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EXISTENCE RESULTS FOR STRONGLY INDEFINITE ELLIPTIC SYSTEMS

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ABSTRACT. In this paper, we show the existence of solutions for the strongly indefinite elliptic system

$$\begin{gathered} -\Delta u = \lambda u + f(x,v) \quad \text{in } \Omega, \\ -\Delta v = \lambda v + g(x,u) \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary, $\lambda_{k_0} < \lambda < \lambda_{k_0+1}$, where λ_k is the kth eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition. Both cases when f, g being superlinear and asymptotically linear at infinity are considered.

1. INTRODUCTION

In this paper, we investigate the existence of solutions for the strongly indefinite elliptic system

$$\begin{gathered} -\Delta u = \lambda u + f(x,v) \quad \text{in } \Omega, \\ -\Delta v = \lambda v + g(x,u) \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\lambda_{k_0} < \lambda < \lambda_{k_0+1}$, where λ_k is the kth eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition.

Problem (1.1) with $\lambda = 0$ was considered in [5, 6], where the existence results for superlinear nonlinearities were established by finding critical points of the functional

$$J(u,v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega F(x,v)\, dx - \int_\Omega G(x,u)\, dx. \tag{1.2}$$

A typical feature of the functional J is that the quadratic part

$$Q(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

is positive definite in an infinite dimensional subspace $E^+ = \{(u,u) : u \in H_0^1(\Omega)\}$ of $H_0^1(\Omega) \times H_0^1(\Omega)$ and negative definite in its infinite dimensional complimentary

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subspace $E^- = \{(u, -u) : u \in H_0^1(\Omega)\}$, that is, J is strongly indefinite. A linking theorem is then used in finding critical points of J.

In the case that λ lies in between higher eigenvalues, the parameter λ affects the definiteness of the corresponding quadratic part

$$Q_\lambda(u, v) = \int_\Omega (\nabla u \nabla v - \lambda uv)\, dx$$

of the associated functional

$$J_\lambda(u, v) = \int_\Omega (\nabla u \nabla v - \lambda uv)\, dx - \int_\Omega F(x, v)\, dx - \int_\Omega G(x, u)\, dx, \tag{1.3}$$

of (1.1) defined on $H_0^1(\Omega) \times H_0^1(\Omega)$. A key ingredient in use of the linking theorem is to find a proper decomposition of $H_0^1(\Omega) \times H_0^1(\Omega)$ into a direct sum of two subspaces so that Q_λ is definite in each subspace. Obviously, Q_λ is neither positive definite in E^+ nor negative definite in E^-. So we need to find out a suitable decomposition of $H_0^1(\Omega) \times H_0^1(\Omega)$.

We first consider the asymptotically linear case. Such a problem has been extensively studied for one equation, see for instance, [4, 10, 11] and references therein. For asymptotically linear elliptic system, we refer readers to [8]. Particularly, in this case, the Ambrosetti-Rabinowtz condition is not satisfied, whence it is hard to show a Palais-Smale sequence is bounded. So one turns to using Cerami condition in critical point theory instead of the Palais-Smale condition, various existence results for asymptotically linear problems are then obtained. By a functional I defined on E satisfies Cerami condition we mean that for any sequence $\{u_n\} \subset E$ such that $|I(u_n)| \leq C$ and $(1 + \|u_n\|)I'(u_n) \to 0$, there is a convergent subsequence of $\{u_n\}$. For the asymptotically linear system (1.1), it is strongly indefinite and the nonlinearities do not fulfill the Ambrosetti-Rabinowitz condition. To handle the problem, we assume:

- (A1) $f, g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x, v) = o(|v|), g(x, u) = o(|u|)$ uniformly for $x \in \Omega$ as $|u|, |v| \to 0$ and $tf(x, t) \geq 0, tg(x, t) \geq 0$.
- (A2) There exist positive constants l, m, such that $\lim_{t\to\pm\infty} \frac{f(x,t)}{t} = l$ and $\lim_{t\to\pm\infty} \frac{g(x,t)}{t} = m$.
- (A3) $\lambda \pm \sqrt{ml} \neq \lambda_k$ for any $k \in \mathbb{N}$.
- (A4) There exists $u_0 \in \operatorname{span}\{\varphi_{k_0+1}, \varphi_{k_0+2}, \dots\}$ with $\int_\Omega |\nabla u_0|^2 - \lambda(u_0)^2\, dx = \frac{1}{2}$ such that
$$\int_\Omega (|\nabla u_0|^2 - \lambda u_0^2)\, dx - \min(l, m) \int_\Omega u_0^2\, dx < 0.$$

Theorem 1.1. *Suppose* (A1)-(A4), *problem* (1.1) *has at least a nontrivial solution.*

Condition (A4) holds, for example, if $\min(l, m) > \lambda_{k_0+1} - \lambda$, we choose $u_0 = \alpha\varphi_{k+1}$ for some $\alpha > 0$, then $\int_\Omega |\nabla u_0|^2 - \lambda u_0^2\, dx - \min(l, m) \int_\Omega u_0^2\, dx = (\lambda_{k_0+1} - \lambda - \min(l, m)) \int_\Omega u_0^2\, dx < 0$.

Theorem 1.1 is proved by the following linking theorem with Cerami condition in [3], which is a generalization of usual one in [2], [9].

Lemma 1.2. *Let E be a real Hilbert space with $E = E_1 \oplus E_2$. Suppose $I \in C^1(E, \mathbb{R})$, satisfies Cerami condition, and*

- (I1) *$I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : E_i \to E_i$ is bounded and selfadjoint, i=1,2.*

(I2) *b' is compact.*
(I3) *There exists a subspace $\tilde{E} \subset E$ and sets $S \subset E, Q \subset \tilde{E}$ and constants $\alpha > \omega$ such that*
(i) *$S \subset E_1$ and $I|_S \geq \alpha$,*
(ii) *Q is bounded and $I|_{\partial Q} \leq \omega$,*
(iii) *S and Q link.*
Then I possesses a critical value $c \geq \alpha$.

Next, we consider superlinear case. We assume that

(B1) $f, g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,v) = o(|v|), g(x,u) = o(|u|)$ uniformly for $x \in \Omega$ as $|u|, |v| \to 0$.
(B2) There exists a constant $\gamma > 2$ such that
$$0 < \gamma F(x,v) \leq v f(x,v), \quad 0 < \gamma G(x,u) \leq u g(x,u),$$
where $F(x,v) = \int_0^v f(x,s)\,ds$ and $G(x,u) = \int_0^u g(x,u)\,ds$.
(B3) There exist $p, q > 1, \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$, constants $a_1, a_2 > 0$, such that $|f(x,v)| \leq a_1 + a_2|v|^q$, $|g(x,u)| \leq a_1 + a_2|u|^p$.

Theorem 1.3. *Assume* (B1)-(B3)*, then* (1.1) *has at least one solution.*

We remark that in [6], it also considered the subcritical superlinear problem
$$\begin{gathered} -\Delta u = \lambda v + f(v) \quad \text{in } \Omega, \\ -\Delta v = \mu u + g(u) \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$
The functional corresponding to (1.4) is no longer positive definite in E^+, but it is negative definite in E^-. It is different from our case.

In section 2, we prove Theorem 1.1. While Theorem 1.3 is showed in section 3.

2. Asymptotically linear case

Let $H := H_0^1(\Omega)$, it can be decomposed as $H = H^1 \oplus H^2$, where $H^1 = \operatorname{span}\{\varphi_{k_0+1}, \varphi_{k_0+2} \dots\}$, $H^2 = \operatorname{span}\{\varphi_1, \varphi_2 \dots \varphi_{k_0}\}$ and φ_k is the eigenfunction related to λ_k. Let P_i be the projection of H on the subspace $H^i, i = 1, 2$, then we define for $u \in H$ a new norm by
$$\|u\|^2 = \int_\Omega |\nabla (P_1 u)|^2 - \lambda (P_1 u)^2\,dx - \int_\Omega |\nabla (P_2 u)|^2 - \lambda (P_2 u)^2\,dx,$$
it is equivalent to the usual norm of $H_0^1(\Omega)$. To find out the subspaces of $H \times H$ such that the quadratic part
$$Q_\lambda(u,v) = \int_\Omega (\nabla u \nabla v - \lambda uv)\,dx$$
of the functional
$$J_\lambda(u,v) = \int_\Omega (\nabla u \nabla v - \lambda uv)\,dx - \int_\Omega F(x,v)\,dx - \int_\Omega G(x,u)\,dx$$
is positive or negative definite on it, we denote
$$\begin{gathered} E_{11} = \{(u,u) : u \in H^1\}, \quad E_{12} = \{(u,-u) : u \in H^1\}, \\ E_{21} = \{(u,u) : u \in H^2\}, \quad E_{22} = \{(u,-u) : u \in H^2\}. \end{gathered}$$

Therefore, $H \times H = E_{11} \oplus E_{12} \oplus E_{21} \oplus E_{22}$. We may write for any $(u,v) \in H \times H$ that

$$(u,v) = (u_{11}, u_{11}) + (u_{12}, -u_{12}) + (u_{21}, u_{21}) + (u_{22}, -u_{22}), \tag{2.1}$$

where

$$u_{11} = P_1(\frac{u+v}{2}) \in H^1, \quad u_{21} = P_2(\frac{u+v}{2}) \in H^2,$$
$$u_{12} = P_1(\frac{u-v}{2}) \in H^1, \quad u_{22} = P_2(\frac{u-v}{2}) \in H^2.$$

It is easy to check that Q_λ is positive definite in $E_{11} \oplus E_{22}$ and negative definite in $E_{12} \oplus E_{21}$, so we denote $E_+ = E_{11} \oplus E_{22}$ and $E_- = E_{12} \oplus E_{21}$ for convenience.

Then

$$J_\lambda(u,v) = \|u_{11}\|^2 + \|u_{22}\|^2 - \|u_{12}\|^2 - \|u_{21}\|^2 - \int_\Omega F(x,v)\,dx - \int_\Omega G(x,u)\,dx, \tag{2.2}$$

it is C^1 on $H \times H$.

Lemma 2.1. *The functional J_λ satisfies the Cerami condition.*

Proof. It is sufficient to show that any Cerami sequence is bounded, a standard argument then implies that the sequence has a convergent subsequence. We argue indirectly. Suppose it were not true, there would exist a Cerami sequence $z_n = \{(u_n, v_n)\} \subset H \times H$ of J_λ such that $\|z_n\| \to \infty$. Let

$$w_n = \frac{z_n}{\|z_n\|} = (\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|}) = (w_n^1, w_n^2),$$

we may assume that

$$(w_n^1, w_n^2) \rightharpoonup (w^1, w^2) \quad \text{in } H \times H, \quad (w_n^1, w_n^2) \to (w^1, w^2) \quad \text{in } L^2(\Omega) \times L^2(\Omega),$$
$$w_n^1 \to w^1, w_n^2 \to w^2 \quad \text{a.e. in } \Omega.$$

We write as the decomposition (2.1) that $u_n = \sum_{i,j=1}^2 u_{ij}^n$ and correspondingly, $w_n^1 = \sum_{i,j=1}^2 w_{ij}^n$. We claim that $(w^1, w^2) \neq (0,0)$. Otherwise, there would hold

$$|\langle J_\lambda'(u_n,v_n), (u_{11}^n, u_{11}^n)\rangle| \le \|J_\lambda'(u_n,v_n)\| \cdot \|(u_{11}^n, u_{11}^n)\| \le \|J_\lambda'(u_n,v_n)\| \cdot \|(u_n,v_n)\| \to 0; \tag{2.3}$$

that is,

$$\|u_{11}^n\|^2 - \int_\Omega f(x,v_n) u_{11}^n\,dx - \int_\Omega g(x,u_n) u_{11}^n\,dx \to 0 \tag{2.4}$$

implying

$$\|w_{11}^n\|^2 - \int_\Omega \frac{f(x,v_n)}{v_n} \frac{v_n}{\|z_n\|} \frac{u_{11}^n}{\|z_n\|}\,dx - \int_\Omega \frac{g(x,u_n)}{u_n} \frac{u_n}{\|z_n\|} \frac{u_{11}^n}{\|z_n\|}\,dx \to 0. \tag{2.5}$$

Therefore,

$$\|w_{11}^n\|^2 \le C \int_\Omega [(w_n^1)^2 + (w_n^2)^2]\,dx + o(1), \tag{2.6}$$

which yields $\|w_{11}^n\| \to 0$. Similarly, $\|w_{12}^n\| \to 0$, $\|w_{21}^n\| \to 0$ and $\|w_{22}^n\| \to 0$ as $n \to \infty$. Consequently, $w_n \to 0$. This contradicts to $\|w_n\| = 1$. Hence, there are three possibilities: (i) $w^1 \neq 0, w^2 \neq 0$; (ii) $w^1 \neq 0, w^2 = 0$; (iii) $w^1 = 0, w^2 \neq 0$. We show next that all these cases will lead to a contradiction. Hence, $\|z_n\|$ is bounded.

In case (i), we claim that (w^1, w^2) satisfies

$$
\begin{gathered}
-\Delta w^1 = \lambda w^1 + l w^2, \quad \text{in } \Omega, \\
-\Delta w^2 = \lambda w^2 + m w^1, \quad \text{in } \Omega, \\
w^1 = w^2 = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.7}
$$

Indeed, let

$$
p_n(x) = \begin{cases} \frac{f(x,v_n(x))}{v_n(x)} & \text{if } v_n(x) \neq 0, \\ 0 & \text{if } v_n(x) = 0, \end{cases} \tag{2.8}
$$

and

$$
q_n(x) = \begin{cases} \frac{g(x,u_n(x))}{u_n(x)} & \text{if } u_n(x) \neq 0, \\ 0 & \text{if } u_n(x) = 0. \end{cases} \tag{2.9}
$$

Since $0 \leq p_n, q_n \leq M$ for some $M > 0$, we may suppose that $p_n \rightharpoonup \varphi$, $q_n \rightharpoonup \psi$ in $L^2(\Omega)$ and $p_n \to \varphi$, $q_n \to \psi$ *a.e* in Ω. The fact $w^1(x) \neq 0$ implies $u_n(x) \to \infty$ and consequently, $q_n(x) \to m$. Similarly, $w^2(x) \neq 0$ yields $v_n(x) \to \infty$ and $p_n(x) \to l$. Hence, $\varphi(x) = l$ if $w^2(x) \neq 0$ and $\psi(x) = m$ if $w^1(x) \neq 0$.

Since $J'_\lambda(u_n, v_n) \to 0$, for any $(\eta_1, \eta_2) \in H \times H$, we have

$$
\int_\Omega \nabla v_n \nabla \eta_1 - \lambda v_n \eta_1 \, dx - \int_\Omega g(x, u_n)\eta_1 \, dx \to 0, \tag{2.10}
$$

$$
\int_\Omega \nabla u_n \nabla \eta_2 - \lambda u_n \eta_2 \, dx - \int_\Omega f(x, v_n)\eta_2 \, dx \to 0. \tag{2.11}
$$

It follows from $\|z_n\| \to \infty$ that

$$
\int_\Omega \nabla w_n^1 \nabla \eta_2 - \lambda w_n^1 \eta_2 \, dx - \int_\Omega p_n(x) w_n^2 \eta_2 \, dx \to 0, \tag{2.12}
$$

$$
\int_\Omega \nabla w_n^2 \nabla \eta_1 - \lambda w_n^2 \eta_1 \, dx - \int_\Omega q_n(x) w_n^1 \eta_1 \, dx \to 0. \tag{2.13}
$$

Noting $p_n w_n^2, q_n w_n^1$ are bounded in $L^2(\Omega)$, we may assume $p_n w_n^2 \rightharpoonup \xi(x), q_n w_n^1 \rightharpoonup \zeta(x)$ in $L^2(\Omega)$ and $p_n w_n^2 \to \xi(x)$, $q_n w_n^1 \to \zeta(x)$ a.e. in Ω. We deduce from the fact $w_n^2 \to w^2$, $w_n^1 \to w^1$, $p_n \to \varphi$ and $q_n \to \psi$ a.e. in Ω that $\xi = \varphi w^2 = l w^2$ and $\zeta = \psi w^1 = m w^1$. Let $n \to \infty$ in (2.12) and (2.13) we see that (w^1, w^2) solves (2.7).

Let $\tilde{w}^2 = \sqrt{\frac{l}{m}} w^2$, then $(w^1, \tilde{w}^2)$ solves

$$
\begin{gathered}
-\Delta w^1 = \lambda w^1 + \sqrt{ml} w^2 \quad \text{in } \Omega, \\
-\Delta \tilde{w}^2 = \lambda \tilde{w}^2 + \sqrt{ml} w^1 \quad \text{in } \Omega, \\
w^1 = \tilde{w}^2 = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.14}
$$

which implies

$$
\begin{gathered}
-\Delta(w^1 + \tilde{w}^2) = (\lambda + \sqrt{ml})(w^1 + \tilde{w}^2) \quad \text{in } \Omega, \\
w^1 + \tilde{w}^2 = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.15}
$$

If $w^1 + \tilde{w}^2 \neq 0$, this contradicts to (A3). If $w^1 + \tilde{w}^2 = 0$, then

$$
\begin{gathered}
-\Delta w^1 = (\lambda - \sqrt{ml}) w^1 \quad \text{in } \Omega, \\
w^1 = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.16}
$$

This again contradicts to (A3).

For case (ii), we derive from (2.12) that $\int_\Omega p_n(x)w_n^2\eta_2\,dx \to 0$ and then w^1 solves

$$\begin{aligned} -\Delta w^1 &= \lambda w^1 \quad \text{in } \Omega,\\ w^1 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.17}$$

which is a contradiction to the assumption that $\lambda_{k_0} < \lambda < \lambda_{k_0+1}$. Similarly, we may rule out case (iii). The proof is complete. □

Next, we show that J_λ has the linking structure. Denote $z_0 = (u_0, u_0)$, where u_0 is given by assumption (A_4), then $\|z_0\|^2 = 1$. Let $[0, s_1 z_0] = \{sz_0 : 0 \le s \le s_1\}$, $M_R = \{z = z^- + \rho z_0 : \|z\| \le R, \rho \ge 0\}$, $\tilde{H} = \text{span}\{z_0\} \oplus E_-$, $S = \partial B_\rho \cap E_+$.

Lemma 2.2. *There exist constants $\alpha > 0$ and $\rho > 0$, such that $J_\lambda(u, v) \ge \alpha$ for $(u, v) \in S$.*

Proof. By (A1) and (A2), for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|F(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p, \quad |G(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p$$

for some $2 < p < \frac{2N}{N-2}$. It implies that for $(u, v) \in S$,

$$J_\lambda(u, v) \ge (\frac{1}{2} - \varepsilon)\|z^+\|^2 - C_\varepsilon\|z^+\|^p. \tag{2.18}$$

The assertion follows. □

Lemma 2.3. *There exists $R > \rho$ such that $J_\lambda(u, v) \le 0$ for $(u, v) \in \partial M_R$.*

Proof. For $z \in \partial M_R$, we write $z = z^- + rz_0$ with $\|z\| = R$, $r > 0$ or $\|z\| < R$ and $r = 0$. If $r = 0$, we have $z = z^-$ and

$$J_\lambda(u, v) = -\frac{1}{2}\|z^-\|^2 - \int_\Omega [F(x, v) + G(x, u)]\,dx \le 0 \tag{2.19}$$

since $F(x, t), G(x, t) \ge 0$.

Suppose now that $r > 0$. We argue by contradiction. Suppose the assertion is not true, we would have a sequence $\{z_n\} \in \partial M_R$, $z_n = \rho_n z_0 + z_n^-$, $\rho_n > 0$, $\|z_n\| = n$ such that $J_\lambda(z_n) > 0$. We write $z_n = (u_n, v_n) = (\rho_n u_0 + \phi_n, \rho_n u_0 + \psi_n)$, then

$$J_\lambda(z_n) = \frac{1}{2}\rho_n^2 - \frac{1}{2}\|z_n^-\|^2 - \int_\Omega F(x, v_n) + G(x, u_n)\,dx > 0, \tag{2.20}$$

that is

$$\frac{J_\lambda(z_n)}{\|z_n\|^2} = \frac{1}{2}(\frac{\rho_n^2}{\|z_n\|^2} - \frac{\|z_n^-\|^2}{\|z_n\|^2}) - \int_\Omega \frac{F(x, v_n) + G(x, u_n)}{\|z_n\|^2}\,dx > 0. \tag{2.21}$$

Since $F, G \ge 0$, then we have $\rho_n \ge \|z_n^-\|$. The fact $\frac{\rho_n^2 + \|z_n^-\|^2}{\|z_n\|^2} = 1$ implies $\frac{1}{2} \le \frac{\rho_n^2}{\|z_n\|^2} \le 1$. Assume $\frac{\rho_n^2}{\|z_n\|^2} \to \rho_0^2 > 0$, hence $\rho_n \to +\infty$. We may also assume $\frac{\phi_n}{\|z_n\|} \rightharpoonup \xi_1$, $\frac{\psi_n}{\|z_n\|} \rightharpoonup \xi_2$ in H and $\frac{\phi_n}{\|z_n\|} \to \xi_1$, $\frac{\psi_n}{\|z_n\|} \to \xi_2$ a.e. in Ω. If $x \in \Omega$ such that $\rho_0 u_0(x) + \xi_1(x) \ne 0$, then $u_n(x) = \rho_n u_0(x) + \phi_n(x) \to \infty$. Similarly, if $x \in \Omega$ such that $\rho_0 u_0(x) + \xi_2(x) \ne 0$, we have $v_n(x) = \rho_n u_0(x) + \psi_n(x) \to \infty$. It follows from

(2.21) that

$$\begin{aligned} 0 &< \frac{1}{2}\frac{\rho_n^2}{\|z_n\|^2} - \frac{1}{2}\frac{\|z_n^-\|^2}{\|z_n\|^2} - \int_\Omega [\frac{F(x,v_n)}{v_n^2}(\frac{v_n}{\|z_n\|})^2 + \frac{G(x,u_n)}{u_n^2}(\frac{u_n}{\|z_n\|})^2]\,dx \\ &\le \frac{1}{2}\frac{\rho_n^2}{\|z_n\|^2} - \frac{1}{2}\frac{\|z_n^-\|^2}{\|z_n\|^2} - \int_{\{\rho_0 u_0+\xi_2\neq 0\}} \frac{F(x,v_n)}{v_n^2}(\frac{v_n}{\|z_n\|})^2\,dx \\ &+ \int_{\{\rho_0 u_0+\xi_1\neq 0\}} \frac{G(x,u_n)}{u_n^2}(\frac{u_n}{\|z_n\|})^2\,dx \end{aligned} \tag{2.22}$$

Let $z = \rho_0 z_0 + \xi^-$ with $\xi^- = (\xi_1, \xi_2)$ and take limit in (2.22), we get

$$\begin{aligned} &\frac{1}{2}(\rho_0^2\|z_0\|^2 - \|\xi^-\|^2) - \frac{l}{2}\int_{\{\rho_0 u_0+\xi_2\neq 0\}} (\rho_0 u_0 + \xi_2)^2\,dx \\ &- \frac{m}{2}\int_{\{\rho_0 u_0+\xi_1\neq 0\}} (\rho_0 u_0 + \xi_1)^2\,dx \ge 0. \end{aligned} \tag{2.23}$$

There are two cases: either $\xi^- = (\xi_1, \xi_2) \in E_{12}$, that is, $\xi_1 = -\xi_2 \in H^1$ or $\xi^- = (\xi_1, \xi_2) \in E_{21}$, that is, $\xi_1 = \xi_2 \in H^2$. In both cases we have $\int_\Omega (u_0\xi_1 + u_0\xi_2)\,dx = 0$. By (2.23), we obtain

$$\begin{aligned} 0 &\le \frac{1}{2}(\rho_0^2\|z_0\|^2 - \|\xi^-\|^2) - \min(l,m)\int_\Omega (\rho_0^2 u_0^2 + \xi_1^2)\,dx \\ &\le \rho_0^2(\int_\Omega |\nabla u_0|^2 - \lambda u_0^2\,dx - \min(l,m)\int_\Omega u_0^2\,dx) - \frac{1}{2}\|\xi^-\|^2 - \min(l,m)\int_\Omega \xi_1^2\,dx \\ &< 0, \end{aligned} \tag{2.24}$$

a contradiction. □

Proof of Theorem 1.1. Let $L(u, v) = (v, u)$, we may check that L is a bounded selfadjoint operator on $H \times H$ and that $E_{11}, E_{12}, E_{21}.E_{22}$ are invariant subspace of L, so both E_+ and E_- are invariant subspace of L. (I1) of Lemma 1.2 then holds. (I2) follows from the Sobolev compact imbeddings; (i) and (ii) in (I3) are consequences of Lemma 2.2 and Lemma 2.3. The proof of (iii) in (I3) can be found in [2] and [9]. The proof of Theorem 1.1 is complete. □

3. Superlinear case

Let $\phi_1, \phi_2, \phi_3, \dots$ be the eigenfunctions of $-\Delta$ in Ω with Dirichlet boundary condition, which consist of the orthogonal basis of $L^2(\Omega)$. We assume that the eigenfunctions are normalized in $L^2(\Omega)$; i.e, $\int_\Omega \phi_i\phi_j\,dx = \delta_{ij}$. Thus,

$$L^2(\Omega) = \{u = \sum_{k=1}^\infty \xi_k\phi_k : \sum_{k=1}^\infty \xi_k^2 < \infty\},$$

and

$$(u, v)_{L^2} = \sum_{k=1}^\infty \xi_k\eta_k,$$

with $u = \sum_{k=1}^\infty \xi_k\phi_k$, $v = \sum_{k=1}^\infty \eta_k\phi_k$. For $u \in L^2(\Omega)$, we define operator $(-\Delta)^{r/2}$ by

$$(-\Delta)^{r/2}u = \sum_{k=1}^\infty \lambda_k^{r/2}\xi_k\phi_k$$

with domain

$$D((-\Delta)^{r/2}) = \Theta^r(\Omega) = \big\{\sum_{k=1}^{\infty}\xi_k\phi_k : \sum_{k=1}^{\infty}\lambda_k^r\xi_k^2 < \infty\big\}$$

for $r \geq 0$. It is proved in [7] that $\Theta^r(\Omega) = H_0^r(\Omega) = H^r(\Omega)$ if $0 < r < \frac{1}{2}$, $\Theta^{1/2}(\Omega) = H_{00}^{1/2}(\Omega)$, $\Theta^r(\Omega) = H_0^r(\Omega)$ if $\frac{1}{2} < r \leq 1$, and $\Theta^r(\Omega) = H^r(\Omega) \cap H_0^1(\Omega)$ if $1 < r \leq 2$. For $r \geq 0$, $\Theta^r(\Omega)$ is a Hilbert space with inner product

$$(u, v)_{\Theta^r(\Omega)} = (u, v)_{L^2} + ((-\Delta)^{r/2}u, (-\Delta)^{r/2}v)_{L^2}.$$

Let

$$E^r(\Omega) = \Theta^r(\Omega) \times \Theta^{2-r}(\Omega), \quad 0 < r < 2,$$

we choose $r > 0$ such that $2 < p + 1 \leq \frac{2N}{N-2r}$ and $2 < q + 1 \leq \frac{2N}{N+2r-4}$. By the Sobolev embedding, the inclusion $E^r(\Omega) \hookrightarrow L^{p+1}(\Omega) \times L^{q+1}(\Omega)$ is compact.

The quadratic form $Q_\lambda(u, v) = \int_\Omega(\nabla u\nabla v - \lambda uv)\,dx$ can be extended to $E^r(\Omega)$ since

$$\int_\Omega \nabla u\nabla v\,dx = \sum_{k=1}^{\infty}\lambda_k\xi_k\eta_k = \sum_{k=1}^{\infty}\lambda_k^{\frac{r}{2}}\xi_k\lambda_k^{1-\frac{r}{2}}\eta_k,$$

it implies

$$\Big|\int_\Omega \nabla u\nabla v\,dx\Big| \leq \{\sum_{k=1}^{\infty}\lambda_k^r\xi_k^2\}^{1/2}\{\sum_{k=1}^{\infty}\lambda_k^{2-r}\eta_k^2\}^{1/2} = \|u\|_{\Theta^r}\|v\|_{\Theta^{2-r}}.$$

A direct calculation shows that for $z \in E^r(\Omega)$,

$$Q_\lambda(z) = \frac{1}{2}(Lz, z)_{E^r},$$

where

$$L = \begin{pmatrix} 0 & (-\Delta)^{1-r} - \lambda(-\Delta)^{-r} \\ (-\Delta)^{r-1} - \lambda(-\Delta)^{r-2} & 0 \end{pmatrix}, \tag{3.1}$$

which is a bounded and self-adjoint operator in $E^r(\Omega)$. In order to determine the spectrum of L, we note that $E^r(\Omega)$ is the direct sum of the spaces $E_k, k = 1, 2, \dots$, where E_k is the two-dimensional subspace of $E^r(\Omega)$, spanned by $(\phi_k, 0)$ and $(0, \phi_k)$. An orthonormal basis of E_k is given by

$$\big\{\frac{1}{\sqrt{2}}(\lambda_k^{-\frac{r}{2}}\phi_k, 0), \frac{1}{\sqrt{2}}(0, \lambda_k^{\frac{r}{2}-1}\phi_k)\big\}.$$

Every E_k is invariant under L, and the restriction of L on E_k is given by the matrix

$$L^k = \begin{pmatrix} 0 & \lambda_k^{1-r} - \lambda\lambda_k^{-r} \\ \lambda_k^{r-1} - \lambda\lambda_k^{r-2} & 0 \end{pmatrix}.$$

The eigenvalue of L^k is $\mu_k^{\pm} = \pm(1 - \lambda\lambda_k^{-1})$. Therefore, $\mu_k^+ < 0$ and $\mu_k^- > 0$ if $k = 1, \dots, k_0$; while $\mu_k^+ > 0$ and $\mu_k^- < 0$ if $k = k_0 + 1, \dots$. Furthermore,

$$\mu_k^{\pm} \to \pm 1 \quad \text{as } k \to \infty.$$

Let $H^+(H^-)$ be the subspace spanned by eigenvectors corresponding to positive (negative) eigenvalues of L_k, then

$$E^r(\Omega) = H^+ \oplus H^-.$$

Both H^+ and H^- are infinite dimensional. Now we introduce an equivalent norm $\|\cdot\|_*$ on $E^r(\Omega)$ by

$$\frac{1}{2}\|z\|_*^2 = (Lz^+, z^+) - (Lz^-, z^-),$$

where $z^\pm \in H^\pm$. Then the functional corresponding to (1.1) is

$$I(z) = \frac{1}{2}(Lz, z)_{E^r(\Omega)} - \Gamma(z)$$

for $z = (u, v) \in E^r(\Omega)$, where

$$\Gamma(z) = \int_\Omega F(x, v)\,dx + \int_\Omega G(x, u)\,dx.$$

Lemma 3.1. *The functional I satisfies the (PS) condition.*

Proof. Let $\{z_n\}$ be a (PS) sequence of I in $E^r(\Omega)$, we need only to show that $\{z_n\}$ is bounded. Since

$$\begin{aligned} M + \varepsilon\|z_n\| &\geq I(z_n) - \frac{1}{2}\langle I'(z_n), z_n\rangle \\ &\geq (\frac{1}{2} - \frac{1}{\gamma})(\int_\Omega |u_n||g(x, u_n)|\,dx + \int_\Omega |v_n||f(x, v_n)|\,dx) - C, \end{aligned} \tag{3.2}$$

we have

$$\int_\Omega |u_n||g(x, u_n)|\,dx + \int_\Omega |v_n||f(x, v_n)|\,dx \leq C + \varepsilon\|z_n\|. \tag{3.3}$$

We write $z_n^\pm = (u_n^\pm, v_n^\pm)$, then

$$\begin{aligned} \|z_n^\pm\|^2 - \varepsilon\|z_n^\pm\| &\leq |\langle Lz_n, z_n^\pm\rangle - I'(z_n)z^\pm| \\ &= |\langle \Gamma'(z_n), z_n^\pm\rangle| \\ &= |\int_\Omega g(x, u_n)u_n^\pm\,dx + \int_\Omega f(x, v_n)v_n^\pm\,dx| \\ &\leq \{\int_\Omega |g(x, u_n)|^{\frac{p+1}{p}}\}^{\frac{p}{p+1}}\|u_n^\pm\|_{L^{p+1}} + \{\int_\Omega |f(x, v_n)|^{\frac{q+1}{q}}\}^{\frac{q}{q+1}}\|v_n^\pm\|_{L^{q+1}} \\ &\leq C\{1 + \{\int_\Omega |g(x, u_n)||u_n|\}^{\frac{p}{p+1}} + \{\int_\Omega |f(x, v_n)||v_n|\}^{\frac{q}{q+1}}\}\|z_n^\pm\|_{E^r} \end{aligned} \tag{3.4}$$

Dividing (3.3) by $\|z_n^\pm\|_{E^r}$, we obtain

$$\|z_n^\pm\|_{E^r} \leq C\{1 + \{\int_\Omega |g(x, u_n)||u_n|\}^{\frac{p}{p+1}} + \{\int_\Omega |f(x, v_n)||v_n|\}^{\frac{q}{q+1}}\}. \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\|z_n^\pm\|_{E^r} \leq C\{1 + \{C + \varepsilon\|z_n\|_{E^r}\}^{\frac{p}{p+1}} + \{C + \varepsilon\|z_n^\pm\|_{E^r}\}^{\frac{q}{q+1}}\}, \tag{3.6}$$

which implies that $\|z_n\|_{E^r}$ is bounded. The proof is complete. □

Proof of Theorem 1.3. The proof will be completed by verifying the conditions in Lemma 1.2. We denote $E^1 = H^+$ and $E^2 = H^-$, $b(z) = \Gamma(z)$ and L is defined by (3.1). Apparently, (I1) and (I2) of Lemma 1.2 hold. Now, we verify (I3).

For $\rho > 0$, let $s_1 > \rho$ and s_2 be positive constants to be specified later. Let $e^\pm$ be the eigenvectors corresponding to the positive eigenvalue and negative eigenvalue of L^1 respectively and set $[0, s_1e^+] = \{se^+ : 0 \leq s \leq s_1\}$, $Q = [0, s_1e^+] \oplus (\bar{B}_{s_2} \cap H^-)$, $\tilde{H} = \text{span}\{e^+\} \oplus H^-$, $S = \partial B_\rho \cap H^+$.

By assumption (B3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G(x,u) \le \varepsilon u^2 + C(\varepsilon)|u|^{p+1}, f(x,v) \le \varepsilon v^2 + C(\varepsilon)|v|^{q+1}, \forall u,v \in \mathbb{R},$$

which implies

$$I(z^+) \ge (\frac{1}{2} - \varepsilon)\|z^+\|^2 - C(\varepsilon)\|z^+\|^{p+1} - C(\varepsilon)\|z^+\|^{q+1}$$

for $z^+ \in E^+$. Thus, we may fix $\rho > 0$ and $\alpha > 0$ such that $I(z) \ge \alpha$ on S. This proves (i) of (I3) in Lemma 1.2.

Next we show that for suitable choices of s_1 and s_2, $I(z) \le 0$ on ∂Q. Note that the boundary of Q in $\tilde{H}$ consists of three parts, i.e, $\partial Q = \{Q \cap \{s = 0\}\} \cup \{Q \cap \{s = s_1\}\} \cup \{[0, s_1 e^+] \oplus (\partial B_{s_2} \cap H^-)\}$. It is obvious that $I(z) \le 0$ on $Q \cap \{s = 0\}$ since $I(z) \le 0$ for $(u,v) \le H^-$ and $\Gamma(z)$ is nonnegative. For the remaining parts of ∂Q, we write $z = z^- + se^+ \in \tilde{H}$, then

$$I(z) = \frac{1}{2}s^2 - \frac{1}{2}\|z^-\|^2 - \Gamma(z^- + se^+). \tag{3.7}$$

We may show as in [6] that

$$\Gamma(z^- + se^+) \ge Cs^\beta - C_1, \tag{3.8}$$

where $\beta = \min\{p+1, q+1\}$. Therefore,

$$I(z^- + se^+) \le \frac{1}{2}s^2 - Cs^\beta + C_1 - \frac{1}{2}\|z^-\|^2. \tag{3.9}$$

Choose s_1 sufficient large such that

$$\psi(s) = \frac{1}{2}s^2 - Cs^\beta + C_1 \le 0 \ \forall s \ge s_1,$$

and then choose s_2 large such that $s_2^2 > 2\max_{s\ge 0}\psi(s)$, then we get $I(z) \le 0$ on ∂Q. This proves (ii) of (I3) in Lemma 1.2. Since S and ∂Q are link. The proof is complete. □

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