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MATRIX ELEMENTS FOR SUM OF POWER-LAW POTENTIALS IN QUANTUM MECHANIC USING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper we derive close form for the matrix elements for $\hat{H} = -\Delta + V$, where V is a pure power-law potential. We use trial functions of the form

$$\psi_n(r) = \sqrt{\frac{2\beta^{\gamma/2}(\gamma)_n}{n!\Gamma(\gamma)}} r^{\gamma-1/2} e^{-\frac{\sqrt{\beta}}{2} r^q} \; {}_pF_1(-n, a_2, \dots, a_p; \gamma; \sqrt{\beta} r^q),$$

for $\beta, q, \gamma > 0$ to obtain the matrix elements for $\hat{H}$. These formulas are then optimized with respect to variational parameters β, q and γ to obtain accurate upper bounds for the given nonsolvable eigenvalue problem in quantum mechanics. Moreover, we write the matrix elements in terms of the generalized hypergeomtric functions. These results are generalization of those found earlier in [2], [8]–[16] for power-law potentials. Applications and comparisons with earlier work are presented.

1. Introduction

In 1998 Hall et al [15] found a closed form expressions for the singular-potential integrals $\langle m \mid r^{-\alpha} \mid n\rangle$ that obtained with respect to the Gol'dman and Krivchenkov eigenfunctions for the singular Hamiltonian with $\hbar^2 = 2m = 1$,

$$\hat{H} = -\frac{d^2}{dr^2} + \beta r^2 + \frac{A}{r^2}, \quad (\beta > 0, A \geq 0). \tag{1.1}$$

We present a variational analysis of the generalized spiked harmonic oscillator Hamiltonian [2, 3, 4], [7]–[19] of the form

$$\hat{H} = -\frac{d^2}{dr^2} + \beta r^2 + \frac{A}{r^2} + \frac{\lambda}{r^\alpha}, \quad 0 \leq r < \infty, \tag{1.2}$$

where α and λ are positive real numbers. By writing $\hat{H} = H^{(0)} + \lambda V$ with $H^{(0)}$ standing for the generalized spiked harmonic-oscillator Hamiltonian, and $V(r) = r^{-\alpha}$, Hall et al [8]–[16] used the basis set

$$\psi_n(r) = T_n r^{\gamma-1/2} e^{\frac{-\sqrt{\beta}}{2} r^2} \; {}_1F_1(-n; \gamma; \sqrt{\beta} r^2), \quad T_n = \sqrt{\frac{2\beta^{\frac{\gamma}{2}}(\gamma)_n}{n!\Gamma(\gamma)}},$$

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$n = 0, 1, 2, 3, \dots$, constructed from the normalized solutions of $H^{(0)}\psi_n = E_n\psi_n$ to evaluate the matrix elements of $\hat{H}$. They found that

$$H_{mn} = \langle m \mid \hat{H} \mid n\rangle = 2\sqrt{\beta}(2n+\gamma)\delta_{mn} + \lambda\langle m \mid r^{-\alpha} \mid n\rangle \quad m, n = 0, 1, 2, \dots, N-1$$

where

$$\begin{aligned}\langle m \mid r^{-\alpha} \mid n\rangle = (-1)^{m+n}\beta^{\frac{\alpha}{4}}\sqrt{\frac{(\gamma)_n(\gamma)_m}{n!m!}}\frac{\Gamma(\gamma-\frac{\alpha}{2})(\frac{\alpha}{2})_n}{(\gamma)_n\Gamma(\gamma)}\\ \times_3 F_2(-m, \gamma-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \gamma, 1-\frac{\alpha}{2}-n; 1).\end{aligned}$$

Now, by considering the problem in a finite dimensional subspace and choosing a suitable trial functions as a linear combination of the form

$$\psi_n(r) = T_n r^{\gamma-1/2} e^{\frac{-\sqrt{\beta}}{2}r^q}\ {}_pF_1(-n, a_2, \dots, a_p; \gamma; \sqrt{\beta}r^q), \quad T_n = \sqrt{\frac{2\beta^{\frac{\gamma}{2}}(\gamma)_n}{n!\Gamma(\gamma)}} \tag{1.3}$$

This basis is more general than the basis used in [2] and [14], because the number of variational parameters for the exponential function and the generalized hypergeometric function play an important rule for improving the numerical results for our eigenvalue problem. Herein, ${}_pF_1$ stands for the hypergeometric function defined by

$${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x) = \sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}(\alpha_i)_k}{\Pi_{j=1}^{q}(\beta_j)_k}\frac{x^k}{k!}; \tag{1.4}$$

where p and q are non-negative integers and $\beta_j (j = 1, 2, \dots, q)$ cannot be a non-positive integer [1]. The expression $(a)_n$ is the Pochhammer symbol defined by the relations

$$\begin{aligned}(a)_n &= a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\\ (a)_0 &= 1\\ (a)_{-n} &= \frac{(-1)^n}{(1-a)_n},\end{aligned}$$

where n is an integer (positive, negative or zero).

2. Variational Technique

In this section we review the well known variational technique for eigenvalue problem in quantum mechanics [5]. We consider a non solvable eigenvalue problem of the form $\hat{H} = -\triangle + V(r)$ in N-dimensional space over a finite subspace in the domain of its angular momentum subspace. Our variational technique is based on forming trial wave function from a linear combination of some of orthonormal basis function $\psi_n(r), n = 1, 2, \dots, D$ such that,

$$\Psi(r) = \sum_{n=1}^{D} c_n\psi_n(r) \tag{2.1}$$

where the c_n are variational parameters. The variational energy E_Ψ we obtain with this trial function is given by,

$$E_\Psi = \frac{\int \Psi^*\hat{H}\Psi d\tau}{\int \Psi^*\Psi d\tau} = \frac{\sum_{n=1}^{D}\sum_{m=1}^{D} c_n^*c_m\int\psi_n^*\hat{H}\psi_m d\tau}{\sum_{n=1}^{D}\sum_{m=1}^{D} c_n^*c_m\int\psi_n^*\psi_m d\tau} \tag{2.2}$$

Now, defining the matrix element of $\hat{H}$ as

$$H_{nm} = \int \psi_n^* \hat{H} \psi_m d\tau \tag{2.3}$$

and using the orthonormal property of the ψ functions, we obtain that

$$E_\Psi = \frac{\sum_{n=1}^{D} \sum_{m=1}^{D} c_n^* c_m H_{nm}}{\sum_{n=1}^{D} \mid c_n \mid^2} \tag{2.4}$$

or

$$E_\Psi \sum_{n=1}^{D} \mid c_n \mid^2 = \sum_{n=1}^{D} \sum_{m=1}^{D} c_n^* c_m H_{nm}$$

We seek the minimum value of E_Ψ as a function of all the c_n. By differentiating with respect to c_i, we obtain

$$\frac{\partial E}{\partial c_i} \sum_{n=1}^{D} \mid c_n \mid^2 + E c_i^* = \sum_{n=1}^{D} c_n^* H_{ni}$$

and setting $\frac{\partial E}{\partial c_i} = 0$, we have

$$\sum_{n=1}^{D} c_n^* H_{ni} - E c_i^* = 0. \tag{2.5}$$

We can derive one equation similar to (2.5) for each possible value of $i, i = 1, \dots, D$, and we may take (2.5) to represent a set of linear equations with each equation characterized by a different value of i. According to the theory of linear algebraic equations, there is a nontrivial solution, if and only if the determinant of the coefficients vanishes or if and only if

$$\begin{vmatrix} H_{11} - E & H_{12} & H_{13} & \cdots & H_{1D} \\ H_{21} & H_{22} - E & H_{23} & \cdots & H_{2D} \\ \vdots & \vdots & \vdots & & \vdots \\ H_{D1} & H_{D2} & H_{D3} & \cdots & H_{DD} - E \end{vmatrix} = 0$$

This determinant is called a *secular determinant*. If we use a basis set that is not orthonormal, we define the matrix element S_{nm} by

$$S_{nm} = \int \psi_n \psi_m d\tau,$$

in this case the equation corresponding to (2.5) and the secular determinant are

$$\sum_{n=1}^{D} c_n H_{in} - E S_{in} c_n = 0 \tag{2.6}$$

and the secular determinant can be written as

$$\begin{vmatrix} H_{11} - ES_{11} & H_{12} - ES_{12} & \cdots & H_{1D} - ES_{1D} \\ H_{21} - ES_{21} & H_{22} - ES_{22} & \cdots & H_{2D} - ES_{2D} \\ \vdots & \vdots & & \vdots \\ H_{D1} - ES_{D1} & H_{D2} - ES_{D2} & \cdots & H_{DD} - ES_{DD} \end{vmatrix} = 0.$$

3. Derivation of the Matrix Elements

We divide the problem of computing the matrix elements for Schrödinger equation into simple parts. In the first lemma we compute the matrix elements for singular potential in three-spatial dimensions, it will be used to compute the matrix elements for the kinetic energy.

Lemma 3.1. *For the generalized spiked harmonic oscillator Hamiltonian* (1.2)*, the expectation values of the operator* $V(x) = r^{-\alpha}$ *with respect to a trial basis* (1.3) *are given by*

$$\begin{aligned}\langle \psi_m \mid r^{-\alpha} \mid \psi_n\rangle &= \frac{1}{q} T_m T_n \beta^{\frac{\alpha-2\gamma}{2q}} \Gamma(\frac{2\gamma-\alpha}{q}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (\frac{2\gamma-\alpha}{q})_k}{(\gamma)_k k!} \\ &\quad \times {}_{p+1} F_1(-n, a_2, \dots, a_p, \frac{2\gamma-\alpha}{q} + k; \gamma; 1)\end{aligned} \tag{3.1}$$

Proof. Using (1.3), it follows that

$$\begin{aligned}\langle \psi_m \mid r^{-\alpha} \mid \psi_n\rangle = \int_0^\infty & r^{2\gamma-\alpha-1} e^{-\sqrt{\beta} r^q} {}_pF_1(-m, a_2, \dots, a_p; \gamma; \sqrt{\beta} r^q) \\ & \times {}_p F_1(-n, a_2, \dots, a_p; \gamma; \sqrt{\beta} r^q) dr\end{aligned} \tag{3.2}$$

Using the series representation (1.4) of the hypergeometric function ${}_pF_1$, yields

$$\begin{aligned}r_{mn}^{-\alpha} &= T_m T_n \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l k! l!} \\ &\quad \times \beta^{\frac{l+k}{2}} \int_0^\infty r^{2\gamma-\alpha+qk+ql-1} e^{-\sqrt{\beta} r^q} dr\end{aligned} \tag{3.3}$$

By restoring to the integral representation of gamma function, we obtain under the condition $\frac{2\gamma-\alpha}{q} + k + l > 0$, that

$$\begin{aligned}& r_{mn}^{-\alpha} \\ &= \frac{1}{q} T_m T_n \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l} \frac{\beta^{\frac{l+k}{2}}}{k! l!} \Gamma(\frac{2\gamma-\alpha}{q} + k + l) \\ &= \frac{1}{q} T_m T_n \beta^{\frac{\alpha-2\gamma}{2q}} \sum_{k=0}^{m} \Big[\sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma-\alpha}{q} + k + l)\Big] \\ &\quad \times \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!}\end{aligned} \tag{3.4}$$

On the other hand, using the definition of the Pochhammer symbols and the series representation of the hypergeometric functions (1.4), the finite sum inside the bracket collapses to

$$\begin{aligned}
&\sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_l l!}\Gamma(\frac{2\gamma-\alpha}{q}+k+l)\\
&=\sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l(\frac{2\gamma-\alpha}{q}+k)_l}{(\gamma)_l l!}\Gamma(\frac{2\gamma-\alpha}{q}+k)\\
&={}_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2\gamma-\alpha}{q}+k;\gamma;1)\Gamma(\frac{2\gamma-\alpha}{q}+k)
\end{aligned}\tag{3.5}$$

Consequently, we arrive at

$$\begin{aligned}
r_{mn}^{-\alpha}&=\frac{1}{q}T_mT_n\beta^{\frac{\alpha-2\gamma}{2q}}\sum_{k=0}^{m}\Gamma(\frac{2\gamma-\alpha}{q}+k)\frac{(-m)_k(a_2)_k\cdots(a_p)_k}{(\gamma)_k k!}\\
&\quad\times{}_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2\gamma-\alpha}{q}+k;\gamma;1)\\
&=\frac{1}{q}T_mT_n\beta^{\frac{\alpha-2\gamma}{2q}}\Gamma(\frac{2\gamma-\alpha}{q})\sum_{k=0}^{m}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(\frac{2\gamma-\alpha}{q})_k}{(\gamma)_k k!}\\
&\quad\times{}_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2\gamma-\alpha}{q}+k;\gamma;1).
\end{aligned}$$

This completes the proof. ☐

For the case of $\gamma>1$ and $\alpha=2$, we have

$$\begin{aligned}
r_{mn}^{-2}&=\frac{1}{q}T_mT_n\beta^{\frac{1-\gamma}{q}}\sum_{k=0}^{m}\frac{(-m)_k(a_2)_k\cdots(a_p)_k}{(\gamma)_k k!}\Gamma(\frac{2(\gamma-1)}{q}+k)\\
&\quad\times{}_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2(\gamma-1)}{q}+k;\gamma;1).
\end{aligned}\tag{3.6}$$

3.1. **Matrix Elements for $V(r)=r^\alpha$.** We now use the suggested basis (1.3) to compute the matrix elements for the power-law potential operators r^α , $\alpha>0$. This kind of computation is important for may problems in the literature, such as Kratzer potential [6]. Whose calculation is achieved by means of following lemma.

Lemma 3.2. *For the generalized spiked harmonic oscillator Hamiltonian* (1.2)*, the expectation values of the operator $V(x)=r^\alpha$ with respect to a trial basis* (1.3) *are given by*

$$\begin{aligned}
\langle\psi_m\mid r^\alpha\mid\psi_n\rangle&=\frac{1}{q}T_mT_n\beta^{-\frac{\alpha+2\gamma}{2q}}\Gamma(\frac{2\gamma+\alpha}{q})\sum_{k=0}^{m}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(\frac{2\gamma+\alpha}{q})_k}{(\gamma)_k(k!}\\
&\quad\times{}_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2\gamma+\alpha}{q}+k;\gamma;1)
\end{aligned}\tag{3.7}$$

Proof. Using (1.3), it immediately follows that

$$\begin{aligned}
\langle\psi_m\mid r^\alpha\mid\psi_n\rangle&=\int_0^\infty r^{2\gamma+\alpha-1}e^{-\sqrt{\beta}r^q}{}_pF_1(-m,a_2,\dots,a_p;\gamma;\sqrt{\beta}r^q)\\
&\quad\times{}_pF_1(-n,a_2,\dots,a_p;\gamma;\sqrt{\beta}r^q)dr
\end{aligned}\tag{3.8}$$

Using the series representation (1.4) of the hypergeometric function ${}_pF_1$, yields

$$\begin{aligned} r_{mn}^{\alpha} &= T_m T_n \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l k! l!} \\ &\quad \times \beta^{\frac{l+k}{2}} \int_0^{\infty} r^{2\gamma+\alpha+qk+ql-1} e^{-\sqrt{\beta} r^q} dr. \end{aligned} \tag{3.9}$$

Using the integral representation of gamma function, we obtain under the condition $\frac{2\gamma+\alpha}{q} + k + l > 0$ that

$$\begin{aligned} r_{mn}^{\alpha} &= T_m T_n \frac{\beta^{-\frac{\alpha+2\gamma}{2q}}}{q} \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l} \frac{1}{k! l!} \\ &\quad \times \Gamma(\frac{2\gamma+\alpha}{q} + k + l) \\ &= T_m T_n \frac{\beta^{-\frac{\alpha+2\gamma}{2q}}}{q} \sum_{k=0}^{m} \Big[\sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma+\alpha}{q} + k + l) \Big] \\ &\quad \times \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \end{aligned} \tag{3.10}$$

On the other hand, by using the definition of the Pochhammer symbols and the series representation of the hypergeometric functions (1.4), the finite sum inside the bracket collapses to

$$\begin{aligned} &\sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma+\alpha}{q} + k + l) \\ &= \sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l (\frac{2\gamma+\alpha}{q} + k)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma+\alpha+2}{q} + k) \\ &= {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma+\alpha}{q} + k; \gamma; 1) \Gamma(\frac{2\gamma+\alpha}{q} + k). \end{aligned} \tag{3.11}$$

Finally, we arrive at

$$\begin{aligned} r_{mn}^{\alpha} &= T_m T_n \frac{\beta^{-\frac{\alpha+2\gamma}{2q}}}{q} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma+\alpha}{q} + k) \\ &\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma+\alpha}{q} + k; \gamma; 1) \\ &= T_m T_n \frac{\beta^{-\frac{\alpha+2\gamma}{2q}}}{q} \Gamma(\frac{2\gamma+\alpha}{q}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (\frac{2\gamma+\alpha}{q})_k}{(\gamma)_k k!} \\ &\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma+\alpha}{q} + k; \gamma; 1). \end{aligned}$$

The proof is complete. $\square$

On the other hand, for the case of $\gamma > 0$ and $\alpha = 2 > 0$, we have that

$$\begin{aligned}\langle \psi_m \mid r^2 \mid \psi_n \rangle = T_m T_n \frac{1}{q} \beta^{-\frac{1+\gamma}{q}} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2(\gamma+1)}{q} + k) \\ \times_{p+1} F_1(-n, a_2, \dots, a_p, \frac{2(\gamma+1)}{q} + k; \gamma; 1).\end{aligned} \tag{3.12}$$

Lemma 3.3. *For $\gamma > 0$, and $m, n = 0, 1, 2, \dots$,*

$$\begin{aligned} S_{mn} &= \langle \psi_m \mid \psi_n \rangle \\ &= T_m T_n \frac{\beta^{-\frac{\gamma}{q}}}{q} \Gamma(\frac{2\gamma}{q}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (\frac{2\gamma}{q})_k}{(\gamma)_k k!} \\ &\quad \times_{p+1} F_1(-n, a_2, \dots, a_p, \frac{2\gamma}{q} + k; \gamma; 1) \end{aligned} \tag{3.13}$$

Proof. Using (1.3), it follows that

$$\begin{aligned} S_{mn} = \int_0^\infty & r^{2\gamma-1} e^{-\sqrt{\beta} r^q} {}_pF_1(-m, a_2, \dots, a_p; \gamma; \sqrt{\beta} r^q) \\ & \times_p F_1(-n, a_2, \dots, a_p; \gamma; \sqrt{\beta} r^q) dr \end{aligned} \tag{3.14}$$

By means of the series representation (1.4) of the hypergeometric function ${}_pF_1$, that

$$\begin{aligned} S_{mn} = T_m T_n \sum_{k=0}^{m} \sum_{l=0}^{n} & \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l} \frac{\beta^{\frac{l+k}{2}}}{k! l!} \\ \times \int_0^\infty & r^{2\gamma + qk + ql - 1} e^{-\sqrt{\beta} r^q} dr. \end{aligned} \tag{3.15}$$

Further, after restoring the integral representation of the gamma function and a change of variables, we obtain for $\frac{2\gamma}{q} + k + l > 0$, that

$$\begin{aligned} S_{mn} &= T_m T_n \frac{\beta^{-\frac{\gamma}{q}}}{q} \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_k (\gamma)_l} \frac{\Gamma(\frac{2\gamma}{q} + k + l)}{k! l!} \\ &= T_m T_n \frac{\beta^{-\frac{\gamma}{q}}}{q} \sum_{k=0}^{m} \Big[\sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma}{q} + k + l) \Big] \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \end{aligned} \tag{3.16}$$

On the other hand, by using the definition of the pochhammer symbols and the series representation of the hypergeometric functions (1.4), the finite sum inside the bracket collapses to

$$\begin{aligned} & \sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma}{q} + k + l) \\ & = \sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l (\frac{2\gamma}{q} + k)_l}{(\gamma)_l l!} \Gamma(\frac{2\gamma}{q} + k) \\ & =_{p+1} F_1(-n, a_2, \dots, a_p, \frac{2\gamma}{q} + k; \gamma; 1) \Gamma(\frac{2\gamma}{q} + k) \end{aligned} \tag{3.17}$$

Consequently,

$$\begin{aligned}
S_{mn} &= T_m T_n \frac{\beta^{-\frac{\gamma}{q}}}{q} \sum_{k=0}^{m} \Gamma(\frac{2\gamma}{q}+k) \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma}{q}+k; \gamma; 1) \\
&= T_m T_n \frac{\beta^{-\frac{\gamma}{q}}}{q} \Gamma(\frac{2\gamma}{q}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k (\frac{2\gamma}{q})_k}{(\gamma)_k k!} \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma}{q}+k; \gamma; 1)
\end{aligned}$$

This completes the proof. ☐

Lemma 3.4. *For $\gamma > 1$, and $m, n = 1, 2, \dots,$ the matrix elements of the generalized spiked harmonic oscillator Hamiltonians* (1.2) *can be written as*

$$\begin{aligned}
&H_{mn} \\
&= T_m T_n [\beta^{\frac{1-\gamma}{q}} \frac{(A-\gamma^2+2\gamma-\frac{3}{4})}{q} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma-2}{q}+k) \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma-2}{q}+k; \gamma; 1) \\
&\quad + B^{-\frac{1+\gamma}{q}} \frac{1}{q} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma+2}{q}+k) \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma+2}{q}+k; \gamma; 1) \\
&\quad - \beta^{\frac{1-\gamma}{q}} \frac{q}{4} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma-2}{q}+2+k) \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma-2}{q}+2+k; \gamma; 1) \\
&\quad + \beta^{\frac{1-\gamma}{q}} (\gamma-1+\frac{q}{2}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma-2}{q}+1+k) \\
&\quad \times {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2\gamma-2}{q}+1+k; \gamma; 1) \\
&\quad + \beta^{\frac{1-\gamma}{q}} (q+2(\gamma-1)) \frac{n a_2 \cdots a_p}{\gamma} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma-2}{q}+1+k) \\
&\quad \times {}_{p+1}F_1(1-n, 1+a_2, \dots, 1+a_p, \frac{2\gamma-2}{q}+1+k; 1+\gamma; 1) \\
&\quad - \beta^{\frac{1-\gamma}{q}} q \frac{n a_2 \cdots a_p}{\gamma} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} \Gamma(\frac{2\gamma-2}{q}+2+k) \\
&\quad \times {}_{p+1}F_1(1-n, 1+a_2, \dots, 1+a_p, \frac{2\gamma-2}{q}+2+k; 1+\gamma; 1) \\
&\quad - \beta^{\frac{1-\gamma}{q}} q \frac{n(-1+n) a_2 (1+a_2) \cdots a_p (1+a_p)}{\gamma(1+\gamma)} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!}
\end{aligned}$$

$$
\begin{aligned}
&\times \Gamma(\frac{2\gamma-2}{q}+2+k)_{p+1}F_1(2-n,2+a_2,\dots,2+a_p,\frac{2\gamma-2}{q}+2+k;2+\gamma;1)\\
&+\lambda\beta^{\frac{\alpha-2\gamma}{2q}}\frac{1}{q}\Gamma(\frac{2\gamma-\alpha}{q})\sum_{k=0}^{m}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(\frac{2\gamma-\alpha}{q})_k}{(\gamma)_k k!}\\
&\times_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2\gamma-\alpha}{q}+k;\gamma;1)
\end{aligned}\tag{3.18}
$$

Proof. By writing the Hamiltonian H in compactified form as

$$
H_{mn}=\prec\psi_m\mid-\frac{d^2}{dr^2}\mid\psi_n\succ+\beta r_{mn}^2+Ar_{mn}^{-2}+\lambda r_{mn}^{-\alpha}
$$

where r_{mn}^2 is given by (3.12) r_{mn}^{-2} is given by (3.6), and $r_{mn}^{-\alpha}$ is given by (3.1). By using (1.3) it is follows that

$$
\begin{aligned}
H_{mn}=-&\int_0^\infty r^{\gamma-\frac12}e^{-\frac{\sqrt{B}}{2}r^q}{}_pF_1(-m,a_2,\dots,a_p;\gamma;\sqrt{B}r^q)\\
&\times\frac{d^2}{dr^2}(r^{\gamma-\frac12}e^{-\frac{\sqrt{B}}{2}r^q}\ {}_pF_1(-n,a_2,\dots,a_p;\gamma;\sqrt{B}r^q))dr\\
&+\beta r_{mn}^2+Ar_{mn}^{-2}+\lambda r_{mn}^{-\alpha}
\end{aligned}\tag{3.19}
$$

We denote the first term on the right-hand side of (3.19) by $I_{mn}$and have by means of the series representation (1.4) of the hypergeometric function ${}_pF_1$, that

$$
\begin{aligned}
I_{mn}=(-\gamma^2+2\gamma-\frac34)&\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{B^{\frac{l+k}{2}}}{k!l!}\\
&\times\int_0^\infty r^{2\gamma-3+qk+ql}e^{-\sqrt{B}r^q}dr\\
&-B\frac{q^2}{4}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{B^{\frac{l+k}{2}}}{k!l!}\\
&\times\int_0^\infty r^{2\gamma-3+2q+qk+ql}e^{-\sqrt{B}r^q}dr\\
&+\sqrt{B}q(\gamma-1+\frac{q}{2})\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{B^{\frac{l+k}{2}}}{k!l!}\\
&\times\int_0^\infty r^{2\gamma-3+q+qk+ql}e^{-\sqrt{B}r^q}dr+\sqrt{B}q(2(\gamma-1)+q)\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m}\\
&\times\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l}\\
&\times\frac{B^{\frac{l+k}{2}}}{k!l!}\int_0^\infty r^{2\gamma-3+q+qk+ql}e^{-\sqrt{B}r^q}dr\\
&-Bq^2\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l}\\
&\times\frac{B^{\frac{l+k}{2}}}{k!l!}\int_0^\infty r^{2\gamma-3+2q+qk+ql}e^{-\sqrt{B}r^q}dr
\end{aligned}
$$

$$
\begin{aligned}
&- Bq^2 \frac{n(-1+n)a_2(1+a_2)\cdots a_p(1+a_p)}{\gamma(1+\gamma)} \\
&\times \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-m)_k(a_2)_k\cdots(a_p)_k(2-n)_l(2+a_2)_l\cdots(2+a_p)_l}{(\gamma)_k(2+\gamma)_l} \\
&\times \frac{B^{\frac{l+k}{2}}}{k!l!}\int_0^\infty r^{2\gamma-3+2q+qk+ql}e^{-\sqrt{B}r^q}\,dr
\end{aligned}
$$

Further, after restoring the integral representation of the gamma function and a change of variables, we obtain for $\frac{2(\gamma-1)}{q}+k+l>0$, $\frac{2(\gamma-1)}{q}+1+k+l>0$ and $\frac{2(\gamma-1)}{q}+2+k+l>0$ that

$$
\begin{aligned}
&I_{mn} \\
&= B^{\frac{1-\gamma}{q}}\frac{(-\gamma^2+2\gamma-\frac{3}{4})}{q}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l} \\
&\quad\times\frac{\Gamma(\frac{2(\gamma-1)}{q}+k+l)}{k!l!} \\
&\quad- B^{\frac{l-\gamma}{q}}\frac{q}{4}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{k!l!} \\
&\quad+ B^{\frac{l-\gamma}{q}}(\gamma-1+\frac{q}{2})\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l} \\
&\quad\times\frac{\Gamma(\frac{2(\gamma-1)}{q}+1+k+l)}{k!l!} + B^{\frac{1-\gamma}{q}}(2(\gamma-1)+q)\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m} \\
&\quad\times\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+1+k+l)}{k!l!} \\
&\quad- B^{\frac{1-\gamma}{q}}q\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l} \\
&\quad\times\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{k!l!} - B^{\frac{1-\gamma}{q}}q\frac{n(-1+n)a_2(1+a_2)\cdots a_p(1+a_p)}{\gamma(1+\gamma)} \\
&\quad\times\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k(a_2)_k\cdots(a_p)_k(2-n)_l(2+a_2)_l\cdots(2+a_p)_l}{(\gamma)_k(2+\gamma)_l} \\
&\quad\times\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{k!l!} \\
&= -B^{\frac{1-\gamma}{q}}\frac{(-\gamma^2+2\gamma-\frac{3}{4})}{q}\sum_{k=0}^{m}\Big[\sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+k+l)}{l!}\Big] \\
&\quad\times\frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!} \\
&\quad- B^{\frac{l-\gamma}{q}}\frac{q}{4}\sum_{k=0}^{m}\Big[\sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{l!}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!} + B^{\frac{l-\gamma}{q}}(\gamma-1+\frac{q}{2})\sum_{k=0}^{m} \\
&\times \Big[\sum_{l=0}^{n} \frac{(-m)_k(a_2)_k\cdots(a_p)_k(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_k(\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+1+k+l)}{l!}\Big] \\
&\times \frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!} \\
&+ B^{\frac{1-\gamma}{q}}(2(\gamma-1)+q)\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m}\Big[\sum_{l=0}^{n}\frac{(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l} \\
&\times \frac{\Gamma(\frac{2(\gamma-1)}{q}+1+k+l)}{k!}\Big]\frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!} \\
&- B^{\frac{1-\gamma}{q}}q\frac{na_2\cdots a_p}{\gamma}\sum_{k=0}^{m}\Big[\sum_{l=0}^{n}\frac{(1-n)_l(1+a_2)_l\cdots(1+a_p)_l}{(\gamma)_k(1+\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{l!}\Big] \\
&\times \frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!} - B^{\frac{1-\gamma}{q}}q\frac{n(-1+n)a_2(1+a_2)\cdots a_p(1+a_p)}{\gamma(1+\gamma)} \\
&\times \sum_{k=0}^{m}\Big[\sum_{l=0}^{n}\frac{(2-n)_l(2+a_2)_l\cdots(2+a_p)_l}{(\gamma)_k(2+\gamma)_l}\frac{\Gamma(\frac{2(\gamma-1)}{q}+2+k+l)}{l!}\Big] \\
&\times \frac{(-m)_k(a_2)_k\cdots(a_p)_k}{k!}
\end{aligned} \tag{3.20}
$$

We denote the finite sum inside the brackets in the expression above as I_{1mn}, I_{2mn}, $\dots$, I_{6mn}. On the other hand, by using of the definition of the pochhammer symbols and the series representation of the hypergeometric functions (1.4), the finite sum inside the bracket collapses to

$$
\begin{aligned}
I_{1mn} &= \sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_l l!}\Gamma(\frac{2(\gamma-1)}{q}+k+l) \\
&= \sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l(\frac{2(\gamma-1)}{q}+k)_l}{(\gamma)_l l!}\Gamma(\frac{2(\gamma-1)}{q}+k) \\
&=_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2(\gamma-1)}{q}+k;\gamma;1)\Gamma(\frac{2(\gamma-1)}{q}+k)
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
I_{2mn} &= \sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l}{(\gamma)_l l!}\Gamma(\frac{2(\gamma-1)}{q}+1+k+l) \\
&= \sum_{l=0}^{n}\frac{(-n)_l(a_2)_l\cdots(a_p)_l(\frac{2(\gamma-1)}{q}+1+k)_l}{(\gamma)_l l!}\Gamma(\frac{2(\gamma-1)}{q}+1+k) \\
&=_{p+1}F_1(-n,a_2,\dots,a_p,\frac{2(\gamma-1)}{q}+1+k;\gamma;1)\Gamma(\frac{2(\gamma-1)}{q}+1+k)
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
I_{3mn} &= \sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l}{(\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 2 + k + l) \\
&= \sum_{l=0}^{n} \frac{(-n)_l (a_2)_l \cdots (a_p)_l (\frac{2(\gamma-1)}{q} + 2 + k)_l}{(\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 2 + k) \\
&= {}_{p+1}F_1(-n, a_2, \dots, a_p, \frac{2(\gamma-1)}{q} + 2 + k; \gamma; 1) \Gamma(\frac{2(\gamma-1)}{q} + 2 + k)
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
I_{4mn} &= \sum_{l=0}^{n} \frac{(1-n)_l (1+a_2)_l \cdots (1+a_p)_l}{(1+\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 1 + k + l) \\
&= \sum_{l=0}^{n} \frac{(1-n)_l (1+a_2)_l \cdots (1+a_p)_l (\frac{2(\gamma-1)}{q} + 1 + k)_l}{(1+\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 1 + k) \\
&= {}_{p+1}F_1(1-n, 1+a_2, \dots, 1+a_p, \frac{2(\gamma-1)}{q} + 1 + k; 1+\gamma; 1) \\
&\quad \times \Gamma(\frac{2(\gamma-1)}{q} + 1 + k)
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
I_{5mn} &= \sum_{l=0}^{n} \frac{(1-n)_l (1+a_2)_l \cdots (1+a_p)_l}{(1+\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 2 + k + l) \\
&= \sum_{l=0}^{n} \frac{(1-n)_l (1+a_2)_l \cdots (1+a_p)_l (\frac{2(\gamma-1)}{q} + 2 + k)_l}{(1+\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 2 + k) \\
&= {}_{p+1}F_1(1-n, 1+a_2, \dots, 1+a_p, \frac{2(\gamma-1)}{q} + 2 + k; 1+\gamma; 1) \\
&\quad \times \Gamma(\frac{2(\gamma-1)}{q} + 2 + k)
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
I_{6mn} &= \sum_{l=0}^{n} \frac{(2-n)_l (2+a_2)_l \cdots (2+a_p)_l}{(2+\gamma)_l} \frac{\Gamma(\frac{2(\gamma-1)}{q} + 2 + k + l)}{l!} \\
&= \sum_{l=0}^{n} \frac{(2-n)_l (2+a_2)_l \cdots (2+a_p)_l (\frac{2(\gamma-1)}{q} + 2 + k)_l}{(2+\gamma)_l l!} \Gamma(\frac{2(\gamma-1)}{q} + 2 + k) \\
&= {}_{p+1}F_1(2-n, 2+a_2, \dots, 2+a_p, \frac{2\gamma-2}{q} + 2 + k; 2+\gamma; 1) \\
&\quad \times \Gamma(\frac{2(\gamma-1)}{q} + 2 + k)
\end{aligned} \tag{3.26}
$$

Substituting (3.21), (3.22), (3.23), (3.24), (3.25) and (3.26) into (3.20) we obtain

$$
I_{mn} = -B^{\frac{1-\gamma}{q}} \frac{(\gamma^2 - 2\gamma + \frac{3}{4})}{q} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{1mn}
$$

$$
\begin{aligned}
&- B^{\frac{1-\gamma}{q}} \frac{q}{4} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{2mn} \\
&+ B^{\frac{1-\gamma}{q}} (\gamma - 1 + \frac{q}{2}) \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{3mn} \\
&+ B^{\frac{1-\gamma}{q}} (q + 2(\gamma-1)) \frac{n a_2 \cdots a_p}{\gamma} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{4mn} \\
&- B^{\frac{1-\gamma}{q}} q \frac{n a_2 \cdots a_p}{\gamma} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{5mn} \\
&- B^{\frac{1-\gamma}{q}} q \frac{n(-1+n) a_2 (1+a_2) \cdots a_p (1+a_p)}{\gamma(1+\gamma)} \sum_{k=0}^{m} \frac{(-m)_k (a_2)_k \cdots (a_p)_k}{(\gamma)_k k!} I_{6mn}
\end{aligned}
$$

This completes the proof. □

In the following tables, we used the derived formulas to compute upper bounds for many problems of interest in the literature. We compare our results with other authors who analyze the spectrum for this kind of problems. One of the advantage and the purpose of such formulas is to provide the researcher with simple method to compute the spectrum for sum of power-law potentials in the literature other than the saving in the time in these computations.

Table 1 shows the eigenvalues of $H = -\frac{d^2}{dr^2} + V(r)$, where $V(r) = r^2 + \frac{\lambda}{r^{2.5}}$ anharmonic oscillator potential for different values of λ using

$$
\psi_n(r) = \sqrt{\frac{2\beta^{\frac{\gamma}{2}} (\gamma)_n}{n!\Gamma(\gamma)}} r^{\gamma - \frac{1}{2}} e^{-\frac{\sqrt{\beta} r^q}{2}} \ {}_3F_1(-n, 0.2, 0.07; \gamma; \sqrt{\beta} r^q)
$$

obtained using the computed matrix elements (3.18) and minimizing with respect to $q, \gamma, \text{and} \beta$. The exponent (n) refer to the dimensions of the matrix used for the variational computations.

λ	q	γ	β	$E^{(3)}$	$E^{(4)}$	$E^{(5)}$	E
100	1.824	11.117	2.4115	17.541 916	17.541 912	17.541 912	17.541 890
10	1.811	4.797	2.154	7.735 317	7.735 256	7.735 254	7.735 111
1	1.801	2.553	1.917	4.318 485	4.318 181	4.318 180	4.317 312
0.5	1.807	2.229	1.8169	3.850 244	3.850 028	3.849 862	3.848 553
0.01	1.95	1.55	1.132	3.037 422	3.037 421	3.037 415	3.036 665
0.001	1.993	1.506	1.017	3.004 046 2	3.004 046 2	3.004 046 1	3.004 014

TABLE 1.

Table 2 Shows a comparison between the results of$E^{HS}$and the results of the present workE^{U} using

$$
\psi_n(r) = \sqrt{\frac{2\beta^{\frac{\gamma}{2}} (\gamma)_n}{n!\Gamma(\gamma)}} r^{\gamma - \frac{1}{2}} e^{-\frac{\sqrt{\beta} r^q}{2}} \ {}_3F_1(-n, 0.2, 0.07; \gamma; \sqrt{\beta} r^q)
$$

and $H = -\frac{d^2}{dr^2} + V(r)$ where $V(r) = r^2 + \frac{\lambda}{r^{2.5}}$ anharmonic oscillator potential for different values of λ, obtained using the computed matrix elements (3.18) and minimizing with respect to $q, \gamma, \text{ and} \beta$. The exponent (n) refer to the dimensions of the matrix used for the variational computations.

λ	q	γ	β	E^{HS}	E^{HS}_m	E^U	E
100	1.824	11.117	2.4115	$17.541\ 890^{(30)}$	$17.542\ 040^{(5)}$	$17.541\ 912^{(5)}$	17.541 890
10	1.811	4.797	2.154	$7.735\ 637^{(30)}$	$7.735\ 596^{(5)}$	$7.735\ 254^{(5)}$	7.735 111
1	1.801	2.553	1.917	$4.323\ 263^{(30)}$	$4.318\ 963^{(5)}$	$4.318\ 141^{(7)}$	4.317 312
0.5	1.807	2.229	1.8169	$3.869\ 547^{(30)}$	$3.850\ 823^{(5)}$	$3.849\ 759^{(7)}$	3.848 553
0.01	1.95	1.55	1.132	$3.039\ 244^{(30)}$	$3.037\ 474^{(5)}$	$3.037\ 399^{(8)}$	3.036 665
0.001	1.993	1.506	1.017	$3.004\ 074^{(30)}$	$3.004\ 047^{(5)}$	$3.004\ 046^{(8)}$	3.004 014

TABLE 2. E^{HS} from [14]. E^{HS}_m from [14] after minimizing with respect to one parameter.

Table 3 shows a comparison between the results of $E^B$and the results of the present work E^U using

$$\psi_n(r) = \sqrt{\frac{2\beta^{\frac{\gamma}{2}}(\gamma)_n}{n!\Gamma(\gamma)}}r^{\gamma-\frac{1}{2}}e^{-\frac{\sqrt{\beta}r^q}{2}}\ {}_3F_1(-n, 0.2, 0.07; \gamma; \sqrt{\beta}r^q)$$

and $H = -\frac{d^2}{dr^2} + V(r)$where $V(r) = r^4 - \lambda r^2$ anharmonic oscillator potential for different values of λ, obtained using the computed matrix elements (3.18) and minimizing with respect to $q, \gamma,$ and β. The exponent (n) refer to the dimensions of the matrix used for the variational computations.

λ	q	γ	β	E^B	E^U	E
2.0	2.928	1.472	0.425	$1.726\ 29^{(10)}$	$1.713\ 36^{(9)}$	1.713 03
1.0	2.023	1.469	0.927	$2.838\ 91^{(10)}$	$2.835\ 34^{(8)}$	2.834 54
0.9	2.616	1.469	0.989	$2.941\ 23^{(10)}$	$2.937\ 73^{(9)}$	2.937 30
0.8	2.595	1.470	1.052	$3.042\ 10^{(10)}$	$3.039\ 06^{(8)}$	3.038 56
0.7	2.574	1.470	1.117	$3.141\ 55^{(10)}$	$3.138\ 86^{(8)}$	3.138 37
0.6	2.536	1.740	1.184	$3.239\ 62^{(10)}$	$3.237\ 24^{(9)}$	3.236 76
0.5	2.536	1.470	1.253	$3.336\ 36^{(10)}$	$3.334\ 12^{(9)}$	3.333 78
0.4	2.518	1.471	1.323	$3.431\ 79^{(10)}$	$3.429\ 94^{(9)}$	3.429 47
0.3	2.500	1.471	1.394	$3.525\ 96^{(10)}$	$3.524\ 02^{(9)}$	3.523 87
0.2	2.484	1.472	1.467	$3.618\ 90^{(10)}$	$3.617\ 47^{(8)}$	3.617 01
0.1	2.468	1.472	1.542	$3.710\ 64^{(10)}$	$3.709\ 39^{(8)}$	3.708 93

TABLE 3. E^B from [4].

Conclusion. In this paper we compute closed formula for the matrix elements for Schrödinger equation using generalized hypergeometric function. We used general basis using special kind of functions obtained from the exact solutions of the Gol'dman and Krivchenkov eigenvalue problem [19]. In fact, for specific values of the constants p and q in (1.3) we retrieve to the old results found in the literature [8, 12, 13, 14, 15]. One of the advantages of our results is to obtain analytical formulas that provide us with accurate bounds for the non-solvable singular eigenvalues problems as well as saves time of the computations using the derived formulas.

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