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MEAN VALUE THEOREMS FOR SOME LINEAR INTEGRAL OPERATORS

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Abstract. In this article we study some mean value results involving linear integral operators on the space of continuous real-valued functions defined on the compact interval $[0,1]$. The existence of such points will rely on some classical theorems in real analysis like Rolle, Flett and others. Our approach is rather elementary and does not use advanced techniques from functional analysis or nonlinear analysis.

1. Introduction and Preliminaries

Mean value theorems play a key role in analysis. The simplest form of the mean value theorem is the next basic result due to Rolle, namely

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$. Then there exists a point $c\in(a,b)$ such that $f'(c)=0$.*

Rolle's theorem has also a geometric interpretation which states that if $f(a)=f(b)$ then there exists a point in the interval (a,b) such that the tangent line to the graph of f is parallel to the x-axis. There is another geometric interpretation as pointed out in [8], namely the polar form of Rolle's theorem. As been noticed in [8], if we take into account the geometric interpretation of Rolle's theorem, one expects that it is possible to relate the slope of the chord connecting $(a,f(a))$ and $(b,f(b))$ with the value of the derivative at some interior point. There are also other mean value theorems like Lagrange, Cauchy, Darboux which are well-known and can be found in any undergraduate Real Analysis course. In 1958, Flett gave a variation of Lagrange's mean value theorem with a Rolle type condition, namely

Theorem 1.2 ([8, 5])**.** *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$, differentiable on (a,b) and $f'(a)=f'(b)$. Then there exists a point $c\in(a,b)$ such that*

$$f'(c)=\frac{f(c)-f(a)}{c-a}.$$

A detailed proof can be found in [5] and some applications are provided too. The same proof appears also in [8]. A slightly different proof which uses Rolle's theorem instead of Fermat's, can be found in [3] and [11]. There is a nice geometric

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interpretation for Theorem 1.2, namely: If the curve $y = f(x)$ has a tangent at each point in $[a, b]$, and if the tangents at $(a, f(a))$ and $(b, f(b))$ are parallel, then there is an intermediate point $c \in (a, b)$ such that the tangent at $(c, f(c))$ passes through the point $(a, f(a))$. Later, Riedel and Sahoo [11] removed the boundary assumption on the derivatives and prove the following

Theorem 1.3 ([11])**.** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a)^2.$$

The proof relies on Theorem 1.2 applied to the auxiliary function $\alpha : [a, b] \to \mathbb{R}$ defined by

$$\alpha(x) = f(x) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(x - a)^2.$$

We leave the details to the reader. We also point out that this theorem is used to extend Flett's mean value theorem for holomorphic functions. In this sense, one can consult [9]. On the other hand, there exists another result due to Flett as it is pointed out in [8] for the second derivative of a function.

Theorem 1.4. *If $f : [a, b] \to \mathbb{R}$ is a twice differentiable function such that $f''(a) = f''(b)$ then there exists $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{(c - a)^2}{2}f''(c).$$

There exists another version of Flett's theorem for the antiderivative:

Theorem 1.5. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that*

$$\int_a^c f(x)dx = (c - a)f(c).$$

Similar to Theorem 1.1 there exists another mean value theorem due to Penner (problem 987 from the Mathematics Magazine, [8]) that we shall apply in the next section.

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with f' be continuous on $[a, b]$ such that there exists $\lambda \in (a, b)$ with $f'(\lambda) = 0$. Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(c) - f(a)}{b - a}.$$

The proof of the above theorem can be found in [8, page 233].

The version of Theorem 1.6 for antiderivative is the following theorem.

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that there is $\lambda \in (a, b)$ such that $f(\lambda) = 0$. Then there is $c \in (a, b)$ such that*

$$\int_a^c f(x)dx = (b - a)f(c).$$

In 1966, Trahan [15] extended Theorem 1.2 by removing the condition $f'(a) = f'(b)$ to the following theorem.

Theorem 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$, differentiable on (a,b) such that*

$$(f'(a) - m)(f'(b) - m) > 0,$$

where $m = \frac{f(b)-f(a)}{b-a}$. Then there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

A proof for the above theorem can be found in [11]. At the 35-th International Symposium on Functional Equations held in Graz in 1997, Zsolt Pales raised a question regarding a generalization of Flett's theorem. An answer to his question was given by Pawlikowska in [6], namely:

Theorem 1.9. *If f possesses a derivative of order n on the interval $[a,b]$, then there exists a point $c \in (a,b)$ such that*

$$f(c) - f(a) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} f^{(k)}(c)(c-a)^k + \frac{(-1)^n}{(n+1)!} \cdot \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}(c-a)^{n+1}.$$

Other generalizations and several new mean value theorems in terms of divided differences are given in [4]. For further reading concerning mean value theorems we recommend [11].

2. Main results

In this section, we shall prove mean value value problems for some function mapping. Our main results are for continuous, real-valued functions defined on the interval the $[0,1]$. We also mention that the results can be easily extended to the interval $[a,b]$. Before proceeding into the main results of the paper, we state and prove a lemmata. We give more proofs to some lemmas, which we consider very instructive. We start with the first two lemmas involving the integrant factor e^{-t}.

Lemma 2.1. *Let $h_1 : [0,1] \to \mathbb{R}$ be a continuous function with $\int_0^1 h_1(x) = 0$. Then there exists $c_1 \in (0,1)$ such that*

$$h_1(c_1) = \int_0^{c_1} h_1(x)dx.$$

First proof. We assume by the way of contradiction, that

$$h_1(t) > \int_0^t h_1(x)dx, \quad \forall t \in [0,1].$$

Now, we consider the auxiliary function $\zeta_1 : [0,1] \to \mathbb{R}$, given by

$$\zeta_1(t) = e^{-t} \int_0^t h_1(x)dx.$$

A simple calculation gives

$$\zeta_1'(t) = e^{-t}\Big(h_1(t) - \int_0^t h_1(t)dt\Big) > 0,$$

and from our assumption we deduce that ζ_1 is strictly increasing. This means that $\zeta_1(0) < \zeta_1(1)$ which is equivalent to $0 < \frac{1}{e}\int_0^1 h_1(x)dx = 0$, a contradiction.

Second proof. Like in the previous proof, let us consider the differentiable function $\gamma_1 = \zeta_1 : [0,1] \to \mathbb{R}$, defined by

$$\gamma_1(t) = e^{-t} \int_0^t h_1(x)dx.$$

A simple calculation yelds

$$\gamma_1'(t) = e^{-t}\Big(h_1(t) - \int_0^t h_1(t)dt\Big).$$

More than that we have $\gamma_1(0) = \gamma_1(1) = 0$, so by applying Theorem 1.1, there exists $c_1 \in (0,1)$ such that $\gamma_1'(c_1) = 0$, i.e.

$$h_1(c_1) = \int_0^{c_1} h_1(x)dx.$$

Third proof. We shall use Theorem 1.7. Indeed from the hypothesis $\int_0^1 h_1(x)dx = 0$, by applying the first mean value theorem for integrals we obtain the existence of $\lambda \in (0,1)$ such that

$$0 = \int_0^1 h_1(x)dx = h_1(\lambda).$$

Now, by Theorem 1.7 there exists $c_1 \in (0,1)$ such that

$$h_1(c_1) = \int_0^{c_1} h_1(x)dx.$$

□

Similarly with Lemma 2.1, we prove the following result.

Lemma 2.2. *Let $h_2 : [0,1] \to \mathbb{R}$ be a continuous function with $h_2(1) = 0$. Then there exists $c_2 \in (0,1)$ such that*

$$h_2(c_2) = \int_0^{c_2} h_2(x)dx.$$

First proof. Let us consider the following auxiliary function $\zeta_2 : [0,1] \to \mathbb{R}$, given by

$$\zeta_2(t) = e^{-t} \int_0^t h_2(x)dx.$$

Suppose by the way of contradiction that $h_2(t) \neq \int_0^t h_2(x)dx, \forall t \in [0,1]$. This means that, without loss of generality, we can assume that

$$h_2(t) > \int_0^t h_2(x)dx, \forall t \in [0,1]. \tag{2.1}$$

A simple calculation of derivatives of the function ζ_2 combined with the inequality above, gives us the following inequality

$$\zeta_2'(t) = e^{-t}\Big(h_2(t) - \int_0^t h_2(t)dt\Big) > 0,$$

so our function ζ_2 is strictly increasing for every $t \in (0,1)$. This means that $\zeta_2(1) > \zeta_2(0)$. It follows immediately that $\frac{1}{e}\int_0^1 h_2(x)dx > 0$. On the other hand, taking into account our assumption, namely (2.1), we deduce in particular, that $h_2(1) > \int_0^1 h_2(x)dx > 0$ which contradicts the hypothesis $h_2(1) = 0$.

Second proof. Let us consider the differentiable function $\gamma_2 : [0,1] \to \mathbb{R}$, defined by

$$\gamma_2(t) = te^{-t} \int_0^t h_2(x)dx.$$

A simple calculation yields

$$\gamma_2'(t) = e^{-t}\Big(\int_0^t h_2(x)dx - t\Big(h_2(t) - \int_0^t h_2(x)dx\Big)\Big).$$

Taking into account that $h_2(1) = 0$, it is clearly that $\gamma_2'(0) = \gamma_2'(1)$, so by Flett's mean value theorem (Theorem 1.2) (see [3]), we deduce the existence of $c_2 \in (0,1)$ such that

$$\gamma_2'(c_2) = \frac{\gamma_2(c_2) - \gamma_2(0)}{c_2}$$

which is equivalent to

$$c_2e^{-c_2}\Big(\int_0^{c_2} h_2(x)dx - c_2\Big(h_2(c_2) - \int_0^{c_2} h_2(x)dx\Big)\Big) = c_2e^{-c_2} \int_0^{c_2} h_2(x)dx,$$

or

$$h_2(c_2) = \int_0^{c_2} h_2(x)dx.$$

□

Following the same idea from lemma 2.1 we state and prove the following result.

Lemma 2.3. *Let $h_3 : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $\int_0^1 h_3(x)dx = 0$. Then there exists $c_3 \in (0,1)$ such that*

$$h_3(c_3) = h_3'(c_3) \int_0^{c_3} h_3(x)dx.$$

Proof. As in the proof of Lemma 2.1, let us consider the differentiable function $\gamma_3 = \zeta_3 : [0,1] \to \mathbb{R}$, defined by

$$\gamma_1(t) = e^{-h_3(t)} \int_0^t h_3(x)dx.$$

A simple calculation yields

$$\gamma_3'(t) = e^{-h_3(t)}\Big(h_3(t) - h_3'(t) \int_0^t h_3(t)dt\Big).$$

More than that we have $\gamma_3(0) = \gamma_3(1) = 0$, so by applying Theorem 1.1, there exists $c_3 \in (0,1)$ such that $\gamma_3'(c_3) = 0$, i.e.

$$h_3(c_3) = h_3'(c_3) \int_0^{c_3} h_3(x)dx.$$

□

In what will follow we prove other technical lemmas without the integrant factor e^{-t}.

Lemma 2.4. *Let $h_4 : [0,1] \to \mathbb{R}$ be a continuous function with $\int_0^1 h_4(x)dx = 0$. Then there exists $c_4 \in (0,1)$ such that*

$$\int_0^{c_4} xh_4(x)dx = 0.$$

First proof. We assume by contradiction that $\int_0^t xh_4(x)dx \neq 0$, for all $t \in (0,1)$. Without loss of generality, let $\int_0^t xh_4(x)dx > 0$, for all $t \in (0,1)$ and let $H_4(t) = \int_0^t h_4(x)dx$. Integrating by parts, we obtain

$$0 < \int_0^t xh_4(x)dx = tH_4(t) - \int_0^t H_4(x)dx, \quad \forall t \in (0,1).$$

Now, by passing to the limit when $t \to 1$, and taking into account that $H_4(1) = 0$, we deduce that

$$\int_0^1 H_4(x)dx \leq 0. \tag{2.2}$$

Now, we consider the differentiable function, $\mu : [0,1] \to \mathbb{R}$ defined by

$$\mu(t) = \begin{cases} \frac{1}{t}\int_0^t H_4(x)dx, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0. \end{cases}$$

It is easy to see $\mu'(t) = \big(tH_4(t) - \int_0^t H_4(x)dx\big)/t^2 > 0$, so μ is increasing on the interval $(0,1)$, so it is increasing on the interval $[0,1]$ (by continuity argument). Because $\mu(0) = 0$, it follows that

$$\int_0^1 H_4(x)dx > 0,$$

which is in contradiction to (2.2). So, there exists $c_4 \in (0,1)$ such that

$$\int_0^{c_4} xh_4(x)dx = 0.$$

Second proof. We consider the differentiable function $\mathcal{H} : [0,1] \to \mathbb{R}$, defined by

$$\mathcal{H}(t) = t\int_0^t h_4(x)dx - \int_0^t xh_4(x)dx$$

with $\mathcal{H}'(t) = \int_0^t h_4(x)dx$. It is clear that $\mathcal{H}'(0) = \mathcal{H}'(1) = \int_0^1 h_4(x)dx = 0$. Applying Flett's mean value theorem (see [3]), there exists $c_4 \in (0,1)$ such that

$$\mathcal{H}'(c_4) = \frac{\mathcal{H}(c_4) - \mathcal{H}(0)}{c_4}$$

or

$$c_4\int_0^{c_4} h_4(x)dx = c_4\int_0^{c_4} h_4(x)dx - \int_0^{c_4} xh_4(x)dx$$

which is equivalent to $\int_0^{c_4} xh_4(x)dx = 0$.

Third proof. Let $\tilde{H}_4(t) = \int_0^t xh_4(x)dx$ which is continuous on $[0,1]$. By L'Hopital rule we derive that $\lim_{t\to 0^+} \tilde{H}_4(t)/t = 0$. Integrating by parts, we obtain

$$\int_0^1 h_4(x)dx = \int_0^1 \frac{xh_4(x)}{x}dx = \frac{\tilde{H}_4(x)}{x}\Big|_0^1 + \int_0^1 \frac{\tilde{H}_4(x)}{x^2}dx = \tilde{H}_4(1) + \int_0^1 \tilde{H}_4(x)x^2dx.$$

Now, since $\int_0^1 h_4(x)dx = 0$, by the equality above $\tilde{H}_4(x)$ cannot be positive or negative for all $x \in (0,1)$. So, by the intermediate value property there exists $c_4 \in (0,1)$ such that $\tilde{H}_4(c_4) = 0$ and thus the conclusion follows. $\square$

Remark 2.5. Using the same idea as in Lemmas 2.1 and 2.2, we define the auxiliary function like in the first solution, we define the auxiliary function $\gamma_4 = \zeta_4 : [0,1] \to \mathbb{R}$, given by

$$\gamma_4(t) = e^{-t} \int_0^t x h_4(x) dx,$$

whose derivative is

$$\gamma_4'(t) = e^{-t}\Big(t h_4(t) - \int_0^t x h_4(x) dx\Big).$$

Since $\gamma_4(0) = \gamma_4(c_3)$ (by Lemma 2.4), by applying Rolle's theorem on the interval $[0, c_4]$, there exists $\tilde{c_4} \in (0, c_4)$ such that $\gamma'(\tilde{c_4}) = 0$, i.e.

$$\tilde{c_4} h_4(\tilde{c_4}) = \int_0^{\tilde{c_4}} x h_4(x) dx.$$

Remark 2.6. As we have seen in the first remark, if we consider the differentiable function $\tilde{\gamma_4} : [0,1] \to \mathbb{R}$ defined by

$$\tilde{\gamma_4}(t) = e^{-h_4(t)} \int_0^t x h_4(x) dx,$$

whose derivative is

$$\tilde{\gamma_4}'(t) = e^{-h_4(t)}\Big(t h_4(t) - h_4'(t) \int_0^t x h_4(x) dx\Big).$$

Now, it is clear that $\tilde{\gamma_4}(0) = \tilde{\gamma_4}(c_4) = 0$ by Lemma 2.4. So, by applying Rolle's theorem there exists $\bar{c_4} \in (0,1)$ such that $\tilde{\gamma_4}'(\bar{c_4}) = 0$ which is equivalent to

$$\bar{c_4} h_4(\bar{c_4}) = h_4'(\bar{c_4}) \int_0^{\bar{c_4}} x h_4(x) dx.$$

In what follows we prove two lemmas starting from the same hypothesis.

Lemma 2.7. *Let $h_5 : [0,1] \to \mathbb{R}$ be a continuous function such that*

$$\int_0^1 h_5(x) dx = \int_0^1 x h_5(x) dx.$$

Then, there exists $c_5 \in (0,1)$ such that $\int_0^{c_5} h_5(x) dx = 0$.

First proof. Consider the differentiable function $\mathcal{I} : [0,1] \to \mathbb{R}$ defined by

$$\mathcal{I}(t) = t \int_0^t h_5(x) dx - \int_0^t x h_5(x) dx.$$

We have

$$\mathcal{I}'(t) = \int_0^t h_5(x) dx.$$

Moreover $\mathcal{I}(0) = \mathcal{I}(1)$, so by Rolle's theorem, there exists $c_5 \in (0,1)$ such that

$$\mathcal{I}'(c_5) = 0 \Leftrightarrow \int_0^{c_5} h_5(x) dx = 0.$$

Second proof. Let $H_5 : [0,1] \to \mathbb{R}$ defined by

$$H_5(t) = \int_0^t h_5(x) dx.$$

Integrating by part and using the hypothesis, we have

$$H_5(1) = \int_0^1 h_5(x)dx = \int_0^1 xh_5(x)dx = \int_0^1 xH_5'(x)dx = H_5(1) - \int_0^1 H_5(x)dx,$$

and we get $\int_0^1 H_5(x)dx = 0$. By the first mean value theorem for integrals we have the existence of $c_5 \in (0,1)$ such that

$$0 = \int_0^1 H_5(x)dx = H_5(c_5)$$

which is equivalent to $\int_0^{c_5} h_5(x)dx = 0$.

Third proof. Let us rewrite the hypothesis in the following

$$\int_0^1 (x-1)h_5(x)dx = 0.$$

The answer is given by the following mean value theorem for integrals (see [1], page 193)

Proposition. *If $\Omega_1, \Omega_2 : [a,b] \to \mathbb{R}$ are two integrable functions and Ω_2 is monotone, then there exists $c_5 \in (a,b)$ such that*

$$\int_a^b \Omega_1(x)\Omega_2(x) = \Omega_2(a)\int_a^{c_5} \Omega_1(x) + \Omega_1(b)\int_{c_5}^b \Omega_2(x)dx.$$

Now, we consider $a = 0, b = 1$ and $\Omega_1(x) = h_5(x)$ and $\Omega_2(x) = x - 1$ which is increasing. By the mean value theorem in Lemma 2.7, there is $c_5 \in (0,1)$ such that

$$0 = \int_0^1 (x-1)f(x)dx = -\int_0^{c_5} f(x)dx$$

equivalent to $\int_0^{c_5} f(x)dx = 0$. □

Lemma 2.8. *Let $h_6 : [0,1] \to \mathbb{R}$ be a continuous function such that*

$$\int_0^1 h_6(x)dx = \int_0^1 xh_6(x)dx.$$

Then, there exists $c_6 \in (0,1)$ such that $\int_0^{c_6} xh_6(x)dx = 0$.

First proof. Let us consider the function $\varphi : [0,1] \to \mathbb{R}$ given by
$H_6(t) = \int_0^t \tilde{h_6}(s)ds$, where

$$\tilde{h_6}(s) = \begin{cases} \frac{1}{s^2}\int_0^s xh_6(x)dx, & \text{if } s \in (0,1] \\ h_6(0)/2, & \text{if } s = 0 \end{cases}$$

Clearly the function $\tilde{h_6}$ is continuous and $H_6(0) = 0$. Next, we compute

$$\begin{aligned} H_6(1) &= \lim_{\epsilon\to 0, \epsilon>0} \int_\epsilon^1 \big(-\frac{1}{s}\big)\Big(\int_0^s xh_6(x)dx\Big)ds \\ &= -\lim_{\epsilon\to 0,\epsilon>0} \frac{1}{s}\int_0^s xh_6(x)dx\big|_\epsilon^1 + \lim_{\epsilon\to 0}\int_\epsilon^1 \big(\frac{1}{s}sh_6(s)\big)ds \\ &= -\int_0^1 xh_6(x)dx + \int_0^1 h_6(x)dx = 0. \end{aligned}$$

By Rolle's theorem there exists $c_6 \in (0,1)$ such that $H_6'(c_6) = 0$; i.e. $\int_0^{c_6} xh_6(x)dx = 0$.

Second proof. Consider the differentiable function $\tilde{H}_6; [0,1] \to \mathbb{R}$ defined by

$$\tilde{H}_6(t) = t\int_0^t h_6(x)dx - \int_0^t xh_6(x)dx.$$

It is obvious that $\tilde{H}_6'(t) = \int_0^t h_6(x)dx$. By Lemma 2.4 there exists $c_6 \in (0,1)$ such that $\tilde{H}_6'(c_6) = \int_0^{c_5} h_6(x)dx = 0$. On the other hand, since $\tilde{H}_6'(0) = \tilde{H}_6'(c_5) = 0$, by Theorem 1.2 there exists $c_6 \in (0, c_5)$ such that

$$\tilde{H}_6'(c_6) = \frac{\tilde{H}_6(c_6) - \tilde{H}_5(0)}{c_6}$$

which is equivalent to $\int_0^{c_6} xh_6(x)dx = 0$. □

Combining Lemmas 2.4 and 2.8, one can easily derive the following result.

Theorem 2.9. *Assume $h_7 : [0,1] \to \mathbb{R}$ is continuous such that*

$$\int_0^1 h_7(x)dx = \int_0^1 xh_7(x)dx.$$

Then there are $c_7, \tilde{c_7} \in (0,1)$ such that

$$h_7(c_7) = \int_0^{c_7} h_7(x)dx,$$
$$\tilde{c_7}h_7(\tilde{c_7}) = \int_0^{\tilde{c_7}} xh_7(x)dx.$$

Proof. Let us define the auxiliary functions $\zeta_7, \tilde{\zeta_7} : [0,1] \to \mathbb{R}$ given by

$$\zeta_7(t) = e^{-t}\int_0^t h_7(x)dx,$$
$$\tilde{\zeta_7}(t) = e^{-t}\int_0^t xh_7(x)dx.$$

It is clear that

$$\zeta_7'(t) = e^{-t}\Big(h_7(t) - \int_0^t h_7(x)dx\Big),$$
$$\tilde{\zeta_7}'(t) = e^{-t}\Big(th_7(t) - \int_0^t xh_7(x)dx\Big).$$

By Lemmas 2.7 and 2.8, we have $\zeta_7(0) = \zeta_7(c_4)$ and $\tilde{\zeta_7}(0) = \tilde{\zeta_7}(c_5)$. By Rolle's theorem applied on the intervals $(0, c_4)$ and $(0, c_5)$ to ζ_7 and $\tilde{\zeta_7}$ we obtain the conclusion. □

Following the proof of Theorem 2.9, instead of e^{-t} in the construction of the auxiliary functions we can put $e^{-f(t)}$ where f is differentiable with continuous derivative. This gives the following result.

Theorem 2.10. *Assume $h_8 : [0,1] \to \mathbb{R}$ is a differentiable function with continuous derivative such that*

$$\int_0^1 h_8(x)dx = \int_0^1 xh_8(x)dx.$$

Then there are $c_8, \tilde{c_8} \in (0,1)$ *such that*
$$h_8(c_8) = h_8'(c_8)\int_0^{c_8} h_8(x)dx,$$
$$\tilde{c_8}h_8(\tilde{c_8}) = \tilde{h_8'}(\tilde{c_8})\int_0^{\tilde{c_8}} xh_8(x)dx.$$

Proof. Let us define the auxiliary functions $\zeta_8, \tilde{\zeta_8} : [0,1] \to \mathbb{R}$ given by
$$\zeta_8(t) = e^{-h_8(t)}\int_0^t h_8(x)dx,$$
$$\tilde{\zeta_8}(t) = e^{-h_8(t)}\int_0^t xh_8(x)dx.$$
It is clear that
$$\zeta_8'(t) = e^{-h_8(t)}\Big(h_8(t) - h_8'(t)\int_0^t h_8(x)dx\Big),$$
$$\tilde{\zeta_8'}(t) = e^{-h_8(t)}\Big(th_8(t) - h_8'(t)\int_0^t xh_8(x)dx\Big).$$
By Lemmas 2.7 and 2.8, we have $\zeta_8(0) = \zeta_8(c_4)$ and $\tilde{\zeta_8}(0) = \tilde{\zeta_8}(c_5)$. By Rolle's theorem applied on the intervals $(0, c_4)$ and $(0, c_5)$ to ζ_8 and $\tilde{\zeta_8}$ we obtain the conclusion. $\square$

Now, we are ready to state and prove the first main result of the paper.

Theorem 2.11. *For two continuous functions* $\varphi, \psi : [0,1] \to \mathbb{R}$*, we define the operators* $T, S \in (C([0,1]))$*, as follows:*
$$(T\varphi)(t) = \varphi(t) - \int_0^t \varphi(x)dx,$$
$$(S\psi)(t) = t\psi(t) - \int_0^t x\psi(x)dx.$$
Let $f, g : [0,1] \to \mathbb{R}$ *be two continuous functions. Then there exist* $c_1, c_2, \tilde{c_4} \in (0,1)$ *such that*
$$\int_0^1 f(x)dx \cdot (Tg)(c_1) = \int_0^1 g(x)dx \cdot (Tf)(c_1),$$
$$(Tf)(c_2) = (Sf)(c_2),$$
$$\int_0^1 f(x)dx \cdot (Sg)(\tilde{c_4}) = \int_0^1 g(x)dx \cdot (Sf)(\tilde{c_4}).$$

Proof. We put
$$h_1(t) = f(t)\int_0^1 g(x)dx - g(t)\int_0^1 f(x)dx,$$
where $f, g : [0,1] \to \mathbb{R}$ are continuous functions and if we apply lemma 2.1, we get
$$\begin{aligned}
&f(c_1)\int_0^1 g(x)dx - g(c_1)\int_0^1 f(x)dx \\
&= \int_0^{c_1} f(x)dx\int_0^1 g(x)dx - \int_0^{c_1} g(x)dx\int_0^1 f(x)dx,
\end{aligned}$$

which is equivalent to

$$\int_0^1 f(x)dx\Big(g(c_1)-\int_0^{c_1} g(x)dx\Big)=\int_0^1 g(x)dx\Big(f(c_1)-\int_0^{c_1} f(x)dx\Big)$$

and equivalent to

$$\int_0^1 f(x)dx\cdot(Tg)(c_1)=\int_0^1 g(x)dx\cdot(Tf)(c_1).$$

For $h_2(t)=(t-1)f(t)$, with $f:[0,1]\to\mathbb{R}$ is a continuous function, we apply lemma 2.2 and we obtain

$$(c_2-1)f(c_2)=\int_0^{c_2}(t-1)f(x)dx$$

which is equivalent to

$$c_2f(c_2)-\int_0^{c_2} xf(x)dx=f(c_2)-\int_0^{c_2} f(x)dx;$$

that is $(Tf)(c_2)=(Sf)(c_2)$.

For the last assertion we do the same thing. We put $h_3(t)=f(t)\int_0^1 g(x)dx-g(t)\int_0^1 f(x)dx$, where $f,g:[0,1]\to\mathbb{R}$ are continuous functions. So, applying the remark 2.5 from Lemma 2.4, we conclude that

$$\begin{aligned}&\tilde{c_3}f(\tilde{c_3})\int_0^1 g(x)dx-\tilde{c_3}g(\tilde{c_3})\int_0^1 f(x)dx\\&=\int_0^{\tilde{c_3}} xf(x)dx\int_0^1 g(x)dx-\int_0^{\tilde{c_3}} xg(x)dx\int_0^1 f(x)dx\end{aligned}$$

which is equivalent to

$$\int_0^1 f(x)dx\Big(\tilde{c_3}g(\tilde{c_3})-\int_0^{\tilde{c_3}} xg(x)dx\Big)=\int_0^1 g(x)dx\Big(\tilde{c_3}f(\tilde{c_3})-\int_0^{\tilde{c_3}} xf(x)dx\Big)$$

or

$$\int_0^1 f(x)dx\cdot(Sg)(\tilde{c_3})=\int_0^1 g(x)dx\cdot(Sf)(\tilde{c_3}).$$

□

In connection with the operators T and S defined in Theorem 2.9, we shall also prove the following result.

Theorem 2.12. *Let $T,S\in(C[0,1])$ be the operators defined as in Theorem 2.9; namely for two continuous functions $\varphi,\psi:[0,1]\to\mathbb{R}$, define*

$$\begin{aligned}(T\varphi)(t)&=\varphi(t)-\int_0^t\varphi(x)dx,\\(S\psi)(t)&=t\psi(t)-\int_0^t x\psi(x)dx.\end{aligned}$$

Let $f, g : [0,1] \to \mathbb{R}$ be two continuous functions. Then there exist $c_7, \tilde{c_7} \in (0,1)$ such that

$$\int_0^1 (1-x)f(x)dx \cdot (Tg)(c_7) = \int_0^1 (1-x)g(x)dx \cdot (Tf)(c_7),$$
$$\int_0^1 (1-x)f(x)dx \cdot (Sg)(\tilde{c_7}) = \int_0^1 (1-x)g(x)dx \cdot (Sf)(\tilde{c_7}).$$

Proof. We put

$$h_7(t) = f(t)\int_0^1 (1-x)g(x)dx - g(t)\int_0^1 (1-x)f(x)dx,$$

where $f, g : [0,1] \to \mathbb{R}$ are continuous functions and if we apply Theorem 2.9, we get

$$\begin{aligned} &f(c_7)\int_0^1 (1-x)g(x)dx - g(c_7)\int_0^1 (1-x)f(x)dx \\ &= \int_0^{c_7} f(x)\int_0^1 (1-x)g(x)dx - \int_0^{c_7} g(x)dx\int_0^1 (1-x)f(x)dx \end{aligned}$$

which is equivalent to

$$\int_0^1 (1-x)f(x)dx\Big(g(c_7) - \int_0^{c_7} g(x)dx\Big) = \int_0^1 (1-x)g(x)dx\Big(f(c_7) - \int_0^{c_7} f(x)dx\Big)$$

or

$$\int_0^1 (1-x)f(x)dx \cdot (Tg)(c_7) = \int_0^1 (1-x)g(x)dx \cdot (Tf)(c_7).$$

For the second part, let us define the function $\tilde{h_7} : [0,1] \to \mathbb{R}$ given by

$$\tilde{h_7}(t) = tf(t)\int_0^1 (1-x)g(x)dx - tg(t)\int_0^1 (1-x)f(x)dx.$$

Again, by Theorem 2.9 there exists $\tilde{c_7} \in (0,1)$ such that

$$\begin{aligned} &\tilde{c_7}f(\tilde{c_7})\int_0^1 (1-x)g(x)dx - \tilde{c_7}g(\tilde{c_7})\int_0^1 (1-x)f(x)dx \\ &= \int_0^{\tilde{c_7}} xf(x)\int_0^1 (1-x)g(x)dx - \int_0^{\tilde{c_7}} xg(x)dx\int_0^1 (1-x)f(x)dx \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_0^1 (1-x)f(x)dx\Big(\tilde{c_7}g(\tilde{c_7}) - \int_0^{\tilde{c_7}} xg(x)dx\Big) \\ &= \int_0^1 (1-x)g(x)dx\Big(\tilde{c_7}f(\tilde{c_7}) - \int_0^{\tilde{c_7}} xf(x)dx\Big) \end{aligned}$$

or

$$\int_0^1 (1-x)f(x)dx \cdot (Sg)(\tilde{c_7}) = \int_0^1 (1-x)g(x)dx \cdot (Sf)(\tilde{c_7}).$$

□

Next, we prove two theorems of the same type for other two operators. Mainly, we concentrate on the following two theorems.

Theorem 2.13. *For two differentiable functions $\xi, \rho : [0,1] \to \mathbb{R}$, with continuous derivatives, we define the operators $R, V \in (C^1([0,1]))$:*

$$(R\xi)(t) = \xi(t) - \xi'(t)\int_0^t \xi(x)dx,$$
$$(V\rho)(t) = t\rho(t) - \rho'(t)\int_0^t x\rho(x)dx.$$

Let $f, g : [0,1] \to \mathbb{R}$ be two differentiable functions with their derivatives being continuous. Then there exist $c_3, \bar{c_4} \in (0,1)$ such that

$$\int_0^1 f(x)dx \cdot (Rg)(c_3) = \int_0^1 g(x)dx \cdot (Rf)(c_3),$$
$$\int_0^1 f(x)dx \cdot (Vg)(\bar{c_4}) = \int_0^1 g(x)dx \cdot (Vf)(\bar{c_4}).$$

Proof. We put $h_3(t) = f(t)\int_0^1 g(x)dx - g(t)\int_0^1 f(x)dx$, where $f, g : [0,1] \to \mathbb{R}$ are continuous functions and if we apply Lemma 2.3, we get

$$\begin{aligned} &f(c_3)\int_0^1 g(x)dx - g(c_3)\int_0^1 f(x)dx \\ &= f'(c_3)\int_0^{c_3} f(x)dx\int_0^1 g(x)dx - g'(c_3)\int_0^{c_3} g(x)dx\int_0^1 f(x)dx, \end{aligned}$$

which is equivalent to

$$\int_0^1 f(x)dx\Big(g(c_3) - g'(c_3)\int_0^{c_3} g(x)dx\Big) = \int_0^1 g(x)dx\Big(f(c_3) - f'(c_3)\int_0^{c_3} f(x)dx\Big)$$

or

$$\int_0^1 f(x)dx \cdot (Rg)(c_3) = \int_0^1 g(x)dx \cdot (Rf)(c_3).$$

Now, for the second part we consider $h_4(t) = f(t)\int_0^1 g(x)dx - g(t)\int_0^1 f(x)dx$ and we apply Remark 2.6 from Lemma 2.4. In this case, there exists $\bar{c_4} \in (0,1)$ such that

$$\begin{aligned} &\bar{c_4}f(\bar{c_4})\int_0^1 g(x)dx - \bar{c_4}g(\bar{c_4})\int_0^1 f(x)dx \\ &= f'(\bar{c_4})\int_0^{\bar{c_4}} f(x)dx\int_0^1 g(x)dx - g'(\bar{c_4})\int_0^{\bar{c_4}} g(x)dx\int_0^1 f(x)dx, \end{aligned}$$

which is equivalent to

$$\int_0^1 f(x)dx\Big(g(c_3) - g'(\bar{c_4})\int_0^{\bar{c_4}} g(x)dx\Big) = \int_0^1 g(x)dx\Big(f(\bar{c_4}) - f'(\bar{c_4})\int_0^{\bar{c_4}} f(x)dx\Big)$$

or

$$\int_0^1 f(x)dx \cdot (Vg)(\bar{c_4}) = \int_0^1 g(x)dx \cdot (Vf)(\bar{c_4}).$$

□

Finally, based on the some ideas we used so far, we prove the following theorem.

Theorem 2.14. *For two differentiable functions, $\xi, \rho : [0,1] \to \mathbb{R}$, with continuous derivatives we define the operators $R, V \in (C^1([0,1]))$:*

$$(R\xi)(t) = \xi(t) - \xi'(t) \int_0^t \xi(x)dx,$$
$$(V\rho)(t) = t\rho(t) - \rho'(t) \int_0^t x\rho(x)dx.$$

Let $f, g : [0,1] \to \mathbb{R}$ be two differentiable functions with continuous derivatives. Then there exist $c_8, \tilde{c_8} \in (0,1)$ such that

$$\int_0^1 (1-x)f(x)dx \cdot (Rg)(c_8) = \int_0^1 (1-x)g(x)dx \cdot (Rf)(c_8),$$
$$\int_0^1 (1-x)f(x)dx \cdot (Vg)(\tilde{c_8}) = \int_0^1 (1-x)g(x)dx \cdot (Vf)(\tilde{c_8}).$$

Proof. We put $h_8(t) = f(t)\int_0^1 (1-x)g(x)dx - g(t)\int_0^1 (1-x)f(x)dx$, where $f, g : [0,1] \to \mathbb{R}$ are continuous functions and if we apply the first part of Theorem 2.10, there exists $c_8 \in (0,1)$ such that

$$\begin{aligned} &f(c_8)\int_0^1 (1-x)g(x)dx - g(c_8)\int_0^1 (1-x)f(x)dx \\ &= f'(c_8)\int_0^{c_8} f(x)dx \int_0^1 (1-x)g(x)dx - g'(c_8)\int_0^{c_8} g(x)dx \int_0^1 (1-x)f(x)dx, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_0^1 (1-x)f(x)dx\Big(g(c_8) - g'(c_8)\int_0^{c_8} g(x)dx\Big) \\ &= \int_0^1 (1-x)g(x)dx\Big(f(c_8) - f'(c_8)\int_0^{c_8} f(x)dx\Big) \end{aligned}$$

and equivalent to

$$\int_0^1 (1-x)f(x)dx \cdot (Tg)(c_8) = \int_0^1 (1-x)g(x)dx \cdot (Tf)(c_8).$$

For the second part, again we consider the function

$$h_8(t) = f(t)\int_0^1 (1-x)g(x)dx - g(t)\int_0^1 (1-x)f(x)dx,$$

where $f, g : [0,1] \to \mathbb{R}$ are continuous functions. So, applying the second part of the Theorem 2.10, we conclude that there exists $\tilde{c_8} \in (0,1)$ such that

$$\begin{aligned} &\tilde{c_8} f(\tilde{c_8})\int_0^1 (1-x)g(x)dx - \tilde{c_8} g(\tilde{c_8})\int_0^1 (1-x)f(x)dx \\ &= f'(\tilde{c_8})\int_0^{\tilde{c_8}} xf(x)dx \int_0^1 (1-x)g(x)dx - g'(\tilde{c_8})\int_0^{\tilde{c_8}} xg(x)dx \int_0^1 (1-x)f(x)dx \end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_0^1 (1-x)f(x)dx\Big(\tilde{c_8}g(\tilde{c_8}) - g'(\tilde{c_8})\int_0^{\tilde{c_8}} xg(x)dx\Big)\\&= \int_0^1 (1-x)g(x)dx\Big(\tilde{c_8}f(\tilde{c_8}) - f'(\tilde{c_8})\int_0^{\tilde{c_8}} xf(x)dx\Big).\end{aligned}$$

Therefore,

$$\int_0^1 (1-x)f(x)dx \cdot (Sg)(\tilde{c_8}) = \int_0^1 (1-x)g(x)dx \cdot (Sf)(\tilde{c_8}).$$

□

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