are easily deduced.

Combining (2.3)–(2.6) and with the aid of the induction on n, we can show that

$$\begin{aligned} 0 \leq u_n(x,t) \leq u_{n+1}(x,t) \leq e^t\|u_0\|_{L^\infty}, \\ 0 \leq v_n(x,t) \leq v_{n+1}(x,t) \leq e^t\|v_0\|_{L^\infty}, \end{aligned} \tag{2.7}$$

for all $x \in \mathbb{R}$, $t \geq 0$, for all $n \in \mathbb{N}$. Thus, u_n, v_n are non-negative in $\mathbb{R} \times [0,+\infty)$. Moreover u_n, v_n are non-decreasing with respect to x and t separately, Lipschitz on $t \in [0,+\infty)$, uniformly bounded with respect to x. In addition, it follows easily that they are lower semicontinuous on x for every $t \in [0,+\infty)$ as well. Thanks to the above properties of u_n, v_n, it is a simple matter to deduce the existence of both u_∞, v_∞ such that

$$\begin{aligned} u_\infty(x,t) &= \lim_{n\to+\infty} u_n(x,t) \quad \Big(= \sup_{n\in\mathbb{N}}[u_n(x,t)]\Big), \\ v_\infty(x,t) &= \lim_{n\to+\infty} v_n(x,t) \quad \Big(= \sup_{n\in\mathbb{N}}[v_n(x,t)]\Big). \end{aligned} \tag{2.8}$$

However, easy computations show that

- (C1) u_∞, v_∞ are non-negative on $\mathbb{R} \times [0,+\infty)$,
- (C2) u_∞, v_∞ are non-decreasing with respect to each of variables,
- (C3) u_∞, v_∞ are bounded on the sets $\mathbb{R} \times [0,+\infty)$, namely $0 \leq u_\infty(x,t) \leq e^t\|u_0\|_{L^\infty}$, $0 \leq v_\infty(x,t) \leq e^t\|v_0\|_{L^\infty}$,
- (C4) u_∞, v_∞ are lower semicontinuous with respect to x, for all fixed $t \in [0,+\infty)$,
- (C5) u_∞, v_∞ are Lipschitz with respect to $t \in [0,+\infty)$ uniformly with respect to $x \in \mathbb{R}$.

Next, it remains to prove that u_∞ and v_∞ satisfy (2.1). As a matter of fact, for every $x \in \mathbb{R}$ and every $t \in [0,+\infty)$, it is evident that the following integrals exist

$$\begin{gathered} \int_0^t u_\infty(x,s)ds, \\ \int_0^t v_\infty(x,s)ds, \\ \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau, s\Big), s\Big)ds, \\ \int_0^t v_\infty\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau, s\Big), s\Big)ds. \end{gathered} \tag{2.9}$$

Applying $(2.2)_{3,4}$ and (2.8), we immediately obtain that

$$\begin{aligned} u_{n+1}(x,t)-u_0(x) &\le \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ v_{n+1}(x,t)-v_0(x) &\le \int_0^t v_\infty\Big(v_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds. \end{aligned} \tag{2.10}$$

We now observe that "$\le$" in (2.10) may be replaced by either "$\ge$" or "$=$". Indeed, by using (2.7), for $n,p\in\mathbb{N}$, the following inequalities obviously hold

$$\begin{aligned} u_{n+p}\Big(v_{n+p}\Big(\int_0^t u_{n+p}(x,s)ds,s\Big),t\Big) &\ge u_n\Big(v_{n+p}\Big(\int_0^t u_{n+p}(x,s)ds,s\Big),t\Big),\\ v_{n+p}\Big(u_{n+p}\Big(\int_0^t v_{n+p}(x,s)ds,s\Big),t\Big) &\ge v_n\Big(u_{n+p}\Big(\int_0^t v_{n+p}(x,s)ds,s\Big),t\Big). \end{aligned} \tag{2.11}$$

Applying the lower semi-continuity of both u_n, v_n, we can deduce from (2.8) and (2.11) that

$$\begin{aligned} &\lim_{p\to+\infty}\int_0^t u_{n+p}\Big(v_{n+p}\Big(\int_0^s u_{n+p}(x,\tau)d\tau,s\Big),s\Big)ds\\ &\ge \int_0^t u_n\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ &\lim_{p\to+\infty}\int_0^t v_{n+p}\Big(u_{n+p}\Big(\int_0^s v_{n+p}(x,\tau)d\tau,s\Big),s\Big)ds\\ &\ge \int_0^t v_n\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds. \end{aligned} \tag{2.12}$$

Clearly,the above inequalities now become

$$\begin{aligned} \lim_{p\to+\infty}\lim\Big(u_{n+p+1}(x,t)-u_0(x)\Big) &\ge \int_0^t u_n\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ \lim_{p\to+\infty}\lim\Big(v_{n+p+1}(x,t)-v_0(x)\Big) &\ge \int_0^t v_n\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds. \end{aligned} \tag{2.13}$$

Combining (2.8) with (2.13), it is easy to check that

$$\begin{aligned} u_\infty(x,t)-u_0(x) &\ge \int_0^t u_n\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ v_\infty(x,t)-v_0(x) &\ge \int_0^t v_n\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds. \end{aligned} \tag{2.14}$$

Letting $n\to+\infty$, (2.14) is rewritten as

$$\begin{aligned} u_\infty(x,t)-u_0(x) &\ge \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ v_\infty(x,t)-v_0(x) &\ge \int_0^t v_\infty\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds. \end{aligned} \tag{2.15}$$

Consequently, (2.10) and (2.15) make it obvious that

$$\begin{aligned} u_\infty(x,t)-u_0(x) &= \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau,s\Big),s\Big)ds,\\ v_\infty(x,t)-v_0(x) &= \int_0^t v_\infty\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau,s\Big),s\Big)ds, \end{aligned} \tag{2.16}$$

or

$$\begin{aligned} u_\infty(x,t) &= u_0(x) + \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau, s\Big), s\Big)ds, \\ v_\infty(x,t) &= v_0(x) + \int_0^t v_\infty\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau, s\Big), s\Big)ds. \end{aligned} \tag{2.17}$$

Finally, it is easily seen from (2.17) that u_∞, v_∞ satisfy (2.1), and on account of (C1)–(C5), the proof of Theorem 2.1 is complete. □

Now, to study the solutions of (1.1), it is convenient to rewrite (2.17) as

$$\begin{aligned} u_\infty(x,t) - \int_0^t u_\infty\Big(v_\infty\Big(\int_0^s u_\infty(x,\tau)d\tau, s\Big), s\Big)ds = u_0(x), \\ v_\infty(x,t) - \int_0^t v_\infty\Big(u_\infty\Big(\int_0^s v_\infty(x,\tau)d\tau, s\Big), s\Big)ds = v_0(x), \end{aligned} \tag{2.18}$$

for all $x \in \mathbb{R}, t \in [0,+\infty)$. The advantage of considering the left sides of the equations in (2.18) lies in the fact that these functions are differential with respect to t for all fixed x, and their derivatives are equal to zero. Moreover, repeated application of (C3) and (C5), for every fixed x, there exist $\frac{\partial}{\partial t}u_\infty(x,t)$, $\frac{\partial}{\partial t}v_\infty(x,t)$ for a.e. t. Differentiating (2.18) with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial t}u_\infty(x,t) &= u_\infty\Big(v_\infty\Big(\int_0^t u_\infty(x,s)ds, t\Big), t\Big), \\ \frac{\partial}{\partial t}v_\infty(x,t) &= v_\infty\Big(u_\infty\Big(\int_0^t v_\infty(x,s)ds, t\Big), t\Big). \end{aligned} \tag{2.19}$$

This leads to the following theorem.

Theorem 2.2. *Under assumptions* (A1)–(A4)*, there exist two functions* $u, v : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ *which satisfy (B1)–(B5), such that*

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= u\Big(v\Big(\int_0^t u(x,s)ds, t\Big), t\Big), \\ \frac{\partial}{\partial t}v(x,t) &= v\Big(u\Big(\int_0^t v(x,s)ds, t\Big), t\Big), \\ u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \end{aligned} \tag{2.20}$$

for $x \in \mathbb{R}$*, a.e.* $t \in [0,+\infty)$.

An interesting point of Theorem 2.2 is that the solutions (u, v) are Lipschitz (with respect to the time variable) although the initial datum is not.

3. Some open problems

In this section, we state some open problems that the readers may find interesting. Since there have been just a modest number of publications related to the system (1.1), many open questions have been left. As a matter of fact, we can study the following:

- The uniqueness of the solution of (1.1) under the above assumptions; it seems hard.
- Numerical solutions of the mentioned systems (see [7]).

- Further problems of more general systems; for example,

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= u\Big(f(u,v,x,t),t\Big),\\ \frac{\partial}{\partial t}v(x,t) &= v\Big(g(u,v,x,t),t\Big),\\ u(x,0) &= u_0(x), v(x,0) = v_0(x). \end{aligned} \tag{3.1}$$

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