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A NEUMANN BOUNDARY-VALUE PROBLEM ON AN UNBOUNDED INTERVAL

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Abstract. We study a Neumann boundary-value problem on the half line for a second order equation, in which the nonlinearity depends on the (unknown) Dirichlet boundary data of the solution. An existence result is obtained by an adapted version of the method of upper and lower solutions, together with a diagonal argument.

1. Introduction

We study the Neumann boundary-value problem on the half line:

$$
\begin{gathered}
y''(x) = f(x, y(x), y(0), y(\infty)) \quad x \in (0, +\infty), \\
y'(0) = v_0, \quad y'(\infty) = 0,
\end{gathered}
\tag{1.1}
$$

where

$$
y(\infty) := \lim_{x\to+\infty} y(x), \quad y'(\infty) := \lim_{x\to+\infty} y'(x)
$$

and $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Note that the nonlinearity f depends on the (unknown) Dirichlet boundary data of the solution. For a bounded interval, this kind of problem has been considered for example in [9], where a problem on a two-ion electro-diffusion model is discussed.

It is worth to observe that problem (1.1) is resonant, since the kernel of the associated linear operator $\mathcal{L}y := y''$ is non-trivial. For the case of a bounded interval, the Neumann problem has been widely studied in the literature; in this situation, the operator $\mathcal{L}$ is Fredholm of index 0, and the problem can be solved by the use of coincidence degree [4].

If the interval is unbounded then $\mathcal{L}$ is not a Fredholm operator anymore, and an alternative method of proof is needed (see e.g. [1, 2, 3, 5, 6, 7, 8]).

In contrast with the above mentioned works, we deal with an extra difficulty, which arises on the fact that the nonlinear term in problem (1.1) depends also on the (unknown) Dirichlet boundary values of the solution. In particular, it is implicitly assumed that the limit value $y(\infty)$ exists, and it is finite; in this sense, the problem is over-determined. We shall prove the existence of solutions in presence of an ordered couple of a lower and an upper solution that converge at infinity to the

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same limit L. More precisely, we shall assume the existence of smooth functions $\alpha, \beta : [0, \infty) \to \mathbb{R}$ such that $\alpha \leq \beta$, and $\alpha'(0) \geq v_0 \geq \beta'(0)$, $\alpha(\infty) = \beta(\infty) = L$ and

$$\alpha''(x) \geq f(x, \alpha(x), u, L), \quad \beta''(x) \leq f(x, \beta(x), u, L)$$

for each $x \in [0, \infty)$ and $u \in [\alpha(0), \beta(0)]$.

Then, we shall proceed in the following way: firstly, we prove the existence of solutions of a mixed boundary-value problem for any bounded interval $[0, N]$; then we apply a diagonal argument in order to obtain a subsequence of $\{u_N\}$ that converges to a solution of the equation. Finally, under an extra assumption on f we shall prove that the Neumann boundary condition at infinity is satisfied.

Our main theorem reads as follows:

Theorem 1.1. *Let (α, β) be an ordered couple of a lower and an upper solution as before. Furthermore, assume that*

$$|f(x, y, u, L)| \leq \varphi(x)$$

for $x \geq x_0$, $\alpha(0) \leq u \leq \beta(0)$, $\alpha(x) \leq y \leq \beta(x)$ and some $\varphi \in L^1(x_0, +\infty)$. Then (1.1) *admits at least one solution y, with $\alpha \leq y \leq \beta$.*

2. Mixed conditions on a bounded interval. Upper and lower solutions

In this section, we consider the problem with mixed conditions

$$\begin{gathered} y''(x) = f(x, y(x), y(0), L) \quad x \in (0, T), \\ y'(0) = v_0, \quad y(T) = y_T, \end{gathered} \tag{2.1}$$

for some arbitrary constants $y_T, L \in \mathbb{R}$ and $T > 0$.

We shall prove the existence of solutions of (2.1) in presence of an ordered couple of a lower and an upper solution. More precisely, we shall assume the existence of some smooth functions $\alpha \leq \beta$ satisfying

$$\alpha''(x) \geq f(x, \alpha(x), u, L), \quad \beta''(x) \leq f(x, \beta(x), u, L) \tag{2.2}$$

for $0 \leq x \leq T$ and $\alpha(0) \leq u \leq \beta(0)$, and

$$\alpha'(0) \geq v_0 \geq \beta'(0), \quad \alpha(T) \leq y_T \leq \beta(T). \tag{2.3}$$

The following theorem is a slight variation of the standard results in the theory of upper and lower solutions; for the sake of completeness we give a simple proof.

Theorem 2.1. *Let $\alpha \leq \beta$ satisfy* (2.2) *and* (2.3). *Then* (2.1) *admits at least one solution y, with $\alpha \leq y \leq \beta$.*

Proof. Let $\lambda > 0$ be a fixed constant and consider, for each $w \in C([0, T])$, the modified linear problem

$$\begin{gathered} y''(x) - \lambda y(x) = f(x, P_x(w(x)), P_0(w(0)), L) - \lambda P_x(w(x)) \\ y'(0) = v_0, \quad y(T) = y_T \end{gathered} \tag{2.4}$$

where $P : [0, T] \times \mathbb{R} \to \mathbb{R}$ is the truncation function

$$P_x(w) := P(x, w) = \begin{cases} \alpha & \text{if } w < \alpha(x) \\ w & \text{if } \alpha(x) \leq w \leq \beta(x) \\ \beta & \text{if } w > \beta(x) \end{cases} \tag{2.5}$$

Problem (2.4) has a unique solution $y \in C^2([0,T])$, given by the integral representation

$$y(x) = y_0(x) + \int_0^T G(x,s)\phi(s)ds,$$

where

$$y_0(x) = \frac{v_0 \sinh(\sqrt{\lambda}(x-T)) + \sqrt{\lambda} y_T \cosh(\sqrt{\lambda}x)}{\sqrt{\lambda}\cosh(\sqrt{\lambda}T)},$$
$$\phi(s) = \phi(w,s) := f(s, P_s(w(s)), P_0(w(0)), L) - \lambda P_s(w(s))$$

and G is the Green function

$$G(x,s) = \begin{cases} \frac{\cosh(\sqrt{\lambda}s)\sinh(\sqrt{\lambda}(x-T))}{\sqrt{\lambda}\cosh(\sqrt{\lambda}T)} & \text{if } 0 \le s < x \\ \frac{\cosh(\sqrt{\lambda}x)\sinh(\sqrt{\lambda}(s-T))}{\sqrt{\lambda}\cosh(\sqrt{\lambda}T)} & \text{if } x \le s \le T. \end{cases}$$

Then, we define the compact operator $K : C([0,T]) \to C([0,T])$ by

$$Kw(x) = y_0(x) + \int_0^T G(x,s)\phi(w,s)ds,$$

with y_0, ϕ and G as above.

It is clear that $\|\phi(w,\cdot)\|_\infty \le M$ where the constant M is independent of w; thus, there exists $R > 0$ such that $\|Kw\|_\infty \le R$ for every $w \in C([0,T])$. From Schauder's Theorem it follows that K has a fixed point $y \in \overline{B_R(0)}$.

Next, we shall prove that $\alpha(x) \le y(x) \le \beta(x)$ for all $x \in [0,T]$, and hence y is a solution of the original problem. It suffices to prove that $\alpha \le y$, since the other inequality is analogous.

By contradiction, suppose there exists $x_0 \in [0,T]$ such that $y(x_0) < \alpha(x_0)$. We may assume that x_0 is an absolute maximum value of the function $\alpha - y$.

We consider the two possible cases:

Case 1. $x_0 \in (0,T)$. From the definition of P, it follows that $P_{x_0}(y(x_0)) = \alpha(x_0)$. Moreover, $\alpha(0) \le P_0(y(0)) \le \beta(0)$ and hence

$$\alpha''(x_0) \ge f(x_0, \alpha(x_0), P_0(y(0)), L).$$

On the other hand,

$$0 \ge (\alpha - y)''(x_0) = \alpha''(x_0) - f(x_0, \alpha(x_0), P_0(y(0)), L) + \lambda(\alpha(x_0) - y(x_0)) > 0,$$

which is a contradiction.

Case 2. $x_0 = 0$ or $x_0 = T$. As $y(T) = y_T \ge \alpha(T)$, it follows that $x_0 = 0$. Moreover, $\alpha'(0) \ge y'(0) = v_0$, and by continuity

$$(\alpha - y)'' = \alpha'' - f(\cdot, \alpha, P_0(y(0)), L) + \lambda(\alpha - y(\cdot)) > 0$$

over some interval $(0,\delta)$ with δ small enough. This implies $(\alpha - y)' > 0$ on $(0,\delta)$, and then $\alpha - y > (\alpha - y)(0)$ on $(0,\delta)$. This contradicts the fact that 0 is an absolute maximum of the function $\alpha - y$. $\square$

3. A diagonal argument for problem (1.1)

In this section we give a proof of Theorem 1.1. From Theorem 2.1, for every $N \in \mathbb{N}$ we consider a solution y_N of

$$\begin{gathered} y_N''(x) = f(x, y_N(x), y_N(0), L) \quad x \in (0, N), \\ y_N'(0) = v_0, \quad y_N(N) = \frac{\alpha(N) + \beta(N)}{2}, \end{gathered} \tag{3.1}$$

such that $\alpha|_{[0,N]} \le y_N \le \beta|_{[0,N]}$. For fixed M, let us observe firstly that if $N \ge M$ then

$$\|y_N''|_{[0,M]}\|_\infty = \sup_{x\in[0,M]}\{|f(x, y_N(x), y_N(0), L)|\} \le C_2$$

for some positive constant C_2 independent of N. Moreover, writing

$$y_N'(x) = v_0 + \int_0^x y_N''(s)\, ds$$

it is also seen that $\|y_N'|_{[0,M]}\|_\infty \le |v_0| + MC_2 := C_1$. Finally, from the equality

$$y_N(x) = y_N(M) - \int_x^M y_N'(s)\, ds$$

we conclude that $\|y_N|_{[0,M]}\|_\infty \le C_0$ for some constant C_0 depending only on M. Hence, from the Arzelá-Ascoli theorem we deduce the existence of a subsequence of $\{y_N\}_{N\ge M}$ that converges on $[0, M]$ for the C^1-norm.

Thus, we may proceed as follows: for $M = 1$, let us choose a subsequence (still denoted $\{y_N\}$) that converges in $C^1([0, 1])$ to some function y^1. Repeating the procedure for $M = 2, 3, \dots$, we may assume that $y_N|_{[0,M]}$ converges to some function y^M in the C^1-sense.

It follows from the construction that $y^{M+1}|_{[0,M]} = y^M$, and this implies that the function $y : [0, +\infty) \to \mathbb{R}$ given by $y(x) = y^M(x)$ if $0 \le x \le M$ is well defined. Moreover, $y'(0) = v_0$, and y_N'' converges uniformly in $[0, M]$ to $f(\cdot, y(\cdot), y(0), L)$.

Thus, for any test function $\xi \in C_0^\infty(0, M)$ we obtain:

$$\begin{aligned} \int_0^M f(x, y, y(0), L)\xi(x)\, dx &= \lim_{N\to\infty} \int_0^M y_N''(x)\xi(x)\, dx \\ &= \lim_{N\to\infty} \int_0^M y_N(x)\xi''(x)\, dx \\ &= \int_0^M y(x)\xi''(x)\, dx, \end{aligned}$$

whence

$$y''(x) = f(x, y(x), y(0), L)$$

in $[0, M]$, in the weak sense. Moreover, it is clear that $y(\infty) = L$, and as y is twice continuously differentiable we deduce that y is a classical solution of the equation. Finally, for each $N \ge x_0$ we may establish a point $x_N \in (N, N + 1)$ such that $y'(x_N) = y(N + 1) - y(N) \to 0$; thus, for $x > x_N$ we have

$$|y'(x) - y'(x_N)| = \Big|\int_{x_N}^x f(t, y(t), y(0), L)\, dt\Big| \le \int_{x_N}^x \varphi(t)\, dt \le \int_{x_N}^{+\infty} \varphi(t)\, dt.$$

As φ is integrable, we conclude that $y'(x) \to 0$ as $x \to \infty$, and so completes the proof.

Example 3.1. Consider the problem

$$y''(x) = 2(x+1)y(x)^4 + g(x, y(x), y(0), y(\infty)) \quad x \in (0, +\infty),$$
$$y'(0) = v_0, \quad y'(\infty) = 0,$$

where $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies

$$g(x, 0, u, 0) = 0, \quad 0 \le g(x, y, u, 0) \le \varphi(x) \quad \text{for } x \ge 0,\ 0 \le y \le \frac{1}{x+1} \text{ and } u \in [0, 1]$$

for some $\varphi \in L^1([0, +\infty))$. Then, if $v_0 \in [-1, 0]$ the assumptions of Theorem 1.1 are fulfilled, taking $\alpha \equiv 0$ and $\beta(x) = \frac{1}{x+1}$.

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