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STABILITY OF CELLULAR NEURAL NETWORKS WITH UNBOUNDED TIME-VARYING DELAYS

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Abstract. In this article, we prove the existence of local solutions and the stability of the equilibrium points for cellular neural networks.

1. Introduction

Let n correspond to the number of units in a neural network, $x_i(t)$ be the state vector of the ith unit at the time t, $a_{ij}(t)$ be the strength of the jth unit on the ith unit at time t, $b_{ij}(t)$ be the strength of the jth unit on the ith unit at time $t-\tau_{ij}(t)$, and $\tau_{ij}(t)\geq 0$ denote the transmission delay of the ith unit along the axon of the jth unit at the time t. It is well known that the cellular neural networks with time-varying delays are described by the differential equations

$$x_i'(t)=-c_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))+I_i(t), \quad (1.1)$$

where $i=1,2,\dots,n$, for any activation functions of signal transmission f_j and g_j. Here $I_i(t)$ denotes the external bias on the ith unit at the time t, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t.

Since the cellular neural networks (CNNs) were introduced by Chua and Yang [3] in 1990, they have been successfully applied to signal and image processing, pattern recognition and optimization. Hence, CNNs have been the object of intensive analysis by numerous authors in recent years. In particular, extensive results on the problem of the existence and stability of the equilibriums and periodic solutions for (1.1) are given out in the literature. We refer the reader to [2, 4, 5, 6, 7, 8, 9] and the references cited therein. Suppose that the following condition

(H0) there exists a constant τ such that

$$\tau=\max_{1\leq i,j\leq n}\big\{\sup_{t\in\mathbb{R}}\tau_{ij}(t)\big\}. \quad (1.2)$$

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Most authors of bibliographies listed above obtained some sufficient conditions for the existence and stability of the equilibriums and periodic solutions for system (1.1). However, to the best of our knowledge, few authors have considered dynamic behaviors of (1.1) without the assumption (H0). Thus, it is worth while to continue to investigate the stability of system (1.1).

The main purpose of this paper is to give new criteria for the stability of the equilibrium of system (1.1). By applying mathematical analysis techniques, without assuming (H0), we derive some sufficient conditions ensuring that the equilibrium of (1.1) is locally stable, which are new and complement of previously known results. An example is provided to illustrate our results. In this paper, for $i, j = 1, 2, \dots, n$, it will be assumed that $c_i(t)$, $I_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ are constant:

$$c_i \equiv c_i(t), \quad I_i \equiv I_i(t), \quad a_{ij} \equiv a_{ij}(t), \quad b_{ij} \equiv b_{ij}(t). \tag{1.3}$$

It will be assumed that

$$\max_{1\le i,j\le n}\{\sup_{t\in\mathbb{R}} \tau_{ij}(t)\} = +\infty,\ \tau_{ij}(t) < t, i,\ j = 1,\ 2,\ \cdots, n. \tag{1.4}$$

We also assume that the following conditions:

(H1) For each $j \in \{1, 2, \dots, n\}$, there exist nonnegative constants $\tilde{L}_j$ and L_j such that

$$|f_j(u) - f_j(v)| \le \tilde{L}_j|u - v|, \quad |g_j(u) - g_j(v)| \le L_j|u - v|, \quad \text{for all } u,\ v \in \mathbb{R}. \tag{1.5}$$

(H2) There exist constants $\eta > 0$ and $\xi_i > 0$, $i = 1, 2, \dots, n$, such that

$$-c_i\xi_i + \sum_{j=1}^{n} |a_{ij}|\tilde{L}_j\xi_j + \sum_{j=1}^{n} |b_{ij}|L_j\xi_j < -\eta < 0, \quad i = 1, 2, \dots, n.$$

Since $c_i(t)$, $I_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ are constant, by using a similar argument as that in the proof of [4, Theorem 2.3], we can easily show the following lemma.

Lemma 1.1. *Let* (H1), (H2) *hold. Then* (1.1) *has at least one equilibrium point.*

The initial conditions associated with (1.1) are of the form

$$x_i(s) = \varphi_i(s), s \in (-\infty,\ 0], \quad i = 1, 2, \dots, n, \tag{1.6}$$

where $\varphi_i(\cdot)$ denotes real-valued bounded continuous function defined on $(-\infty, 0]$.

For $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, we define the norm

$$\|Z(t)\|_\xi = \max_{i=1,2,\dots,n} |\xi_i^{-1}x_i(t)|. \tag{1.7}$$

The remaining part of this paper is organized as follows. In Section 2, we present sufficient conditions to ensure that the equilibrium of system (1.1) is locally stable. In Section 3, we give some examples and remarks to illustrate our results obtained in the previous sections.

2. Main Results

Theorem 2.1. *Assume* (H1), (H2) *hold. Suppose that* $Z^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ *is the equilibrium of* (1.1). *Then,* Z^* *is locally stable, namely, for all* $\varepsilon > 0$, *there exists a constant* $\delta > 0$ *such that for every solution* $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ *of* (1.1) *with any initial value* $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$ *such that*

$$\|\varphi - Z^*\| = \max_{1\le j\le n}\{\sup_{-\infty\le t\le 0} |\varphi_j - Z_j^*|\} < \delta,$$

there holds
$$|x_i(t) - x_i^*| < \varepsilon, \quad \textit{for all } t \ge 0,\ i = 1, 2, \dots, n.$$

Proof. Let $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be a solution of system (1.1) with any initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$, and define
$$u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T = Z(t) - Z^*.$$
Then
$$u_i'(t) = -c_i u_i(t) + \sum_{j=1}^n a_{ij}[f_j(x_j(t)) - f_j(x_j^*)] + \sum_{j=1}^n b_{ij}[g_j(x_j(t - \tau_{ij}(t))) - g_j(x_j^*)], \tag{2.1}$$
where $i = 1, 2, \dots, n$. Let i_t be an index such that
$$\xi_{i_t}^{-1}|u_{i_t}(t)| = \|u(t)\|_\xi. \tag{2.2}$$
Calculating the upper right derivative of $|u_{i_s}(s)|$ along (2.1), in view of (2.1) and (H1), we have
$$\begin{aligned}
D^+(|u_{i_s}(s)|)\Big|_{s=t} &= \operatorname{sign}(u_{i_t}(t))\Big\{ - c_{i_t}u_{i_t}(t) + \sum_{j=1}^n a_{i_t j}[f_j(x_j(t)) - f_j(x_j^*)] \\
&\quad + \sum_{j=1}^n b_{i_t j}[g_j(x_j(t - \tau_{i_t j}(t))) - g_j(x_j^*)]\Big\} \\
&\le -c_{i_t}|u_{i_t}(t)|\xi_{i_t}^{-1}\xi_{i_t} + \sum_{j=1}^n a_{i_t j}\tilde{L}_j|u_j(t)|\xi_j^{-1}\xi_j \\
&\quad + \sum_{j=1}^n b_{i_t j}L_j|u_j(t - \tau_{i_t j}(t))|\xi_j^{-1}\xi_j.
\end{aligned} \tag{2.3}$$
Let
$$M(t) = \max_{s \le t}\{\|u(s)\|_\xi\}. \tag{2.4}$$
It is obvious that $\|u(t)\|_\xi \le M(t)$, and $M(t)$ is non-decreasing. Now, we consider two cases.

Case (i).
$$M(t) > \|u(t)\|_\xi \quad \text{for all } t \ge 0. \tag{2.5}$$
We claim that $M(t) \equiv M(0)$ is constant for all $t \ge 0$. By way of contradiction, assume that this is not the case. Consequently, there exists $t_1 > 0$ such that $M(t_1) > M(0)$. Since
$$\|u(t)\|_\xi \le M(0) \quad \text{for all } t \le 0.$$
There must exist $\beta \in (0,\ t_1)$ such that
$$\|u(\beta)\|_\xi = M(t_1) \ge M(\beta),$$
which contradicts (2.5). This contradiction implies that $M(t)$ is constant and
$$\|u(t)\|_\xi < M(t) = M(0) \quad \text{for all } t \ge 0. \tag{2.6}$$

Case (ii). There is a point $t_0 \geq 0$ such that $M(t_0) = \|u(t_0)\|_\xi$. Then, using the (2.1) and (2.3), for all $\varepsilon > 0$, we obtain

$$\begin{aligned} D^+(|u_{i_s}(s)|)\Big|_{s=t_0} &\leq -c_{i_{t_0}}|u_{i_{t_0}}(t_0)|\xi_{i_{t_0}}^{-1}\xi_{i_{t_0}} + \sum_{j=1}^n a_{i_{t_0}j}\tilde{L}_j|u_j(t_0)|\xi_j^{-1}\xi_j \\ &\quad + \sum_{j=1}^n b_{i_{t_0}j}L_j|u_j(t_0 - \tau_{i_{t_0}j}(t_0))|\xi_j^{-1}\xi_j \\ &\leq \big[- c_{i_{t_0}}\xi_{i_{t_0}} + \sum_{j=1}^n a_{i_{t_0}j}\tilde{L}_j\xi_j + \sum_{j=1}^n b_{i_{t_0}j}L_j\xi_j\big]M(t_0) \\ &< -\eta M(t_0) + \eta \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon\,. \end{aligned}$$

In addition, if $M(t_0) \geq \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon$, then $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0 + \delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$\|u(t_0)\|_\xi = M(t_0) < \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon. \tag{2.7}$$

Furthermore, for any $t > t_0$, by the same approach used in the proof of (2.7), we have

$$\|u(t)\|_\xi < \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon, \quad \text{if } M(t) = \|u(t)\|_\xi. \tag{2.8}$$

On the other hand, if $M(t) > \|u(t)\|_\xi, t > t_0$. We can choose $t_0 \leq t_3 < t$ such that

$$M(t_3) = \|u(t_3)\|_\xi < \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon, \quad M(s) > \|u(s)\|_\xi \quad \text{for all } s \in (t_3, t].$$

Using a similar argument as in the proof of Case (i), we can show that $M(s) \equiv M(t_3)$ is constant for all $s \in (t_3, t]$, which implies that

$$\|u(t)\|_\xi < M(t) = M(t_3) = \|u(t_3)\|_\xi < \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon.$$

In summary, for all $t \geq 0$, we obtain

$$\|u(t)\|_\xi < \max\big\{M(0), \min_{1\leq j\leq n}\{\xi_j^{-1}\}\varepsilon\big\}. \tag{2.9}$$

Hence, for $\varepsilon > 0$, set

$$\delta = \frac{\min_{1\leq j\leq n}\{\xi_j^{-1}\}\epsilon}{\max_{1\leq j\leq n}\{\xi_j^{-1}\}} > 0.$$

Then, for every solution $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of (1.1) with any initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$ and

$$\|\varphi - Z^*\| = \max_{1\leq j\leq n}\big\{\sup_{-\infty\leq t\leq 0}|\varphi_j - Z_j^*|\big\} < \delta,$$

in view of (2.9), we have $|x_i(t) - x_i^*| < \varepsilon$, for all $t \geq 0$, $i = 1, 2, \dots,\ n$. This completes the proof. $\square$

3. An Example

To illustrate the results obtained in previous sections, consider the CNNs, with unbounded time-varying delays,

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + \frac{1}{4}f_1(x_1(t)) + \frac{1}{36}f_2(x_2(t)) + \frac{1}{4}g_1(x_1(t-\frac{1}{2}|t\sin t|)) \\
&\quad + \frac{1}{36}g_2(x_2(t-\frac{1}{3}|t\sin t|)) + 1, \\
x_2'(t) &= -x_2(t) + f_1(x_1(t)) + \frac{1}{4}f_2(x_2(t)) + g_1(x_1(t-\frac{1}{4}|t\sin t|)) \\
&\quad + \frac{1}{4}g_2(x_2(t-\frac{1}{5}|t\sin t|)) + 2,
\end{aligned}
\tag{3.1}
$$

where $f_1(x) = f_2(x) = g_1(x) = g_2(x) = \arctan x$. Note that

$$
c_1 = c_2 = L_1 = L_2 = \tilde{L}_1 = \tilde{L}_2 = 1, \quad a_{11} = b_{11} = \frac{1}{4},
$$
$$
a_{12} = b_{12} = \frac{1}{36}, \quad a_{21} = b_{21} = 1, \quad a_{22} = b_{22} = \frac{1}{4}.
$$

Then

$$
d_{ij} = \frac{1}{c_i}(a_{ij}\tilde{L}_j + b_{ij}L_j) \quad i,j = 1,2, \quad D = (d_{ij})_{2\times 2} = \begin{pmatrix} 1/2 & 1/18 \\ 2 & 1/2 \end{pmatrix}.
$$

Hence, we have $\rho(D) = \frac{5}{6} < 1$. Therefore, it follows from theory of M-matrix in [1] that there exist constants $\eta > 0$ and $\xi_i > 0, i = 1,2$, such that for all $t > 0$, there holds

$$
-c_i\xi_i + \sum_{j=1}^{2}|a_{ij}|\tilde{L}_j\xi_j + \sum_{j=1}^{2}|b_{ij}|L_j\xi_j < -\eta < 0, \quad i = 1,2,
$$

which implies that (3.1) satisfied (H1) and (H2). Hence, from Lemma 1.1 and Theorem 2.1, system (3.1) has at least one equilibrium Z^*, and Z^* is locally stable.

We remark that since (3.1) is a cellular neural networks with unbounded time-varying delays, the results in [2, 4, 5, 6, 7, 8, 9] can not be applied to prove that the equilibrium point is locally stable. Thus, the results of this paper are essentially new.

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