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PERTURBED FUNCTIONAL AND NEUTRAL FUNCTIONAL EVOLUTION EQUATIONS WITH INFINITE DELAY IN FRÉCHET SPACES

SELMA BAGHLI, MOUFFAK BENCHOHRA

Abstract. This article shows sufficient conditions for the existence of mild solutions, on the positive half-line, for two classes of first-order functional and neutral functional perturbed differential evolution equations with infinite delay. Our main tools are: the nonlinear alternative proved by Avramescu for the sum of contractions and completely continuous maps in Fréchet spaces, and the semigroup theory.

1. Introduction

In this paper, we study the existence of mild solutions, defined on the positive semi-infinite real interval $J := [0, +\infty)$, for two classes of first-order perturbed functional and neutral functional differential evolution equations with infinite delay in Fréchet spaces. Firstly, in Section 3, we study the following partial perturbed evolution equation with infinite delay

$$y'(t) = A(t)y(t) + f(t, y_t) + g(t, y_t), \quad \text{a.e. } t \in J, \tag{1.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.2}$$

where $f, g : J \times \mathcal{B} \to E$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from a real Banach space $(E, |\cdot|)$ into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t < +\infty$.

For any continuous function y defined on $(-\infty, +\infty)$ and any $t \ge 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t. We assume that the histories y_t belongs to some abstract *phase space* $\mathcal{B}$, to be specified later.

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In Section 4, we consider the following perturbed neutral evolution equation with infinite delay

$$\frac{d}{dt}[y(t) - h(t, y_t)] = A(t)y(t) + f(t, y_t) + g(t, y_t), \quad \text{a.e. } t \in J, \tag{1.3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.4}$$

where $A(\cdot)$, f, g and ϕ are as in (1.1)–(1.2) and $h : J \times \mathcal{B} \to E$ is a given function. Finally in Section 5, we give two examples to demonstrate our results.

Functional and partial functional differential equations have been used for modelling the evolution of physical, biological and economic systems in which the response of the system depends not only on the current state, but also on the past history of the system. For more details on this topic, see for example the books of Kolmanovskii and Myshkis [34], Hale and Verduyn Lunel [26] and Wu [37], and the references therein. In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on the finite interval $[-r, 0]$ for $r > 0$, endowed with the uniform norm topology. Some results in this case can be found in the books by Ahmed [4, 5], Heikkila and Lakshmikantham [27], and Pazy [35] and the references therein.

When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, introduced by Hale and Kato in [25]; see also Corduneanu and Lakshmikantham [21], Kappel and Schappacher [33] and Schumacher [36]. For a detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [26], Hino *et al.* [32] and Wu [37].

Many publications are developed for study of (1.1) with $A(t) = A$. We refer the reader to the books by [27] and the pioneer Hino and Murakami paper [31] and the papers by Adimy *et al* [1, 2, 3], Balachandran *et al.* [10, 11], Benchohra and Gorniewicz [16], Benchohra *et al* [17, 18], Ezzinbi [23], Henriquez [28] and Hernandez [29, 30], where existence and uniqueness, among other things, are derived. In a series of papers, Belmekki *et al* [12, 13, 14, 15] considered some classes of semilinear perturbed functional differential problems where existence of solutions are given over a bounded interval $[0, b]$.

When A depends on time, Arara *et al* [6] considered a control multivalued problem on the bounded interval $[0, b]$. Recently, Baghli and Benchohra [8, 9] provided uniqueness results for some classes of partial and neutral functional differential evolution equations on the semiinfinite interval $J = [0, +\infty)$ with local and nonlocal conditions when the delay is finite. Our main purpose in this paper is to extend some results from finite delay and those considered on a bounded interval to partial and neutral *perturbed* evolution equations.

Sufficient conditions are established to obtain the existence of mild solutions, which are fixed points of the appropriate operators. We apply a recent nonlinear alternative given by Avramescu in [7], combined with the semigroup theory [4, 35].

2. Preliminaries

In this section, we introduce notation, definitions and theorems to be used later. Let $C([0, \infty); E)$ be the space of continuous functions from $[0, \infty)$ to E and $B(E)$

be the space of all bounded linear operators from E to E, with the norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : [0, +\infty) \to E$ is Bochner integrable if $|y|$ is Lebesgue integrable. (For details on the Bochner integral properties, see Yosida [38]).

Let $L^1([0, +\infty), E)$ be the Banach space of measurable functions $y : [0, +\infty) \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\, dt.$$

Consider the space

$$B_{+\infty} = \{y : (-\infty, +\infty) \to E : y|_J \in C(J, E),\ y_0 \in \mathcal{B}\},$$

where $y|_J$ is the restriction of y to J.

In this paper, we will employ the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [25] and follow the terminology used in [32]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ to E, and satisfying the following axioms:

(A1) If $y : (-\infty, b) \to E$ with $b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold:
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $y(t)$ with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(A2) For the function $y(.)$ in (A1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.1.

- Condition (ii) in (A1) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
- Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
- From the equivalence of (ii), we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$. This implies necessarily that $\phi(0) = \psi(0)$.

Next we present some examples of phase spaces. For more details we refer to the book by Hino *et al* [32].

Example 2.2. Let BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E. Let BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E. Let

$$C^\infty := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\}.$$

$$C^0 := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\}\,.$$

The space C^0 is endowed with the uniform norm $\|\phi\| = \sup\{|\phi(\theta)| : \theta \le 0\}$.

Then the spaces BUC, C^∞ and C^0 satisfy conditions (A1)–(A3). BC satisfies (A1), (A3) but not (A2).

Example 2.3. Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \{\phi \in C((-\infty,0],E) : \frac{\phi(\theta)}{g(\theta)} \text{is bounded on} (-\infty, 0]\},$$

$$C_g^0 := \{\phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\}$$

endowed with the uniform norm $\|\phi\| = \sup\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0\}$.

Also we assume that

(G1) For all $a > 0$, $\sup_{0 \leq t \leq a} \sup\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \leq -t\} < \infty$.

Then the spaces C_g and C_g^0 satisfy condition (A3). They satisfy conditions (A1) and (A2) if (G1) holds.

Example 2.4. For each constant γ, we define the space

$$C_\gamma := \{\phi \in C((-\infty,0],E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E\}$$

endowed with the norm $\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}$. Then in the space C_γ, assumptions (A1)–(A3) are satisfied.

Definition 2.5. A function $f : J \times \mathcal{B} \to E$ is said to be an L^1-Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t,.) : \mathcal{B} \to E$ is continuous;
(ii) for each $y \in \mathcal{B}$ the function $f(.,y) : J \to E$ is measurable;
(iii) for every positive integer k there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that $|f(t,y)| \leq h_k(t)$ for all $\|y\|_{\mathcal{B}} \leq k$ and almost all $t \in J$.

In what follows, we assume that $\{A(t),\, t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.6. A family of bounded linear operators $\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : E \to E$ for $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$ is called an evolution system if the following properties are satisfied :

(1) $U(t,t) = I$ where I is the identity operator in E,
(2) $U(t,s)U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,
(3) $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)y$ is continuous.
(4) $U(t,s)$ is a compact operator for $0 < s < t < +\infty$.

More details on evolution systems and their properties can be found in the books by Ahmed [4], Engel and Nagel [22], and Pazy [35].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that F is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \text{for all } y \in Y.$$

With X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$,

$int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \dots \quad \text{for every } x \in X.$$

Definition 2.7 ([7]). A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that:

$$\|f(x) - f(y)\|_n \le k_n \|x - y\|_n \quad \text{for all } x, y \in X.$$

We use the following nonlinear alternative, due to Avramescu, has been has a version on Banach spaces by Burton-Kirk [19, 20].

Theorem 2.8 (Avramescu Nonlinear Alternative [7]). *Let X be a Fréchet space and let $A, B : X \to X$ be two operators satisfying:*

1. *A is a compact operator,*
2. *B is a contraction.*

Then either one of the following statements holds:

- (S1) *The operator $A + B$ has a fixed point;*
- (S2) *The set $\{x \in X, x = \lambda A(x) + \lambda B(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in (0,1)$.*

3. Perturbed Evolution Equations

Before stating and proving the main result, we give the definition of mild solution of the semilinear perturbed evolution (1.1)–(1.2).

Definition 3.1. We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of (1.1)–(1.2) if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$ and y satisfies the integral equation

$$y(t) = U(t,0)\phi(0) + \int_0^t U(t,s)[f(s,y_s) + g(s,y_s)]\,ds \quad \text{for each } t \in [0,+\infty). \tag{3.1}$$

We introduce the following hypotheses:

- (H1) $U(t,s)$ is compact for $t - s > 0$ and there exists a constant $\widehat{M} \ge 1$ such that
$$\|U(t,s)\|_{B(E)} \le \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$
- (H2) There exists a function $p \in L^1_{\rm loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ and such that:
$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{B}}) \quad \text{for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$
- (H3) There exists a function $\eta \in L^1_{\rm loc}(J,\mathbb{R}_+)$ such that:
$$|g(t,u) - g(t,v)| \le \eta(t)\|u - v\|_{\mathcal{B}} \quad \text{for a.e. } t \in J \text{ and all } u, v \in \mathcal{B}.$$

Theorem 3.2. *Suppose that hypotheses* (H1)–(H3) *are satisfied and*

$$\int_{\alpha_n}^{+\infty} \frac{ds}{s + \psi(s)} > K_n \widehat{M} \int_0^n \max(p(s), \eta(s))ds\,ds \quad \textit{for each } n \in \mathbb{N} \tag{3.2}$$

with

$$\alpha_n = K_n \widehat{M} \int_0^n |g(s,0)|ds + (K_n \widehat{M} H + M_n)\|\phi\|_{\mathcal{B}}.$$

Then (1.1)–(1.2) *has a mild solution.*

Proof. Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\|y\|_n := \sup\{e^{-\tau L_n^*(t)}|y(t)| : t \in [0,n]\}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)\,ds$ and $\bar{l}_n(t) = \widehat{M}K_n\eta(t)$. Then $C(B_{+\infty};E)$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

We transform (1.1)–(1.2) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\phi(0) + \int_0^t U(t,s)f(s,y_s)\,ds & \\ + \int_0^t U(t,s)g(s,y_s)\,ds, & \text{if } t \in J. \end{cases} \tag{3.3}$$

Clearly, the fixed points of the operator N are mild solutions of (1.1)–(1.2).

For $\phi \in \mathcal{B}$, we define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{3.4}$$

It is obvious that y satisfies (3.1) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s)f(s, z_s + x_s)\,ds + \int_0^t U(t,s)g(s, z_s + x_s)\,ds \quad \text{for } t \in J.$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}$. Define the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)f(s, z_s + x_s)\,ds \quad \text{for } t \in J, \tag{3.5}$$

$$G(z)(t) = \int_0^t U(t,s)g(s, z_s + x_s)\,ds \quad \text{for } t \in J. \tag{3.6}$$

Obviously the operator N having a fixed point is equivalent to $F+G$ having a fixed point. That $F+G$ has a fixed point will be proved in several steps. First we show that F is continuous and compact.

Step 1: F is continuous. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_k \to z$ in $B^0_{+\infty}$. Then

$$\begin{aligned} |F(z_k)(t) - F(z)(t)| &= \Big| \int_0^t U(t,s)[f(s, z_{k_s} + x_s) - f(s, z_s + x_s)]\,ds \Big| \\ &\le \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_{k_s} + x_s) - f(s, z_s + x_s)|\,ds \\ &\le \widehat{M} \int_0^t |f(s, z_{k_s} + x_s) - f(s, z_s + x_s)|\,ds \to 0 \quad \text{as } k \to +\infty. \end{aligned}$$

Thus F is continuous.

Step 2: F maps bounded sets of $B^0_{+\infty}$ into bounded sets. For any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B^0_{+\infty} : \|z\|_n \le d\}$ one has $\|F(z)\|_n \le \ell$.

Let $z \in B_d$. By the hypotheses (H1) and (H2), we have for each $t \in J$

$$\begin{aligned} |F(z)(t)| &= \Big| \int_0^t U(t,s) f(s, z_s + x_s)\, ds \Big| \\ &\le \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)|\, ds \\ &\le \widehat{M} \int_0^t p(s) \psi(\|z_s + x_s\|_{\mathcal{B}})\, ds. \end{aligned}$$

Using the assumption (A1), we get

$$\begin{aligned} \|z_s + x_s\|_{\mathcal{B}} &\le \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\ &\le K(s)|z(s)| + M(s)\|z_0\|_{\mathcal{B}} + K(s)|x(s)| + M(s)\|x_0\|_{\mathcal{B}} \\ &\le K_n |z(s)| + K_n \|U(s,0)\|_{B(E)} |\phi(0)| + M_n \|\phi\|_{\mathcal{B}} \\ &\le K_n |z(s)| + K_n \widehat{M} |\phi(0)| + M_n \|\phi\|_{\mathcal{B}} \\ &\le K_n |z(s)| + K_n \widehat{M} H \|\phi\|_{\mathcal{B}} + M_n \|\phi\|_{\mathcal{B}} \\ &\le K_n |z(s)| + (K_n \widehat{M} H + M_n) \|\phi\|_{\mathcal{B}}. \end{aligned}$$

Set

$$c_n := (K_n \widehat{M} H + M_n)\|\phi\|_{\mathcal{B}}, \quad D_n := K_n d + c_n.$$

Then

$$\|z_s + x_s\|_{\mathcal{B}} \le K_n |z(s)| + c_n \le D_n. \tag{3.7}$$

Using the nondecreasing character of ψ, we get

$$|F(z)(t)| \le \widehat{M} \psi(D_n) \int_0^t p(s)\, ds.$$

Thus

$$\|F(z)\|_{+\infty} \le \widehat{M} \psi(D_n) \|p\|_{L^1} := \ell.$$

Step 3: F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| &\le \Big| \int_0^{\tau_1} [U(\tau_2, s) - U(\tau_1, s)] f(s, z_s + x_s)\, ds \Big| \\ &\quad + \Big| \int_{\tau_1}^{\tau_2} U(\tau_2, s) |f(s, z_s + x_s)|\, ds \Big| \\ &\le \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} |f(s, z_s + x_s)|\, ds \\ &\quad + \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} |f(s, z_s + x_s)|\, ds. \end{aligned}$$

Using $\|z_s + x_s\|_{\mathcal{B}} \le D_n$ in (3.7) and the nondecreasing character of ψ, we get

$$\begin{aligned} &|F(z)(\tau_2) - F(z)(\tau_1)| \\ &\le \psi(D_n) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s) ds + \widehat{M} \psi(D_n) \int_{\tau_1}^{\tau_2} p(s) ds. \end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $U(t,s)$ is a strongly continuous operator and the compactness of $U(t,s)$ for $t > s$ implies the continuity in the uniform operator topology (see [5, 35]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_d$ we define

$$F_\epsilon(z)(t) = U(t, t-\epsilon) \int_0^{t-\epsilon} U(t-\epsilon, s) f(s, z_s + x_s)\, ds.$$

Since $U(t,s)$ is a compact operator, the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) : z \in B_d\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$|F(z)(t) - F_\epsilon(z)(t)| \leq \int_{t-\epsilon}^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)| ds.$$

Using $\|z_s + x_s\|_{\mathcal{B}} \leq D_n$ in (3.7) and the nondecreasing character of ψ, we get

$$|F(z)(t) - F(z)_\epsilon(t)| \leq \widehat{M} \psi(D_n) \int_{t-\epsilon}^t p(s) ds.$$

Therefore the set $Z(t) = \{F(z)(t) : z \in B_d\}$ is totally bounded. Hence the set $\{F(z)(t) : z \in B_d\}$ is relatively compact E. So we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4: G is a contraction mapping. Let $z, \overline{z} \in B^0_{+\infty}$, then using (H1) and (H3) for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} |g(s, z_s + x_s) - g(s, \overline{z}_s + x_s)|\, ds \\
&\leq \int_0^t \widehat{M} \eta(s) \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}}\, ds \\
&\leq \int_0^t \widehat{M} \eta(s) \|z_s - \overline{z}_s\|_{\mathcal{B}}\, ds.
\end{aligned}$$

Using (A1), we obtain

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \widehat{M} \eta(s) (K(s)|z(s) - \overline{z}(s)| + M(s) \|z_0 - \overline{z}_0\|_{\mathcal{B}})\, ds \\
&\leq \int_0^t \widehat{M} K_n \eta(s) |z(s) - \overline{z}(s)|\, ds \\
&\leq \int_0^t [\bar{l}_n(s) e^{\tau L^*_n(s)}][e^{-\tau L^*_n(s)} |z(s) - \overline{z}(s)|]\, ds \\
&\leq \int_0^t \Big[\frac{e^{\tau L^*_n(s)}}{\tau}\Big]'\, ds \|z - \overline{z}\|_n \\
&\leq \frac{1}{\tau} e^{\tau L^*_n(t)} \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|G(z) - G(\overline{z})\|_n \leq \frac{1}{\tau} \|z - \overline{z}\|_n.$$

So, the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5: For applying Theorem 2.8, we must check (S2): i.e. it remains to show that the set

$$\mathcal{E} = \{z \in B^0_{+\infty} : z = \lambda F(z) + \lambda G\big(\frac{z}{\lambda}\big) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $z \in \mathcal{E}$. By (H1)–(H3), we have for each $t \in [0,n]$

$$\begin{aligned}
|z(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)| ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |g(s, z_s + x_s) - g(s,0) + g(s,0)| ds \\
&\leq \widehat{M} \int_0^t p(s) \psi\left(\|z_s + x_s\|_{\mathcal{B}}\right) ds \\
&\quad + \widehat{M} \int_0^t \eta(s) \|z_s + x_s\|_{\mathcal{B}} ds + \widehat{M} \int_0^t |g(s,0)| ds.
\end{aligned}$$

Using (3.7) we get

$$\|z_s + x_s\|_{\mathcal{B}} \leq K_n |z(s)| + c_n.$$

The nondecreasing character of ψ gives

$$\begin{aligned}
|z(t)| &\leq \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n) ds \\
&\quad + \widehat{M} \int_0^t \eta(s) (K_n |z(s)| + c_n) ds + \widehat{M} \int_0^t |g(s,0)| ds.
\end{aligned}$$

Then

$$\begin{aligned}
K_n |z(t)| + c_n &\leq K_n \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n) ds \\
&\quad + K_n \widehat{M} \int_0^t \eta(s) (K_n |z(s)| + c_n) ds + K_n \widehat{M} \int_0^t |g(s,0)| ds + c_n.
\end{aligned}$$

Set

$$\alpha_n := K_n \widehat{M} \int_0^t |g(s,0)| ds + c_n,$$

thus

$$\begin{aligned}
K_n |z(t)| + c_n \leq & K_n \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n) ds \\
& + K_n \widehat{M} \int_0^t \eta(s) (K_n |z(s)| + c_n) ds + \alpha_n.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\{K_n |z(s)| + c_n : 0 \leq s \leq t\}, \quad 0 \leq t < +\infty.$$

Let $t^\star \in [0,t]$ be such that

$$\mu(t) = K_n |z(t^\star)| + c_n,$$

by the previous inequality, we have

$$\mu(t) \leq K_n \widehat{M} \int_0^t p(s) \psi(\mu(s)) ds + K_n \widehat{M} \int_0^t \eta(s) \mu(s) ds + \alpha_n$$

for $t \in [0,n]$. Let us denote the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \leq v(t) \quad \text{for all } t \in [0,n].$$

From the definition of v, we have $v(0) = \alpha_n$ and

$$v'(t) = K_n \widehat{M} p(t) \psi(\mu(t)) + K_n \widehat{M} \eta(t) \mu(t) \quad \text{a.e. } t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq K_n\widehat{M}p(t)\psi(v(t)) + K_n\widehat{M}\eta(t)v(t) \quad \text{a.e. } t \in [0,n].$$

This implies that for each $t \in [0,n]$ and using (3.2), we get

$$\begin{aligned}\int_{\alpha_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq K_n\widehat{M}\int_0^t \max(p(s),\eta(s))ds \\ &\leq K_n\widehat{M}\int_0^n \max(p(s),\eta(s))ds \\ &< \int_{\alpha_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant N_n such that $v(t) \leq N_n$ and hence $\mu(t) \leq N_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq N_n$. This shows that the set $\mathcal{E}$ is bounded. Then statement $(S2)$ in Theorem 2.8 does not hold. The nonlinear alternative of Avramescu implies that $(S1)$ holds, we deduce that the operator $F+G$ has a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty,+\infty)$ is a fixed point of the operator N, which is the mild solution of (1.1)–(1.2). □

4. Perturbed Neutral Evolution Equations

In this section, we give an existence result for the perturbed neutral evolution problem with infinite delay (1.3)–(1.4). Firstly we define the mild solution.

Definition 4.1. We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of (1.3)–(1.4) if $y(t) = \phi(t)$ for all $t \in (-\infty,0]$ and y satisfies the integral equation

$$\begin{aligned}y(t) &= U(t,0)[\phi(0) - h(0,\phi)] + h(t,y_t) + \int_0^t U(t,s)A(s)h(s,y_s)ds \\ &\quad + \int_0^t U(t,s)[f(s,y_s) + g(s,y_s)]\,ds \quad \text{for each } t \in [0,+\infty).\end{aligned} \tag{4.1}$$

In what follows we need the following assumptions:

(H4) There exists a constant $\overline{M}_0 > 0$ such that
$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all } t \in J.$$

(H5) There exists a constant $0 < L < \frac{1}{\overline{M}_0 K_n}$ such that
$$|A(t)h(t,\phi)| \leq L(\|\phi\|_{\mathcal{B}} + 1) \quad \text{for all } t \in J,\ \phi \in \mathcal{B}.$$

(H6) There exists a constant $L_* > 0$ such that
$$|A(s)h(s,\phi) - A(\overline{s})h(\overline{s},\overline{\phi})| \leq L_*(|s-\overline{s}| + \|\phi-\overline{\phi}\|_{\mathcal{B}})$$
for all $s,\overline{s} \in J$ and $\phi,\overline{\phi} \in \mathcal{B}$.

Theorem 4.2. *Suppose that hypotheses* (H1)–(H6) *are satisfied and*

$$\int_{\zeta_n}^{+\infty} \frac{ds}{s+\psi(s)} > \frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}\int_0^n \max(L,\eta(s),p(s))ds \quad \textit{for each } n \in \mathbb{N} \tag{4.2}$$

with

$$\zeta_n := \frac{K_n}{1-\overline{M}_0LK_n}\Big[\overline{M}_0L\big(1+\widehat{M}+c_n+\widehat{M}\|\phi\|_{\mathcal{B}}\big) + \widehat{M}Ln + \widehat{M}\int_0^t |g(s,0)|\,ds\Big] + c_n,$$

and $c_n := (K_n\widehat{M}H + M_n)\|\phi\|_{\mathcal{B}}$. *Then* (1.3)–(1.4) *has a mild solution.*

Proof. Consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)[\phi(0) - h(0,\phi)] + h(t, y_t) \\ + \int_0^t U(t,s)A(s)h(s,y_s)ds \\ + \int_0^t U(t,s)[f(s,y_s) + g(s,y_s)]ds, & \text{if } t \in J. \end{cases} \tag{4.3}$$

Note that the fixed points of the operator $\widetilde{N}$ are mild solutions of (1.3)–(1.4). For $\phi \in \mathcal{B}$, we define the function $x : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{4.4}$$

It is obvious that y satisfies (4.1) if and only if z satisfies $z_0 = 0$. For $t \in J$, we get

$$\begin{aligned} z(t) &= h(t, z_t + x_t) - U(t,0)h(0,\phi) + \int_0^t U(t,s)A(s)h(s, z_s + x_s)ds \\ &+ \int_0^t U(t,s)f(s, z_s + x_s)ds + \int_0^t U(t,s)g(s, z_s + x_s)ds. \end{aligned}$$

Define the operators $\widetilde{F}, \widetilde{G} : B^0_{+\infty} \to B^0_{+\infty}$ by

$$\widetilde{F}(z)(t) = \int_0^t U(t,s)f(s, z_s + x_s)ds \tag{4.5}$$

and

$$\begin{aligned} \widetilde{G}(z)(t) &= h(t, z_t + x_t) - U(t,0)h(0,\phi) + \int_0^t U(t,s)A(s)h(s, z_s + x_s)ds \\ &+ \int_0^t U(t,s)g(s, z_s + x_s)ds. \end{aligned} \tag{4.6}$$

Obviously the operator $\widetilde{N}$ having a fixed point is equivalent to $\widetilde{F} + \widetilde{G}$ having a fixed point. The proof that $\widetilde{F} + \widetilde{G}$ has a fixed point is done in several steps.

Step 1: $\widetilde{F}$ is continuous and compact. This can be shown as we did for F in Section 3.

Step 2: $\widetilde{G}$ is a contraction mapping. Let $z, \overline{z} \in B^0_{+\infty}$, then using (H1), (H3)–(H6) for each $t \in [0, n]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| \\
&\leq |h(t, z_t + x_t) - h(t, \overline{z}_t + x_t)| \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |A(s)[h(s, z_s + x_s) - h(s, \overline{z}_s + x_s)]|\, ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |g(s, z_s + x_s) - g(s, \overline{z}_s + x_s)|\, ds \\
&\leq \|A^{-1}(s)\| |A(t)h(t, z_t + x_t) - A(t)h(t, \overline{z}_t + x_t)| \\
&\quad + \int_0^t \widehat{M} |A(s)h(s, z_s + x_s) - A(s)h(s, \overline{z}_s + x_s)|\, ds \\
&\quad + \int_0^t \widehat{M} |g(s, z_s + x_s) - g(s, \overline{z}_s + x_s)|\, ds \\
&\leq \overline{M}_0 L_* \|z_t + x_t - \overline{z}_t - x_t\|_{\mathcal{B}} + \int_0^t \widehat{M} L_* \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}}\, ds \\
&\quad + \int_0^t \widehat{M} \eta(s) \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}}\, ds \\
&\leq \overline{M}_0 L_* \|z_t - \overline{z}_t\|_{\mathcal{B}} + \int_0^t \widehat{M} [L_* + \eta(s)] \|z_s - \overline{z}_s\|_{\mathcal{B}}\, ds.
\end{aligned}$$

Using (A1), we obtain

$$\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leq \overline{M}_0 L_* \big(K(t)|z(t) - \overline{z}(t)| + M(t)\|z_0 - \overline{z}_0\|_{\mathcal{B}}\big) \\
&\quad + \int_0^t \widehat{M}[L_* + \eta(s)] \big(K(s)|z(s) - \overline{z}(s)| + M(s)\|z_0 - \overline{z}_0\|_{\mathcal{B}}\big)\, ds \\
&\leq \overline{M}_0 L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \widehat{M} K_n [L_* + \eta(s)] |z(s) - \overline{z}(s)|\, ds.
\end{aligned}$$

Let $\bar{l}_n(t) = \widehat{M} K_n [L_* + \eta(t)]$ for the family seminorms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$. Then

$$\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leq \overline{M}_0 L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \bar{l}_n(s)] |z(s) - \overline{z}(s)|\, ds \\
&\leq \big[\overline{M}_0 L_* K_n e^{\tau L_n^*(t)}\big] \big[e^{-\tau L_n^*(t)} |z(t) - \overline{z}(t)|\big] \\
&\quad + \int_0^t \big[\bar{l}_n(s) e^{\tau L_n^*(s)}\big] \big[e^{-\tau L_n^*(s)} |z(s) - \overline{z}(s)|\big] ds \\
&\leq \overline{M}_0 L_* K_n e^{\tau L_n^*(t)} \|z - \overline{z}\|_n + \int_0^t \big[\frac{e^{\tau L_n^*(s)}}{\tau}\big]'\, ds \|z - \overline{z}\|_n \\
&\leq \overline{M}_0 L_* K_n e^{\tau L_n^*(t)} \|z - \overline{z}\|_n + \frac{1}{\tau} e^{\tau L_n^*(t)} \|z - \overline{z}\|_n \\
&\leq \big[\overline{M}_0 L_* K_n + \frac{1}{\tau}\big] e^{\tau L_n^*(t)} \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|\widetilde{G}(z) - \widetilde{G}(\overline{z})\|_n \leq [\overline{M}_0 L_* K_n + \frac{1}{\tau}] \|z - \overline{z}\|_n.$$

So, for $[\overline{M}_0 L_* K_n + \frac{1}{\tau}] < 1$, the operator $\widetilde{G}$ is a contraction for all $n \in \mathbb{N}$.
Step 3: The set

$$\widetilde{\mathcal{E}} = \big\{z \in B^0_{+\infty} : z = \lambda \widetilde{F}(z) + \lambda \widetilde{G}\big(\frac{z}{\lambda}\big) \quad \text{for some } 0 < \lambda < 1\big\}$$

is bounded. Let $z \in \widetilde{\mathcal{E}}$. Then, we have

$$\begin{aligned}
|z(t)| &\leq |h(t, z_t + x_t)| + \|U(t,0)\|_{B(E)} |h(0,\phi)| \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |A(s) h(s, z_s + x_s)|\, ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)|\, ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} |g(s, z_s + x_s) - g(s,0) + g(s,0)|\, ds.
\end{aligned}$$

By (A1) and (H1)–(H6), we have

$$\begin{aligned}
|z(t)| &\leq \|A^{-1}(s)\| |A(t) h(t, z_t + x_t)| + \widehat{M} \|A^{-1}(s)\| |A(t) h(0,\phi)| \\
&\quad + \widehat{M} \int_0^t |A(s) h(s, z_s + x_s)|\, ds + \widehat{M} \int_0^t f(s, z_s + x_s)\, ds \\
&\quad + \widehat{M} \int_0^t |g(s, z_s + x_s) - g(s,0)|\, ds + \widehat{M} \int_0^t |g(s,0)|\, ds \\
&\leq \overline{M}_0 L (\|z_t + x_t\|_{\mathcal{B}} + 1) + \widehat{M}\overline{M}_0 L (\|\phi\|_{\mathcal{B}} + 1) \\
&\quad + \widehat{M} L \int_0^t (\|z_s + x_s\|_{\mathcal{B}} + 1)\, ds + \widehat{M} \int_0^t p(s) \psi(\|z_s + x_s)\|_{\mathcal{B}})\, ds \\
&\quad + \widehat{M} \int_0^t \eta(s) \|z_s + x_s\|_{\mathcal{B}}\, ds + \widehat{M} \int_0^t |g(s,0)|\, ds \\
&\leq \overline{M}_0 L \|z_t + x_t\|_{\mathcal{B}} + \overline{M}_0 L + \widehat{M}\overline{M}_0 L + \widehat{M} L n + \widehat{M}\overline{M}_0 L \|\phi\|_{\mathcal{B}} \\
&\quad + \widehat{M} \int_0^t |g(s,0)|\, ds + \widehat{M} L \int_0^t \|z_s + x_s\|_{\mathcal{B}}\, ds \\
&\quad + \widehat{M} \int_0^t p(s) \psi(\|z_s + x_s)\|_{\mathcal{B}})\, ds + \widehat{M} \int_0^t \eta(s) \|z_s + x_s\|_{\mathcal{B}}\, ds.
\end{aligned}$$

Using $\|z_s + x_s\|_{\mathcal{B}} \leq K_n |z(s)| + c_n$ in (3.7) and the nondecreasing character of ψ, we get

$$\begin{aligned}
|z(t)| &\leq \overline{M}_0 L (K_n |z(t)| + c_n) + \overline{M}_0 L + \widehat{M}\overline{M}_0 L + \widehat{M} L n + \widehat{M}\overline{M}_0 L \|\phi\|_{\mathcal{B}} \\
&\quad + \widehat{M} \int_0^t |g(s,0)|\, ds + \widehat{M} L \int_0^t (K_n |z(s)| + c_n)\, ds \\
&\quad + \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n)\, ds + \widehat{M} \int_0^t \eta(s) (K_n |z(s)| + c_n)\, ds.
\end{aligned}$$

Then

$$\begin{aligned}(1-\overline{M}_0LK_n)|z(t)| &\le \overline{M}_0L(c_n+1+\widehat{M}[1+\|\phi\|_{\mathcal{B}}])+\widehat{M}Ln\\&\quad+\widehat{M}\int_0^t|g(s,0)|\,ds+\widehat{M}L\int_0^t(K_n|z(s)|+c_n)\,ds\\&\quad+\widehat{M}\int_0^t p(s)\psi(K_n|z(s)|+c_n)\,ds\\&\quad+\widehat{M}\int_0^t\eta(s)(K_n|z(s)|+c_n)\,ds.\end{aligned}$$

Set

$$\zeta_n:=\frac{K_n}{1-\overline{M}_0LK_n}\Big[\overline{M}_0L\big(1+\widehat{M}+c_n+\widehat{M}\|\phi\|_{\mathcal{B}}\big)+\widehat{M}Ln+\widehat{M}\int_0^t|g(s,0)|\,ds\Big]+c_n.$$

Thus

$$\begin{aligned}K_n|z(t)|+c_n &\le \zeta_n+\frac{\widehat{M}LK_n}{1-\overline{M}_0LK_n}\int_0^t(K_n|z(s)|+c_n)\,ds\\&\quad+\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}\int_0^t p(s)\psi(K_n|z(s)|+c_n)\,ds\\&\quad+\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}\int_0^t\eta(s)(K_n|z(s)|+c_n)\,ds.\end{aligned}$$

Consider the function μ defined by

$$\mu(t):=\sup\{K_n|z(s)|+c_n:0\le s\le t\},\quad 0\le t<+\infty.$$

Let $t^\star\in[0,t]$ be such that $\mu(t)=K_n|z(t^\star)|+c_n$, by the previous inequality, we have

$$\begin{aligned}\mu(t)&\le\zeta_n+\frac{\widehat{M}LK_n}{1-\overline{M}_0LK_n}\int_0^t\mu(s)\,ds+\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}\int_0^t\eta(s)\mu(s)\,ds\\&\quad+\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}\int_0^t p(s)\psi(\mu(s))\,ds\quad\text{for } t\in[0,n].\end{aligned}$$

Let us denote the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t)\le v(t)\quad\text{for all } t\in[0,n].$$

From the definition of v, we have $v(0)=\zeta_n$ and

$$v'(t)=\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}[L\mu(t)+\eta(t)\mu(t)+p(t)\psi(\mu(t))]\quad\text{a.e. } t\in[0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t)\le\frac{\widehat{M}K_n}{1-\overline{M}_0LK_n}[Lv(t)+\eta(t)v(t)+p(t)\psi(v(t))]\quad\text{a.e. } t\in[0,n].$$

This implies that for each $t \in [0, n]$ and using the condition (4.2), we get

$$\begin{aligned}\int_{\zeta_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq \frac{\widehat{M}K_n}{1-\overline{M}_0 L K_n}\int_0^t \max(L, \eta(s), p(s))ds \\ &\leq \frac{\widehat{M}K_n}{1-\overline{M}_0 L K_n}\int_0^n \max(L, \eta(s), p(s))ds \\ &< \int_{\zeta_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0, n]$, there exists a constant $\widetilde{N}_n$ such that $v(t) \leq \widetilde{N}_n$ and hence $\mu(t) \leq \widetilde{N}_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \widetilde{N}_n$. This shows that the set $\widetilde{\mathcal{E}}$ is bounded. Then the statement (S2) in Theorem 2.8 does not hold. The nonlinear alternative of Avramescu implies that (S1) holds, we deduce that the operator $\widetilde{F} + \widetilde{G}$ has a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator $\widetilde{N}$, which is the mild solution of (1.3)–(1.4). □

5. Applications

To illustrate the previous results, we give in this section two applications.

Example 5.1. Consider the model

$$\begin{gathered}\frac{\partial v}{\partial t}(t,\xi) = a(t,\xi)\frac{\partial^2 v}{\partial \xi^2}(t,\xi) + \int_{-\infty}^0 P(\theta) r(t, v(t+\theta, \xi))d\theta \\ + \int_{-\infty}^0 Q(\theta) s(t, v(t+\theta,\xi))d\theta, \quad t \in [0,+\infty),\ \xi \in [0,\pi] \\ v(t,0) = v(t,\pi) = 0 \quad t \in [0,+\infty) \\ v(\theta,\xi) = v_0(\theta,\xi) \quad -\infty < \theta \leq 0,\ \xi \in [0,\pi],\end{gathered} \tag{5.1}$$

where $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t; $P, Q : (-\infty, 0] \to \mathbb{R}$; $r, s : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ and $v_0 : (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi], \mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous, } w'' \in E,\ w(0) = w(\pi) = 0\}$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H1) (see [24]).

For the phase space $\mathcal{B}$, we choose the well known space $BUC(\mathbb{R}^-, E)$, the space of uniformly bounded continuous functions endowed with the norm

$$\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| \quad \text{for } \varphi \in \mathcal{B}.$$

If we put for $\varphi \in BUC(\mathbb{R}^-, E)$ and $\xi \in [0,\pi]$,

$$\begin{gathered}
y(t)(\xi) = v(t,\xi), \quad t \in [0,+\infty),\ \xi \in [0,\pi], \\
\phi(\theta)(\xi) = v_0(\theta,\xi), \quad -\infty < \theta \le 0,\ \xi \in [0,\pi], \\
f(t,\varphi)(\xi) = \int_{-\infty}^0 P(\theta) r(t, \varphi(\theta)(\xi)) d\theta, \quad -\infty < \theta \le 0,\ \xi \in [0,\pi], \\
g(t,\varphi)(\xi) = \int_{-\infty}^0 Q(\theta) s(t, \varphi(\theta)(\xi)) d\theta, \quad -\infty < \theta \le 0,\ \xi \in [0,\pi].
\end{gathered}$$

Then, (5.1) takes the abstract partial perturbed evolution form (1.1)–(1.2). To show the existence of mild solutions to (5.1), we assume the following hypotheses:

- The function s is Lipschitz continuous with respect to its second argument. Let $lip(s)$ denote the Lipschitz constant of s.
- There exist $p \in L^1([0,+\infty), \mathbb{R}^+)$ and a nondecreasing continuous function $\psi : [0,+\infty) \to (0,\infty)$ such that
$$|r(t,u)| \le p(t)\psi(|u|), \quad \text{for } t \in [0,+\infty),\ u \in \mathbb{R}.$$
- P and Q are integrable on $(-\infty, 0]$.

By the dominated convergence theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. Moreover the mapping g is Lipschitz continuous in its second argument, in fact, we have

$$|g(t,\varphi_1) - g(t,\varphi_2)| \le \mathrm{lip}(s) \int_{-\infty}^0 |Q(\theta)|\, d\theta |\varphi_1 - \varphi_2|, \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

On the other hand, for $\varphi \in \mathcal{B}$ and $\xi \in [0,\pi]$ we have

$$|f(t,\varphi)(\xi)| \le \int_{-\infty}^0 |p(t)P(\theta)| \psi(|(\varphi(\theta))(\xi)|) d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$|f(t,\varphi)| \le p(t) \int_{-\infty}^0 |P(\theta)| d\theta \psi(|\varphi|), \quad \text{for } \varphi \in \mathcal{B}.$$

Proposition 5.2. *Under the above assumptions, if we assume that condition* (3.2) *in Theorem 3.2 is true, $\varphi \in \mathcal{B}$, then the problem* (5.1) *has a mild solution which is defined in* $(-\infty, +\infty)$.

Example 5.3. Consider the model

$$\begin{aligned}
&\frac{\partial}{\partial t}\big[v(t,\xi) - \int_{-\infty}^0 T(\theta) u(t, v(t+\theta,\xi)) d\theta\big] \\
&= a(t,\xi) \frac{\partial^2 v}{\partial \xi^2}(t,\xi) + \int_{-\infty}^0 P(\theta) r(t, v(t+\theta,\xi)) d\theta \\
&\quad + \int_{-\infty}^0 Q(\theta) s(t, v(t+\theta,\xi)) d\theta \quad t \in [0,+\infty), \quad \xi \in [0,\pi] \\
&v(t,0) = v(t,\pi) = 0, \quad t \in [0,+\infty) \\
&v(\theta,\xi) = v_0(\theta,\xi), \quad -\infty < \theta \le 0,\ \xi \in [0,\pi],
\end{aligned} \tag{5.2}$$

where $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t; $T,P,Q:(-\infty,0]\to\mathbb{R}$; $u,r,s:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ and $v_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions.

Consider $E=L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w=a(t,\xi)w''$ with domain

$$D(A)=\{w\in E: w,\ w' \text{ are absolutely continuous },w''\in E,\ w(0)=w(\pi)=0\}$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumptions (H1) and (H4) (see [24]).

For the phase space $\mathcal{B}$, we choose the well known space $BUC(\mathbb{R}^-,E):$ the space of uniformly bounded continuous functions endowed with the norm

$$\|\varphi\|=\sup_{\theta\le 0}|\varphi(\theta)|\quad \text{for } \varphi\in\mathcal{B}.$$

If we put for $\varphi\in BUC(\mathbb{R}^-,E)$ and $\xi\in[0,\pi]$,

$$\begin{gathered}
y(t)(\xi)=v(t,\xi),\quad t\in[0,+\infty),\ \xi\in[0,\pi],\\
\phi(\theta)(\xi)=v_0(\theta,\xi),\quad -\infty<\theta\le 0,\ \xi\in[0,\pi],\\
h(t,\varphi)(\xi)=\int_{-\infty}^0 T(\theta)u(t,\varphi(\theta)(\xi))d\theta,\quad -\infty<\theta\le 0,\ \xi\in[0,\pi],\\
f(t,\varphi)(\xi)=\int_{-\infty}^0 P(\theta)r(t,\varphi(\theta)(\xi))d\theta,\quad -\infty<\theta\le 0,\ \xi\in[0,\pi]\\
g(t,\varphi)(\xi)=\int_{-\infty}^0 Q(\theta)s(t,\varphi(\theta)(\xi))d\theta,\quad -\infty<\theta\le 0,\xi\in[0,\pi].
\end{gathered}$$

Then, (5.2) takes the abstract neutral perturbed evolution form (1.3)–(1.4). To show the existence of the mild solution to (5.2), we assume the following hypotheses:

- the functions u and s are Lipschitz with respect to its second argument, and constants $\operatorname{lip}(u)$ and $\operatorname{lip}(s)$ respectively.
- There exist $p\in L^1([0,+\infty),\mathbb{R}^+)$ and a nondecreasing continuous function $\psi:[0,+\infty)\to(0,\infty)$ such that
$$|r(t,u)|\le p(t)\psi(|u|),\quad \text{for } t\in[0,+\infty),\ u\in\mathbb{R}.$$
- T, P and Q are integrable on $(-\infty,0]$.

By the dominated convergence theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. Moreover the mapping h and g are Lipschitz continuous in its second argument, in fact, we have

$$|g(t,\varphi_1)-g(t,\varphi_2)|\le \operatorname{lip}(s)\int_{-\infty}^0|Q(\theta)|d\theta|\varphi_1-\varphi_2|,\qquad \text{for } \varphi_1,\varphi_2\in\mathcal{B},$$

$$|h(t,\varphi_1)-h(t,\varphi_2)|\le \overline{M}_0L_*\operatorname{lip}(u)\int_{-\infty}^0|T(\theta)|d\theta|\varphi_1-\varphi_2|,\quad \text{for } \varphi_1,\varphi_2\in\mathcal{B}.$$

On the other hand, for $\varphi\in\mathcal{B}$ and $\xi\in[0,\pi]$ we have

$$|f(t,\varphi)(\xi)|\le\int_{-\infty}^0|p(t)P(\theta)|\psi(|(\varphi(\theta))(\xi)|)d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$|f(t,\varphi)|\le p(t)\int_{-\infty}^0|P(\theta)|d\theta\psi(|\varphi|),\quad \text{for } \varphi\in\mathcal{B}.$$

Proposition 5.4. *Under the above assumptions, if we assume that condition (4.2) in Theorem 4.2 is true, $\varphi \in \mathcal{B}$, then* (5.2) *has a mild solution which is defined in* $(-\infty, +\infty)$.

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