

**INFINITE MULTIPLICITY OF POSITIVE SOLUTIONS FOR
SINGULAR NONLINEAR ELLIPTIC EQUATIONS WITH
CONVECTION TERM AND RELATED SUPERCRITICAL
PROBLEMS**

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ABSTRACT. In this article, we consider the singular nonlinear elliptic problem

$$\begin{aligned} -\Delta u &= g(u) + h(\nabla u) + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Under suitable assumptions on g , h , f and Ω that allow a singularity of g at the origin, we obtain infinite multiplicity results. Moreover, we state infinite multiplicity results for related boundary blow up supercritical problems and for supercritical elliptic problems with Dirichlet boundary condition.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In 1869, Lane [19] introduced the equation

$$-\Delta u = u^p \tag{1.1}$$

for p a nonnegative real number and $u > 0$ in a Ball of radius R in \mathbb{R}^3 , with Dirichlet boundary conditions. Lane was interested in computing both the temperature and the density of mass on the surface of the sun. Today the problem (1.1) is named Lane-Emden-Fowler equation [9, 11]. Singular Lane-Emden-Fowler equations ($p < 0$) has been considered in a remarkable pioneering paper by Fulks and Maybe [12]. Nonlinear singular elliptic equations arise in applications, for example in glacial advance [26], ecology [13], in transport of coal slurries down conveyor belts [7], micro-electromechanical system device [10] etc.

Nonlinear singular elliptic equations have been studied intensively during the last 40 years, for a detailed review out of our scope in this article, see Hernández and Mancebo [21], and the recent book by Ghergu and Rădulescu [17]. Multiplicity is a question with few results. Apparently, the first multiplicity result for the problem

$$\begin{aligned} -\Delta u &= K(x)u^{-p} + u^q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

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where Ω is smooth bounded domain and

$$0 < p < 1 < q \leq \frac{N+2}{N-2},$$

was obtained by Yijing et al. [27] using variational methods. Similar results concerning the existence of at least two solutions $K(x) \equiv \lambda$, it can be encountered in [20, 23, 28] with similar restrictions on p, q . A related multiplicity result is stated by Adimurthi and Giacomoni [1] for singular critical problems in domains of \mathbb{R}^2 , allowing $0 < p < 3$. In dimension $N = 1$ results on multiplicity can be found, for example, in Agarwal and O'Reagan [5]. For strong singularities, Aranda and Godoy [3] stated the following theorem.

Theorem 1.1 ([3]). *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $g : (0, \infty) \rightarrow (0, \infty)$ is non increasing locally Hölder continuous function (that may be singular at the origin) ;
- (2) f is locally Hölder continuous, $\inf_{s>0} f(s)/s > 0$ and $\lim_{s \rightarrow \infty} f(s)/s^p < \infty$ for some $p \in (1, \frac{N}{N-2}]$;
- (3) Ω is a strictly convex domain in \mathbb{R}^N .

Then the problem

$$\begin{aligned} -\Delta u &= g(u) + \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has at least two positive solutions for λ positive and small enough and that $\lambda = 0$ is a bifurcation point from infinity for this problem.

Our first result in this article is as follows.

Theorem 1.2. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $g : (0, \infty) \rightarrow (0, \infty)$ is non increasing locally Hölder continuous function (that may be singular at the origin);
- (2) f is continuous, nonnegative and non decreasing function with $f(0) = 0$;
- (3) $f(\xi_i) \geq \beta \xi_i$, $f(\eta_i) \leq \alpha \eta_i$ with

$$\xi_1 < \eta_1 < \dots < \xi_i < \eta_i < \xi_{i+1} < \dots < \xi_m, \quad m \leq \infty;$$

- (4) $\beta C(\Omega)(\int_K \varphi_1) \varphi_1 \geq 1$, on $K \subset \Omega$ compact where φ_1 , λ_1 are the principal eigenfunction an principal eigenvalue of the operator $-\Delta$ ($-\Delta \varphi_1 = \lambda_1 \varphi_1$) with Dirichlet boundary conditions;
- (5) $v + \alpha \eta_i e \leq \eta_i$, where

$$\begin{aligned} -\Delta v &= g(v) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta e &= 1 \quad \text{in } \Omega, \\ e &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then the problem

$$\begin{aligned} -\Delta u &= g(u) + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

has $m \leq \infty$ nonnegative classical solutions. Moreover the problem

$$\begin{aligned} -\Delta u &= g(u) + f(u) \quad \text{in } \Omega, \\ u &= \epsilon \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

has $2m - 1 \leq \infty$ nonnegative classical solutions for all $\epsilon > 0$.

The behavior of the function f in Theorem 1.2 is closely related to a similar nonlinearity studied by Kielhöfer and Maier in [24]. Under our best knowledge this is the first result on infinite multiplicity for nonlinear singular equations. Hernández, Mancebo and Vega obtained the following theorem.

Theorem 1.3 ([22]). *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold: $-1 < q < 1$, $p < q$ and $\lambda > 0$. Then the problem*

$$\begin{aligned} -\Delta u &= \lambda u^{-q} - u^{-p} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

has a unique nonnegative classical solution.

Our second Theorem is related to multiplicity of a nonlinear eigenvalue problem.

Theorem 1.4. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $0 < p < q$;
- (2) f is continuous, nonnegative and non decreasing function with $f(0) = 0$;
- (3) $f(\xi_i) \geq \beta\xi_i$, $f(\eta_i) \leq \alpha\eta_i$ with

$$\xi_1 < \eta_1 < \cdots < \xi_i < \eta_i < \xi_{i+1} < \cdots < \xi_m \leq \left(\frac{q\lambda}{p}\right)^{\frac{1}{q-p}}$$

- (4) $\beta C(\Omega)(\int_K \varphi_1)\varphi_1 \geq 1$, on $K \subset \Omega$ compact;
- (5) $\lambda^{\frac{1}{q+1}}v + \alpha\eta_i e \leq \eta_i$, where

$$\begin{aligned} -\Delta v &= v^{-q} \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta e &= 1 \quad \text{in } \Omega, \\ e &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- (6) $\lambda^{\frac{1}{1+q}}\|v\|_\infty \leq \left(\frac{q\lambda}{p}\right)^{\frac{1}{q-p}}$.

Then the problem

$$\begin{aligned} -\Delta u &= \lambda u^{-q} - u^{-p} + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

has m nonnegative classical solutions. Moreover the problem

$$\begin{aligned} -\Delta u &= \lambda u^{-q} - u^{-p} + f(u) \quad \text{in } \Omega, \\ u &= \epsilon \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

has $2m - 1$ nonnegative classical solutions for all $\epsilon > 0$ small enough.

Existence and nonexistence results for singular nonlinear elliptic equations with convection term have been stated by Zhang [29], Zhang and Yu [30], Ghergu and Rădulescu [15, 16]. Multiplicity for singular Lane-Emden-Fowler equation with convection term is a topic essentially open. A result was stated by Aranda and Lami Dozo in [4]:

Theorem 1.5 ([4]). *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $0 < p < \frac{1}{N}$, $1 < q < \frac{N+1}{N-1}$ and $0 < s < \frac{2}{N}$;
- (2) $w \in L^\infty(\Omega)$, $w > 0$;
- (3) $0 \leq \nu < C \left\{ \frac{\int_\Omega w \varphi_1 dx \int_\Omega \varphi_1^2 dx}{\int_\Omega \varphi_1 dx} \right\}^{p-1}$ where φ_1 , λ_1 are the principal eigenfunction an principal eigenvalue of the operator $-\Delta$ ($-\Delta \varphi_1 = \lambda_1 \varphi_1$) with Dirichlet boundary conditions and C is a constant depending only in Ω , q , λ_1 .

Then there exist $0 < \lambda^{**} \leq \lambda^* < \infty$ such that for all $\lambda \in (0, \lambda^{**})$, the problem

$$\begin{aligned} -\Delta u &= u^{-p} + \lambda(u^q + \nu|\nabla u|^s) + w(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

admits at least two solutions and no solutions for $\lambda > \lambda^*$. Furthermore there is bifurcation at infinity at $\lambda = 0$.

Our third result in this article, it is concerned with infinite multiplicity for nonlinear elliptic equations with strong singularity and convection term.

Theorem 1.6. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) g and f satisfies conditions (1)–(4) of Theorem 1.2;
- (2) h is a locally Hölder continuous function on \mathbb{R}^N and $0 \leq h(\nabla u) \leq b_1|\nabla u|^s + b_0$, $0 < s < 1$;
- (3) $\eta_i \geq b_1(\eta_i \|\nabla e\|_{L^\infty(\Omega)})^s + b_0 + g(\epsilon) + \alpha \eta_i$ for all i , where

$$\begin{aligned} -\Delta e &= 1 \quad \text{in } \Omega, \\ e &= \epsilon \quad \text{on } \partial\Omega. \end{aligned}$$
- (4) $\epsilon + \exp(d(\Omega)) \leq 2$ where $d(\Omega)$ is the distance between two parallel planes containing Ω .

Then the problem

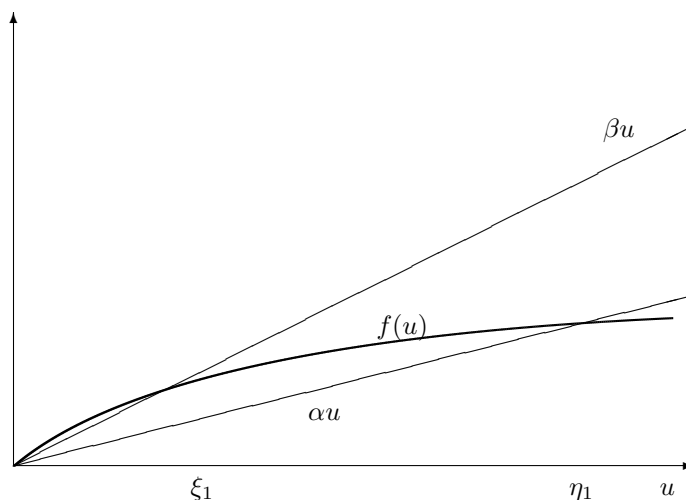
$$\begin{aligned} -\Delta u &= g(u) + h(\nabla u) + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.8}$$

has $m \leq \infty$ nonnegative classical solutions.

Remark 1.7. Condition (4) indicates a deep relation between the domain, the convection term and multiplicity.

The existence of at least three solutions, for singular nonlinear elliptic problems, using variational methods is a difficult task. Next, we apply a classical compensated compactness technique from the calculus of variations, for derive our fourth Theorem.

Theorem 1.8. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$ and $0 < p < 1$. Suppose the following conditions hold:*

FIGURE 1. Double resonance of f

- (1) f is continuous, nonnegative and non decreasing function with $f(0) = 0$;
 (2) $f(\xi_i) \geq \beta\xi_i$, $f(\eta_i) \leq \alpha\eta_i$ with

$$\xi_1 < \eta_1 < \cdots < \xi_i < \eta_i < \xi_{i+1} < \cdots < \xi_m;$$

- (3) With κ , we denote the indicator function of a compact set $K \subset \Omega$. We assume that the problem

$$\begin{aligned} -\Delta u + u|\nabla u|^2 &= \beta\xi_i\kappa && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has at least a solution $u \in \mathcal{W}^{2,r}(\Omega)$, $r > N$ with $u \geq \xi_i\kappa$;

- (4) $v(x) + \alpha\eta_i e(x) < \eta_i$ for all $x \in \bar{\Omega}$, where

$$\begin{aligned} -\Delta v &= v^{-p} && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta e &= 1 && \text{in } \Omega, \\ e &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then the problem

$$\begin{aligned} -\Delta u + u|\nabla u|^2 &= u^{-p} + f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

has m solutions in $H_0^{1,2}(\Omega)$.

Remark 1.9. Condition (3) indicates again a complex relation between domain, convection term and multiplicity.

For large solutions Ghergu et al. stated the following result.

Theorem 1.10 ([14]). *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $f \in C^1[0, \infty)$, $f' \geq 0$, $f(0) = 0$ and $f > 0$ on $(0, \infty)$;
- (2) $\int_1^\infty [F(t)]^{-2/a} dt < \infty$, where
- (3) $\frac{F(t)}{f^{2/a}} \rightarrow 0$ as $t \rightarrow 0$;
- (4) $\mathbf{p}, \mathbf{q} \in C^{0,\gamma}(\overline{\Omega})$ are nonnegative functions such that for every $x_0 \in \Omega$ with $\mathbf{p}(x_0) = 0$, there exists a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega}_0 \subset \Omega$ and $\mathbf{p} > 0$ on $\partial\Omega_0$;
- (5) $0 < a < 2$.

Then the problem

$$\begin{aligned} \Delta u + \mathbf{q}(x)|\nabla u|^a &= \mathbf{p}(x)f(u) && \text{in } \Omega, \\ u &= \infty && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

has a nonnegative solution.

Related to the above Theorem, we have our fifth result:

Theorem 1.11. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose: $f(s) = s^2 \tilde{f}(\frac{1}{s})$, satisfies (2)–(5) of Theorem 1.2 with $g(u) = u^{2-p}$, $p > 2$. Then the problem*

$$\begin{aligned} \Delta v &= \frac{2}{v} |\nabla v|^2 + v^p + \tilde{f}(u) && \text{in } \Omega, \\ u &= \infty && \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

has $m \leq \infty$ nonnegative classical solutions. Moreover the problem

$$\begin{aligned} \Delta v &= \frac{2}{v} |\nabla v|^2 + v^p + \tilde{f}(u) && \text{in } \Omega, \\ u &= M && \text{on } \partial\Omega, \end{aligned} \tag{1.12}$$

has $2m - 1 \leq \infty$ nonnegative classical solutions for all $M > 0$ big enough.

Theorem 1.12. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $f(s) = s^2 \tilde{f}(\frac{1}{s})$ satisfies (2)–(4) of Theorem 1.2;
- (2) $2 < q < p$;
- (3) $0 < \epsilon \leq (\frac{q-2}{p-2})^{\frac{1}{p-q}}$;
- (4) $v + \alpha \eta_i e \leq \eta_i$, where

$$\begin{aligned} -\Delta e &= 1 && \text{in } \Omega, \\ e &= \frac{1}{\epsilon} && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta v &= v^{2-q} - v^{2-p} && \text{in } \Omega, \\ v &= \frac{1}{\epsilon} && \text{on } \partial\Omega. \end{aligned}$$

Then the problem

$$\begin{aligned} -\Delta z + \frac{2}{z} |\nabla z|^2 + \tilde{f}(z) + z^q &= z^p && \text{in } \Omega, \\ z &= \epsilon && \text{on } \partial\Omega, \end{aligned} \tag{1.13}$$

has $2m - 1 \leq \infty$ nonnegative classical solutions.

2. PRELIMINARIES

It is our purpose in this section to prove some preliminary results. Let us start with some known facts about the Laplacian operator and solution properties of nonlinear singular elliptic equations.

Lemma 2.1 ([8, 6, Uniform Hopf maximum principle.]). *Let Ω be a smooth and bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose that*

$$\begin{aligned} -\Delta u &= h && \text{on } \Omega, \\ u &= 0 && \text{in } \partial\Omega, \end{aligned}$$

with $u \in \mathcal{W}_0^{1,1}(\Omega)$ and $h \geq 0$, $h \in L^1(\Omega)$. Then

$$u \geq C \left(\int_{\Omega} h \varphi_1 \right) \varphi_1,$$

where $C = C(\Omega)$ depends only on Ω and

$$\begin{aligned} -\Delta \varphi_1 &= \lambda_1 \varphi_1 && \text{in } \Omega, \\ \varphi_1 &> 0 && \text{in } \Omega, \\ \varphi_1 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Remark 2.2. The proof of Lemma 2.1 given by Brezis and Cabre in [6] relies on the superharmonicity of the laplacian operator.

Theorem 2.3 ([3]). *Let P be the positive cone in $L^\infty(\Omega)$. Let $S_\epsilon : P \rightarrow P$ be the solution operator for the problem*

$$\begin{aligned} -\Delta u &= g(u) + w && \text{in } \Omega, \\ u &= \epsilon && \text{on } \partial\Omega, \end{aligned}$$

gives $S_\epsilon(w) = u$ where $\epsilon \geq 0$ and $g : (0, \infty) \rightarrow (0, \infty)$ is nonincreasing locally Hölder continuous function (that may be singular at the origin). Then $S_\epsilon : P \rightarrow P$ is a continuous, non decreasing and compact map with $S_{\epsilon_0}(w) \leq S_{\epsilon_1}(w)$ for $\epsilon_0 < \epsilon_1$.

Lemma 2.4. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions of the problem*

$$\begin{aligned} -\Delta u - g(u) - h(\nabla u) &\geq -\Delta v - g(v) - h(\nabla v) && \text{in } \Omega, \\ u &\geq v \geq 0 && \text{on } \partial\Omega. \end{aligned}$$

Then $u \geq v$ on Ω .

Proof. Indeed suppose $v > u$ somewhere and consider the non empty open set

$$\Omega_\delta = \{x \in \Omega | v(x) > u(x) + \delta, \delta > 0\}.$$

Since $u, v \in C^2(\Omega)$, we have

$$\begin{aligned} -\Delta(u + \delta) - h(\nabla(u + \delta)) &= g(u) + q \\ &\geq g(v) + r \\ &= -\Delta v - h(\nabla v) && \text{on } \Omega_\delta, \end{aligned}$$

with $q, r \in C(\overline{\Omega_\delta})$ and $\overline{\Omega_\delta} \subset \Omega$. Also $u + \delta = v$ on $\partial\Omega_\delta$ and so the comparison Theorem 10.1 [18] implies $u + \delta \geq v$ on Ω_δ . It follows $\Omega_\delta = \emptyset$ a contradiction. \square

Lemma 2.5. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $g : (0, \infty) \rightarrow (0, \infty)$ is non increasing locally Hölder continuous function (that may be singular at the origin);
- (2) h is locally Hölder continuous function on \mathbb{R}^N with $0 \leq h(\nabla u) \leq b_1 |\nabla u|^s + b_0$, $0 < s < 1$;
- (3) w is a nonnegative locally Hölder continuous function on $\bar{\Omega}$.

Then the problem

$$\begin{aligned} -\Delta u &= g(u) + h(\nabla u) + w(x) \quad \text{in } \Omega, \\ u &= \epsilon \geq 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

has a unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Proof. Let $g_j : \mathbb{R} \rightarrow \mathbb{R}$ be a non increasing and locally Hölder continuous function defined by

$$g_j(s) = \begin{cases} g(s) & \text{if } s \geq \frac{1}{j}, \\ C_j & \text{if } s \leq \frac{1}{j+1}. \end{cases}$$

Using [18, Theorem 15.11], the problem

$$\begin{aligned} -\Delta u &= g_j(u) + h(\nabla u) + w(x) \quad \text{on } \Omega, \\ u &= \epsilon > 0 \quad \text{in } \partial\Omega, \end{aligned}$$

has a classical solution. From

$$\begin{aligned} \Delta u_{j-1} + g_j(u_{j-1}) + h(\nabla u_{j-1}) + w(x) &\geq \Delta u_{j-1} + g_{j-1}(u_{j-1}) + h(\nabla u_{j-1}) + w(x) \\ &= 0 \\ &= \Delta u_j + g_j(u_j) + h(\nabla u_j) + w(x) \quad \text{in } \Omega, \end{aligned}$$

$u_{j-1} = u_j = \epsilon$ on $\partial\Omega$, using [18, Theorem 10.1], we infer $u_{j-1} \leq u_j$ in Ω . Therefore for j big enough there exists a unique $u_\epsilon = u_j$ solution of

$$\begin{aligned} -\Delta u_\epsilon &= g(u_\epsilon) + h(\nabla u_\epsilon) + w(x) \quad \text{in } \Omega, \\ u_\epsilon &= \epsilon \quad \text{on } \partial\Omega. \end{aligned}$$

If $\epsilon_0 < \epsilon_1$, for j big enough, we have

$$\Delta u_{\epsilon_0} + g_j(u_{\epsilon_0}) + h(\nabla u_{\epsilon_0}) + w(x) = \Delta u_{\epsilon_1} + g_j(u_{\epsilon_1}) + h(\nabla u_{\epsilon_1}) + w(x),$$

on Ω , $u_{\epsilon_0} < u_{\epsilon_1}$ in $\partial\Omega$, using Theorem 10.1 [18], we deduce $u_{\epsilon_0} < u_{\epsilon_1}$ on Ω . From the inequalities

$$\begin{aligned} -\Delta u_\epsilon &= g(u_\epsilon) + h(\nabla u_\epsilon) + w(x) \\ &\geq g(u_1) \\ &= -\Delta \mathcal{M} \quad \text{in } \Omega, \\ u_\epsilon &= \epsilon > 0 = \mathcal{M} \quad \text{on } \partial\Omega, \end{aligned}$$

we obtain $u_\epsilon > r$ on Ω . [18, Theorem 15.8] implies

$$-\Delta u_\epsilon = g(u_\epsilon) + h(\nabla u_\epsilon) + w(x) \leq g(\mathcal{M}) + C,$$

on $\bar{\Omega}' \subset \Omega$. Using [18, Theorem 9.11], we have

$$\|u_\epsilon\|_{\mathcal{W}^{2,r}(\Omega')} \leq C(\|u_\epsilon\|_{L^r(\Omega')} + C) \leq C(\|u_1\|_{L^r(\Omega')} + C),$$

with $r > N$. By the Sobolev imbedding [18, Theorem 7.26] $u_\epsilon \rightarrow u$, in $C^{1,\gamma}(\Omega')$. A standard bootstrap argument implies that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a classical solution of problem (2.1). The unicity follows from Lemma 2.4. \square

Lemma 2.6. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose the following conditions hold:*

- (1) $g : (0, \infty) \rightarrow (0, \infty)$ is non increasing locally Hölder continuous function (that may be singular at the origin);
- (2) h is locally Hölder continuous function on \mathbb{R}^N with $0 \leq h(\nabla u) \leq b_1 |\nabla u|^s + b_0$, $0 < s < 1$;
- (3) w is a nonnegative locally Hölder continuous function on Ω and continuous on $\bar{\Omega}$.

Then the problem

$$\begin{aligned} -\Delta u &= g(u) + h(\nabla u) + w(x) && \text{in } \Omega, \\ u &= \epsilon \geq 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

has a unique solution $u_\epsilon \in C^2(\Omega) \cap C(\bar{\Omega})$. Moreover if $0 < \epsilon$ then $u_0 < u_\epsilon$.

Proof. Let $w_\vartheta : \Omega \rightarrow \mathbb{R}^+$ be a nonnegative locally Hölder continuous function on $\bar{\Omega}$ defined by

$$w_\vartheta(x) = \begin{cases} w(x) & \text{if } d(x, \partial\Omega) \geq \vartheta, \\ 0 & \text{if } d(x, \partial\Omega) \leq \frac{\vartheta}{2}. \end{cases}$$

Let us consider the problems

$$\begin{aligned} -\Delta u_\vartheta &= g(u_\vartheta) + h(\nabla u_\vartheta) + w_\vartheta(x) && \text{in } \Omega, \\ u_\vartheta &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta u_{\epsilon,\vartheta} &= g(u_\vartheta) + h(\nabla u_\vartheta) + w_\vartheta(x) && \text{in } \Omega, \\ u_{\epsilon,\vartheta} &= \epsilon && \text{on } \partial\Omega. \end{aligned}$$

Let us suppose that $w_{\vartheta_1} < w_{\vartheta_0}$ if $\vartheta_0 < \vartheta_1$. Using Lemma 2.4, we obtain $u_{\epsilon,\vartheta_1} < u_{\epsilon,\vartheta_0}$ and $u_{\vartheta_1} < u_{\vartheta_0}$ on Ω . By construction, we know that $u_\vartheta \leq u_{\epsilon,\vartheta}$. By Lemma 2.5, the problem

$$\begin{aligned} -\Delta v_\epsilon &= g(v_\epsilon) + h(\nabla v_\epsilon) + \|w\|_{L^\infty(\Omega)} && \text{on } \Omega, \\ v_\epsilon &= \epsilon && \text{in } \partial\Omega, \end{aligned}$$

has a unique classical solution v_ϵ . Using the comparison [18, Theorem 10.1], we conclude that $u_{\epsilon,\vartheta} \leq v_\epsilon$. It follows that $u_\vartheta(x) \nearrow u_0(x) \leq v_\epsilon(x)$ for all $x \in \Omega$ and for $\vartheta \searrow 0$.

Using [18, Theorem 15.8], the standard bootstrap argument and Lemma 2.4, we infer that the problem (2.2) with $\epsilon = 0$ has a unique solution $u_0 \in C^2(\Omega) \cap C(\bar{\Omega})$.

Similar arguments also yield $u_{\epsilon,\vartheta}(x) \nearrow u_\epsilon(x) \leq v_\epsilon(x)$, for all $x \in \Omega$ and for $\vartheta \searrow 0$. Therefore, we get that the problem (2.2) has a unique solution $u_\epsilon \in C^2(\Omega) \cap C(\bar{\Omega})$, for $\epsilon > 0$. \square

Lemma 2.7. *Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$. Suppose that $0 < p < 1$, $0 < \delta$ and $0 \leq w(x) \in L^\infty(\Omega)$. Then the problem*

$$\begin{aligned} -\Delta u_{\epsilon,\delta} + \frac{u_{\epsilon,\delta} |\nabla u_{j,\epsilon,\delta}|^2}{1 + \delta u_{\epsilon,\delta} |u_{\epsilon,\delta}|^2} &= u_{\epsilon,\delta}^{-p} + w(x) \quad \text{in } \Omega, \\ u_{\epsilon,\delta} &= \epsilon \quad \text{on } \partial\Omega, \end{aligned} \quad (2.3)$$

has a unique solution $u_{\epsilon,\delta} \in \mathcal{W}^{2,r}(\Omega) \cap C(\overline{\Omega})$ for all $r > 1$, satisfying:

- (a) Let P be the positive cone in $L^\infty(\Omega)$. Let $\mathcal{S}_{\epsilon,\delta} : P \rightarrow P$ be the solution operator for the problem (2.3), gives $\mathcal{S}_{\epsilon,\delta}(w) = u_{\epsilon,\delta}$. Then $\mathcal{S}_{\epsilon,\delta} : P \rightarrow P$ is continuous, compact and non decreasing map.
- (b) If $\epsilon_0 < \epsilon_1$, then $\mathcal{S}_{\epsilon_0,\delta}(w) \leq \mathcal{S}_{\epsilon_1,\delta}(w)$.
- (c) If $\delta_0 < \delta_1$, then $\mathcal{S}_{\epsilon,\delta_0}(w) \leq \mathcal{S}_{\epsilon,\delta_1}(w)$.
- (d) $\inf_{\Omega'} u_{\epsilon,\delta} \geq C$, $\overline{\Omega'} \subset \Omega$, where $C = C(p, \Omega', w(x))$ is a constant independent of ϵ , δ and $C(\alpha, \Omega', 0) > 0$.
- (e) For $0 < \epsilon, \delta < 1$, we have $\|u_{\epsilon,\delta} - \epsilon\|_{H_0^{1,2}(\Omega)} \leq C$, where C is a constant independent of ϵ and δ .

Remark 2.8. Items (a)–(c) contain the monotone and compactness properties of approximate solutions. Item (d) is a uniform Harnack inequality. Item (e) contains a uniform bound necessary for the compensated compactness technique.

Proof of Lemma 2.7. Let $g_j : \mathbb{R} \rightarrow \mathbb{R}$ be a non increasing and locally Hölder continuous function defined by

$$g_j(s) = \begin{cases} s^{-p} & \text{if } s \geq \frac{1}{j}, \\ C_j & \text{if } s \leq \frac{1}{j+1}. \end{cases}$$

Using a standard argument involving L^r estimates [18, Theorem 9.10], Sobolev imbedding [18, Theorem 7.26], [18, Theorem 10.1] and the Schauder fixed point Theorem, we deduce that the problem

$$\begin{aligned} -\Delta u_{j,\epsilon,\delta} + \frac{u_{j,\epsilon,\delta} |\nabla u_{j,\epsilon,\delta}|^2}{1 + \delta u_{j,\epsilon,\delta} |u_{j,\epsilon,\delta}|^2} &= g_j(u_{j,\epsilon,\delta}) + w(x) \quad \text{in } \Omega, \\ u_{j,\epsilon,\delta} &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

has a unique solution $u_{j,\epsilon,\delta} \in \mathcal{W}^{2,r}(\Omega) \cap C(\overline{\Omega})$ for all $r > 1$. If $w \in L^r(\Omega)$, $r > N$, then by [18, Theorem 7.26], $u_{j,\epsilon,\delta} \in \mathcal{W}^{2,r}(\Omega) \hookrightarrow C^{1,\gamma}(\overline{\Omega})$ for some $\gamma > 0$. Calling

$$b_\delta(u, \nabla u) = \frac{u |\nabla u|^2}{1 + \delta u |\nabla u|^2},$$

we deduce

$$\begin{aligned} & -\Delta u_{j,\epsilon,\delta} + b_\delta(u_{j,\epsilon,\delta}, \nabla u_{j,\epsilon,\delta}) - g_{j+1}(u_{j,\epsilon,\delta}) \\ & \leq -\Delta u_{j,\epsilon,\delta} + b_\delta(u_{j,\epsilon,\delta}, \nabla u_{j,\epsilon,\delta}) - g_j(u_{j,\epsilon,\delta}) \\ & = w(x) \\ & = -\Delta u_{j+1,\epsilon,\delta} + b_\delta(u_{j+1,\epsilon,\delta}, \nabla u_{j+1,\epsilon,\delta}) - g_{j+1}(u_{j+1,\epsilon,\delta}) \end{aligned}$$

in Ω and $u_{j+1,\epsilon,\delta} = u_{j,\epsilon,\delta} = \epsilon$ on $\partial\Omega$. Using Theorem 10.1 [18], we obtain that $u_{j+1,\epsilon,\delta} > u_{j,\epsilon,\delta}$ in Ω . Moreover from

$$-\Delta u_{j,\epsilon,\delta} + b_\delta(u_{j+1,\epsilon,\delta}, \nabla u_{j+1,\epsilon,\delta}) = g_j(u_{j,\epsilon,\delta}) + w(x) \geq -\Delta \epsilon + b_\delta(\epsilon, \nabla \epsilon) \quad \text{in } \Omega,$$

and $u_{j,\epsilon,\delta} = \epsilon$ on $\partial\Omega$, using again [18, Theorem 10.1], we conclude $u_{j,\epsilon,\delta} > \epsilon$ on Ω . Letting $u_{\epsilon,\delta} = \lim_{j \rightarrow \infty} u_{j,\epsilon,\delta}$, we have

$$\begin{aligned} -\Delta u_{\epsilon,\delta} + b_\delta(u_{\epsilon,\delta}, \nabla u_{\epsilon,\delta}) &= u_{\epsilon,\delta}^{-p} + w(x) \quad \text{in } \Omega, \\ u_{\epsilon,\delta} &= \epsilon \quad \text{on } \partial\Omega. \end{aligned}$$

Using standard Nemytskii mappings properties and Sobolev Imbedding Theorems, we demonstrate the continuity and compactness of the map $\mathcal{S}_{\delta,\epsilon}$. This states (a).

Comparison [18, Theorem 10.1] implies if $\epsilon_0 < \epsilon_1$ then $\mathcal{S}_{\epsilon_0,\delta}(w) = u_{\epsilon_0,\delta} < u_{\epsilon_1,\delta} = \mathcal{S}_{\epsilon_1,\delta}(w)$ in Ω . This establishes (b).

We demonstrate now (c). We suppose that the set

$$\widehat{\Omega} = \{x \in \Omega : \mathcal{S}_{\epsilon,\delta_1}(w(x)) < \mathcal{S}_{\epsilon,\delta_0}(w(x))\},$$

is nonempty for $\delta_1 > \delta_0 > 0$. It follows that

$$\begin{aligned} &-\Delta \mathcal{S}_{\epsilon,\delta_1}(w) + \frac{\mathcal{S}_{\epsilon,\delta_1}(w)|\nabla \mathcal{S}_{\epsilon,\delta_1}(w)|^2}{1 + \delta_0 \mathcal{S}_{\epsilon,\delta_1}(w)|\nabla \mathcal{S}_{\epsilon,\delta_1}(w)|^2} \\ &\geq -\Delta \mathcal{S}_{\epsilon,\delta_1}(w) + \frac{\mathcal{S}_{\epsilon,\delta_1}(w)|\nabla \mathcal{S}_{\epsilon,\delta_1}(w)|^2}{1 + \delta_1 \mathcal{S}_{\epsilon,\delta_1}(w)|\nabla \mathcal{S}_{\epsilon,\delta_1}(w)|^2} \\ &= (\mathcal{S}_{\epsilon,\delta_1}(w))^{-p} + w(x) \\ &\geq (\mathcal{S}_{\epsilon,\delta_0}(w))^{-p} + w(x) \\ &= -\Delta \mathcal{S}_{\epsilon,\delta_0}(w) + \frac{\mathcal{S}_{\epsilon,\delta_0}(w)|\nabla \mathcal{S}_{\epsilon,\delta_0}(w)|^2}{1 + \delta_0 \mathcal{S}_{\epsilon,\delta_0}(w)|\nabla \mathcal{S}_{\epsilon,\delta_0}(w)|^2} \quad \text{on } \widehat{\Omega}, \end{aligned}$$

and $\mathcal{S}_{\epsilon,\delta_1}(w(x)) = \mathcal{S}_{\epsilon,\delta_0}(w(x))$ on $\partial\widehat{\Omega}$. Using Theorem 10.1 [18], we infer $\mathcal{S}_{\epsilon,\delta_1}(w) > \mathcal{S}_{\epsilon,\delta_0}(w)$ on $\widehat{\Omega}$. This contradiction implies $\mathcal{S}_{\epsilon,\delta_1}(w) \leq \mathcal{S}_{\epsilon,\delta_0}(w)$. We also have

$$\begin{aligned} -\Delta u_{\epsilon,\delta} + b_\delta(u_{\epsilon,\delta}, \nabla u_{\epsilon,\delta}) &\geq -\Delta u_{\epsilon,\delta} + b_\delta(u_{\epsilon,\delta}, \nabla u_{\epsilon,\delta}) - w(x) \\ &= u_{\epsilon,\delta}^{-p} \\ &\geq u_{1,\delta}^{-p} \\ &= -\Delta \omega_\delta + b_\delta(\omega_\delta, \nabla \omega_\delta) \quad \text{in } \Omega, \end{aligned}$$

and $u_{\epsilon,\delta} = \epsilon > 0 = \omega_\delta$ on $\partial\Omega$. Therefore $u_{\epsilon,\delta} > \omega_\delta$ on Ω . By definition

$$\begin{aligned} -\Delta u_{1,\delta} + b_\delta(u_{1,\delta}, \nabla u_{1,\delta}) &= u_{1,\delta}^{-p} + w(x) \quad \text{in } \Omega \\ u_{1,\delta} &= 1 \quad \text{on } \partial\Omega. \end{aligned}$$

So, we have

$$\begin{aligned} -\Delta u_{1,\delta} - u_{1,\delta}^{-p} &\leq w(x) = -\Delta u_1 - u_1^{-p} \quad \text{in } \Omega \\ u_{1,\delta} &= 1 = u_1 \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore, $u_{1,\delta} \leq u_1$ in Ω . Similarly

$$\begin{aligned} -\Delta \omega_\delta + b_\delta(\omega_\delta, \nabla \omega_\delta) &= u_{1,\delta}^{-p} \\ &\geq u_1^{-p} \\ &= -\Delta \mathcal{O}_\delta + b_\delta(\mathcal{O}_\delta, \nabla \mathcal{O}_\delta) \quad \text{in } \Omega, \end{aligned}$$

and $\omega_\delta \geq \mathcal{O}_\delta$ in $\partial\Omega$. Then, we obtain $\omega_\delta \geq \mathcal{O}_\delta$. For $a \in \Omega$, we define

$$\mathcal{V}(x) = C(C - |x - a|^2).$$

It follows that, for C small enough

$$\begin{aligned} -\Delta \mathcal{O}_\delta + b_\delta(\mathcal{O}_\delta, \nabla \mathcal{O}_\delta) &= g(u_1) \\ &\geq C_1 \\ &\geq -\Delta \mathcal{V} + \mathcal{V} |\nabla \mathcal{V}|^2 \\ &\geq -\Delta \mathcal{V} + \frac{\mathcal{V} |\nabla \mathcal{V}|^2}{1 + \delta \mathcal{V} |\nabla \mathcal{V}|^2} \\ &= -\Delta \mathcal{V} + b_\delta(\mathcal{V}, \nabla \mathcal{V}) \quad \text{in } B_{\sqrt{C}}(a) \subset \Omega, \end{aligned}$$

and $\mathcal{O}_\delta \geq 0 = \mathcal{V}$ on $\partial B_{\sqrt{C}}(a)$. Therefore, we deduce $\mathcal{O}_\delta \geq \mathcal{V}$ in $B_{\sqrt{C}}(a)$. We conclude

$$u_{\epsilon, \delta} \geq \omega_\delta \geq \mathcal{O}_\delta \geq \mathcal{V} \quad \text{in } B_{\sqrt{C}}(a) \subset \Omega.$$

This states (d).

Now we consider (e):

$$\begin{aligned} \|u_{\epsilon, \delta} - \epsilon\|_{H_0^{1,2}}^2 &= \int_{\Omega} |\nabla(u_{\epsilon, \delta} - \epsilon)|^2 dx \\ &\leq \int_{\Omega} u_{\epsilon, \delta}^{-p} (u_{\epsilon, \delta} - \epsilon) dx + \int_{\Omega} w(u_{\epsilon, \delta} - \epsilon) dx \\ &\leq \int_{\Omega} (u_{\epsilon, \delta} - \epsilon)^{-p} (u_{\epsilon, \delta} - \epsilon) dx + \|w\|_{H^{-1}} \|u_{\epsilon, \delta} - \epsilon\|_{H_0^{1,2}} \\ &\leq \int_{\Omega} (\mathcal{S}_{1,1}(w) - \epsilon)^{1-p} dx + \|w\|_{H^{-1}} \|u_{\epsilon, \delta} - \epsilon\|_{H_0^{1,2}}. \end{aligned}$$

This states (e). □

3. PROOFS OF MAINS RESULTS

Proof of Theorem 1.2. Let P be the positive cone in $L^\infty(\Omega)$. Let $A : P \rightarrow P$ be the solution operator for the problem

$$\begin{aligned} -\Delta z &= g(z) + f(u) \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

gives $A(u) = z$. Using Theorem 2.3, we infer that $A : P \rightarrow P$ is a well defined, continuous, non decreasing and compact map. Let us to denote with κ the indicator function of the set K . Using hypothesis (3°), we get

$$-\Delta A(\xi_i \kappa) = g(A(\xi_i \kappa)) + f(\xi_i \kappa) \geq \beta \xi_i \kappa.$$

We will denote by $u = (-\Delta)^{-1}h$ the solution operator of the problem $-\Delta u = h$ in Ω and $u = 0$ on $\partial\Omega$. It follows that

$$\begin{aligned} -\Delta(A(\xi_i \kappa) - (-\Delta)^{-1}\beta \xi_i \kappa) &\geq 0 \quad \text{in } \Omega, \\ A(\xi_i \kappa) - (-\Delta)^{-1}\beta \xi_i \kappa &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

From the maximum principle, Lemma 2.1 and (4), we infer

$$A(\xi_i \kappa) \geq (-\Delta)^{-1}\beta \xi_i \kappa \geq \beta \xi_i C(\Omega) \left(\int_{\Omega} \kappa \varphi_1 \right) \varphi_1 \geq \xi_i \kappa.$$

Now $A(0) < A(\eta_i)$ and so, simple calculation yields

$$\begin{aligned} -\Delta A(\eta_i) &= g(A(\eta_i)) + f(\eta_i) \\ &\leq g(A(0)) + \alpha\eta_i \\ &\leq -\Delta(A(0) + \alpha\eta_i(-\Delta)^{-1}1). \end{aligned}$$

It now follows from maximum principle that

$$A(\eta_i) \leq A(0) + \alpha\eta_i(-\Delta)^{-1}1,$$

so that from hypothesis (5), $A(0) + \alpha\eta_i(-\Delta)^{-1}1 \leq \eta_i$. Then $A(\eta_i) \leq \eta_i$, and hence $A[\xi_i\kappa, \eta_i] \subset [\xi_i\kappa, \eta_i]$. Thus, there are a solution $u_i \in [\xi_i\kappa, \eta_i]$ of the fixed point equation $A(u_i) = u_i$. Now by Amann's "three solution Theorem" [2], problem (1.4) has $2m - 1$ solutions. \square

Proof of Theorem 1.4. Let us to define the function

$$g_0(s) = \begin{cases} \lambda s^{-q} - s^{-p} & \text{if } s < \left(\frac{q\lambda}{p}\right)^{\frac{1}{q-p}} \\ \lambda\left(\frac{q\lambda}{p}\right)^{\frac{-q}{q-p}} - \left(\frac{q}{p}\right)^{\frac{-p}{q-p}} & \text{if } s \geq \left(\frac{q\lambda}{p}\right)^{\frac{1}{q-p}}. \end{cases}$$

If v is a solution of

$$\begin{aligned} -\Delta v &= v^{-q} \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then $z = \lambda^{\frac{1}{1+q}}v$, solves

$$\begin{aligned} -\Delta z &= \lambda z^{-q} \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

From $\lambda^{\frac{1}{1+q}}\|v\|_\infty \leq \left(\frac{q\lambda}{p}\right)^{\frac{1}{q-p}}$, it follows that

$$\begin{aligned} -\Delta z - \lambda z^{-q} + z^{-p} &\geq 0 = -\Delta u_0 - g_0(u_0) \quad \text{in } \Omega, \\ z &= 0 = u_0 \quad \text{in } \partial\Omega; \end{aligned}$$

therefore, $z \geq u_0$ on Ω . Observe that, we can apply Theorem 1.2 for

$$\begin{aligned} -\Delta u &= g_0(u) + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

\square

Proof of Theorem 1.6. Let P the positive cone in the space $C_{loc}^\gamma(\Omega) \cap C(\overline{\Omega})$. Let $H_\epsilon : P \rightarrow P$ be solution operator of the problem

$$\begin{aligned} -\Delta z &= g(z) + h(\nabla z) + f(u) \quad \text{in } \Omega, \\ z &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

gives $H_\epsilon(u) = z$. Now we use the operator A introduced in the proof of Theorem 1.2. By Lemma 2.6, $A^2(\xi_i\kappa)$ belongs to the domain of H_0 . Moreover

$$\begin{aligned} -\Delta H_0(A^2(\xi_i\kappa)) - g(H_0(A^2(\xi_i\kappa))) &= h(\nabla H_0(A^2(\xi_i\kappa))) + f(A^2(\xi_i\kappa)) \\ &\geq f(A(\xi_i\kappa)) \\ &= -\Delta A^2(\xi_i\kappa) - g(A^2(\xi_i\kappa)), \end{aligned}$$

in Ω and $H_0(A^2(\xi_i\kappa)) = A^2(\xi_i\kappa) = 0$ on $\partial\Omega$. It follows from Lemma 2.4 with $h \equiv 0$,

$$H_0(A^2(\xi_i\kappa)) \geq A^2(\xi_i\kappa).$$

In particular, Lemma 2.6 implies $H_0(\eta_i) \leq H_\epsilon(\eta_i)$. Then by hypothesis (1) we have

$$\begin{aligned} -\Delta H_\epsilon(\eta_i) - h(\nabla H_\epsilon(\eta_i)) &= g(H_\epsilon(\eta_i)) + f(\eta_i) \\ &\leq g(\epsilon) + \alpha\eta_i \\ &= -\Delta v_\epsilon - h(\nabla v_\epsilon) \quad \text{in } \Omega, \end{aligned}$$

and $H_\epsilon(\eta_i) = v_\epsilon = \epsilon$ on $\partial\Omega$. Using the [18, Theorem 10.1], we obtain

$$H_\epsilon(\eta_i) \leq v_\epsilon \quad \text{on } \Omega.$$

Consider the auxiliary problem

$$\begin{aligned} -\Delta e &= 1 \quad \text{in } \Omega, \\ e &= \epsilon \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, by hypothesis (3),

$$\begin{aligned} -\Delta\eta_i e &= \eta_i \\ &\geq b_1(\eta_i \|\nabla e\|_{L^\infty(\Omega)})^s + b_0 + g(\epsilon) + \alpha\eta_i \\ &\geq h(\eta_i \nabla e) + g(\epsilon) + \alpha\eta_i \quad \text{on } \Omega, \end{aligned}$$

and so, one obtains

$$\begin{aligned} -\Delta v_\epsilon - h(\nabla v_\epsilon) &\leq -\Delta\eta_i e - h(\eta_i \nabla e) \quad \text{in } \Omega, \\ v_\epsilon &\leq \eta_i e \quad \text{on } \partial\Omega. \end{aligned}$$

Using again [18, Theorem 10.1], we get $v_\epsilon \leq \eta_i e$. From [18, Theorem 3.7] and hypothesis (4) it follows that

$$\sup_{\Omega} e \leq \epsilon + \exp(d(\Omega)) - 1 \leq 1.$$

Then we obtain

$$H_0(\eta_i) \leq H_\epsilon(\eta_i) \leq v_\epsilon \leq \eta_i e \leq \eta_i.$$

Now we consider the non decreasing sequence $\{u_k\} \subset C_{loc}^{2,\gamma}(\Omega) \cap C(\bar{\Omega})$, defined by $u_k = H_0^k(A^2(\xi_i\kappa))$. We deduce

$$A^2(\xi_i\kappa) \leq H_0^k(A^2(\xi_i\kappa)) \leq H_0(\eta_i), \quad (3.1)$$

and by definition

$$\begin{aligned} -\Delta u_k &= g(u_k) + h(\nabla u_k) + f(u_{k-1}) \quad \text{in } \Omega, \\ u_k &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

On the other hand, by (3.1) there holds

$$\|u_k\|_{L^\infty(\Omega')} \leq C, \quad \|g(u_k)\|_{L^\infty(\Omega')} \leq C,$$

where $\bar{\Omega}' \subset \Omega$ and C is a constant independent of k . [18, Theorem 15.8] implies

$$\|\nabla u_k\|_{L^\infty(\Omega')} \leq C.$$

where C is a constant independent of k . By [18, Theorem 9.11],

$$\|u_k\|_{W^{2,r}(\Omega')} \leq C(\|u_k\|_{L^r(\Omega')} + C) \leq C,$$

with $r > N$. By the Sobolev imbedding [18, Theorem 7.26], $u_k \rightarrow u$ in $C^{1,\gamma}(\Omega')$. A standard bootstrap argument implies that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a classical solution of problem (1.8) in the interval $[A^2(\xi_i\kappa), \eta_i]$. \square

Proof of Theorem 1.8. Let P the positive cone in the space $L^\infty(\Omega)$. Let $\mathcal{H}_\epsilon : P \rightarrow P$ be the solution operator of the problem

$$\begin{aligned} -\Delta z &= z^{-p} + f(u) \quad \text{in } \Omega, \\ z &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

gives $\mathcal{H}_\epsilon(u) = z$. Using Theorem 2.3, we deduce that $\mathcal{H}_\epsilon : P \rightarrow P$ is a well defined, continuous, non increasing and compact map with $\mathcal{H}_{\epsilon_0}(u) \leq \mathcal{H}_{\epsilon_1}(u)$ for $\epsilon_0 < \epsilon_1$.

Let $\mathcal{T}_\epsilon : P \rightarrow P$ be the solution operator of the problem

$$\begin{aligned} -\Delta z + \frac{z|\nabla z|^2}{1 + \epsilon z|\nabla z|^2} &= z^{-p} + f(u) \quad \text{in } \Omega, \\ z &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

gives $\mathcal{T}_\epsilon(u) = z$. By Lemma 2.7, we infer that $\mathcal{T}_\epsilon : P \rightarrow P$ is a well defined, continuous, non increasing and compact map with $\mathcal{T}_{\epsilon_0}(u) \leq \mathcal{T}_{\epsilon_1}(u)$ for $\epsilon_0 < \epsilon_1$. From

$$\begin{aligned} -\Delta \mathcal{T}_\epsilon(u) &\leq -\Delta \mathcal{T}_\epsilon(u) + \frac{\mathcal{T}_\epsilon(u)|\nabla \mathcal{T}_\epsilon(u)|^2}{1 + \epsilon \mathcal{T}_\epsilon(u)|\nabla \mathcal{T}_\epsilon(u)|^2} \\ &= u^{-p} + f(u) \\ &= -\Delta \mathcal{H}_\epsilon(u) \quad \text{in } \Omega, \end{aligned}$$

and $\mathcal{T}_\epsilon(u) = \mathcal{H}_\epsilon(u)$ on $\partial\Omega$ implies $\mathcal{T}_\epsilon(u) \leq \mathcal{H}_\epsilon(u)$ in P .

$$\begin{aligned} -\Delta(\mathcal{H}_\epsilon(\eta_i)) &= (\mathcal{H}_\epsilon(\eta_i))^{-p} + f(\eta_i) \\ &\leq (\mathcal{H}_\epsilon(0))^{-p} + \alpha\eta_i \\ &= -\Delta(\mathcal{H}_\epsilon(0) + \alpha\eta_i(-\Delta)^{-1}1) \quad \text{in } \Omega. \end{aligned}$$

From $-\Delta(-\Delta)^{-1}1 = 1$ in Ω and $(-\Delta)^{-1}1 = 0$ on $\partial\Omega$, we implies $\mathcal{H}_\epsilon(\eta_i) = \mathcal{H}_\epsilon(0) + \alpha\eta_i(-\Delta)^{-1}1$ on $\partial\Omega$. Therefore, $\mathcal{H}_\epsilon(\eta_i) \leq \mathcal{H}_\epsilon(0) + \alpha\eta_i(-\Delta)^{-1}1$ in Ω . By condition (4), for ϵ small enough

$$\mathcal{H}_\epsilon(\eta_i) < \eta_i \tag{3.2}$$

By hypothesis (3), we have

$$\begin{aligned} -\Delta \mathbf{u} + \mathbf{u}|\nabla \mathbf{u}|^2 &= \beta\xi_i\kappa \\ &= -\Delta \mathbf{v} + \frac{\mathbf{v}|\nabla \mathbf{v}|^2}{1 + \epsilon \mathbf{v}|\nabla \mathbf{v}|^2} \\ &\leq -\Delta \mathbf{v} + \mathbf{v}|\nabla \mathbf{v}|^2 \quad \text{in } \Omega, \end{aligned}$$

and $\mathbf{u} = 0 < \epsilon = \mathbf{v}$ on $\partial\Omega$. Therefore [18, Theorem 10.1] implies $\mathbf{v} \geq \mathbf{u} \geq \xi_i\kappa$. From

$$\begin{aligned} -\Delta \mathcal{T}_\epsilon(\xi_i\kappa) + \frac{\mathcal{T}_\epsilon(\xi_i\kappa)|\nabla \mathcal{T}_\epsilon(\xi_i\kappa)|^2}{1 + \epsilon \mathcal{T}_\epsilon(\xi_i\kappa)|\nabla \mathcal{T}_\epsilon(\xi_i\kappa)|^2} &= (\mathcal{T}_\epsilon(\xi_i\kappa))^{-p} + f(\xi_i\kappa) \\ &\geq \beta\xi_i\kappa \\ &= -\Delta \mathbf{v} + \frac{\mathbf{v}|\nabla \mathbf{v}|^2}{1 + \epsilon \mathbf{v}|\nabla \mathbf{v}|^2} \quad \text{in } \Omega, \end{aligned}$$

and $\mathcal{T}_\epsilon(\xi_i\kappa) = \epsilon = \mathbf{v}$ on $\partial\Omega$ implies $\mathcal{T}_\epsilon(\xi_i\kappa) \geq \mathbf{v} \geq \mathbf{u} \geq \xi_i\kappa$. It follows that $\mathcal{T}_\epsilon[\xi_i\kappa, \eta_i] \subset [\xi_i\kappa, \eta_i]$, and so there exists a fixed point $u_{i,\epsilon}$ of \mathcal{T}_ϵ in $[\xi_i\kappa, \eta_i]$ for ϵ small enough. By a compensated compactness method, the ‘‘Murat’s lemma’’, the ‘‘Fatou lemma technique’’ of Freshe (see [25, Theorem 3.4]) and Lemma 2.7, letting

$u_i = \lim_{\epsilon \searrow 0} u_{i,\epsilon}$, we have $u_i \in [\xi_i \kappa, \eta_i]$ belongs to $H_0^{1,2}(\Omega)$. Moreover, u_i solves problem (1.9). \square

Proof of Theorem 1.11. Using the identity

$$\Delta\left(\frac{1}{u}\right) = \frac{2}{u^3}|\nabla u|^2 - \frac{1}{u^2}\Delta u$$

If u solves the equation

$$\begin{aligned} -\Delta u &= u^{2-p} + f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then

$$\Delta\left(\frac{1}{u}\right) = u\frac{2}{u^4}|\nabla u|^2 + u^{-p} + \frac{1}{u^2}f(u)$$

Calling $z = \frac{1}{u}$, we conclude

$$\begin{aligned} -\Delta z &= \frac{2}{z}|\nabla z|^2 + z^p + z^2f\left(\frac{1}{z}\right) \quad \text{in } \Omega, \\ z &= \infty \quad \text{on } \partial\Omega. \end{aligned}$$

Using the hypothesis of Theorem 1.2, we conclude the proof. \square

Proof of Theorem 1.12. Let us to define the function

$$g_0(s) = \begin{cases} s^{2-q} - s^{2-p} & \text{if } s > \left(\frac{q-2}{p-2}\right)^{\frac{1}{q-p}} \\ \left(\frac{q-2}{p-2}\right)^{2-q} - \left(\frac{q-2}{p-2}\right)^{2-p} & \text{if } s \leq \left(\frac{q-2}{p-2}\right)^{\frac{1}{q-p}} \end{cases}$$

Using Lemma 2.3 with $g(u) = g_0(u + \frac{1}{\epsilon})$, we can define the solution operator $z = H_{1/\epsilon}(h)$ of the problem

$$\begin{aligned} -\Delta z &= g_0(z) + h \quad \text{in } \Omega, \\ z &= \frac{1}{\epsilon} \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, this operator is well defined $H_{1/\epsilon} : \{h \in L^\infty(\Omega) | h \geq 0\} \rightarrow \{z \in C(\overline{\Omega}) | z \geq 0\}$, it is continuous, non decreasing and compact. Therefore, we can define $z = A_{1/\epsilon}(u) : \{u \in L^\infty(\Omega) | h \geq 0\} \rightarrow \{z \in C(\overline{\Omega}) | z \geq 0\}$, the continuous, increasing and compact solution operator of the problem

$$\begin{aligned} -\Delta z &= g_0(z) + f(u) \quad \text{in } \Omega \\ z &= \frac{1}{\epsilon} \quad \text{on } \partial\Omega. \end{aligned}$$

If κ is the indicator function of the set K , as in the proof of Theorem 1.2, we deduce

$$A_{1/\epsilon}(\xi_i \kappa) \geq \xi_i \kappa.$$

From $A_{1/\epsilon}(0) < A_{1/\epsilon}(\eta_i)$, we get $g(A_{1/\epsilon}(\eta_i)) < g(A_{1/\epsilon}(0))$. Therefore,

$$\begin{aligned} -\Delta A_{1/\epsilon}(\eta_i) &= g(A_{1/\epsilon}(\eta_i)) + f(\eta_i) \\ &\leq g(A_{1/\epsilon}(0)) + \alpha \eta_i \\ &= -\Delta(A_{1/\epsilon}(0) + \alpha \eta_i e) \end{aligned}$$

Where

$$\begin{aligned} -\Delta e &= 1 && \text{in } \Omega, \\ z &= \frac{1}{\epsilon} && \text{on } \partial\Omega. \end{aligned}$$

The maximum principle implies

$$A_{1/\epsilon}(\eta_i) \leq A_{1/\epsilon}(0) + \alpha\eta_i e$$

By condition (4), it holds $A_{1/\epsilon}(0) + \alpha\eta_i e \leq \eta_i$. Therefore $A_{1/\epsilon}[\xi_i\kappa, \eta_i] \subset [\xi_i\kappa, \eta_i]$. By construction $A_{1/\epsilon} : P \rightarrow P$ where P is the positive cone in $L^\infty(\Omega)$. The interior of P is nonempty, so we deduce the existence of $2m - 1$ different solutions of the equation

$$A_{1/\epsilon}(u) = u \tag{3.3}$$

If $\frac{1}{\epsilon} \geq \left(\frac{q-2}{p-2}\right)^{\frac{1}{q-p}}$, then a solution of equation (3.3), solves the problem

$$\begin{aligned} -\Delta u &= u^{2-q} - u^{2-p} + f(u) && \text{in } \Omega, \\ u &= \frac{1}{\epsilon} && \text{on } \partial\Omega. \end{aligned}$$

From

$$\Delta\left(\frac{1}{u}\right) = \frac{2}{u^3}|\nabla u|^2 - \frac{1}{u^2}\Delta u,$$

We infer

$$\Delta\left(\frac{1}{u}\right) = u\frac{2}{u^4}|\nabla u|^2 + u^{-q} - u^{-p} + \frac{1}{u^2}f(u).$$

If we define $z = \frac{1}{u}$, we have

$$\begin{aligned} \Delta z &= \frac{2}{z}|\nabla z|^2 + z^q - z^p + z^2f\left(\frac{1}{z}\right) && \text{in } \Omega, \\ z &= \epsilon && \text{on } \partial\Omega, \end{aligned}$$

and we complete the proof □

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REFERENCES

- [1] Adimurthi and J. Giacomoni. *Multiplicity of positive solutions for a singular and critical elliptic problem in \mathbb{R}^2* . Communications Contemp. Math., 5(8)(2006), 621-656.
- [2] H. Amann. *On the number of solutions of nonlinear equations in ordered Banach spaces*. J. Funct. Analysis 14 (1973) 346-384.
- [3] C. C. Aranda and T. Godoy. *Existence and Multiplicity of positive solutions for a singular problem associated to the p -laplacian operator*. Electron. J. Diff. Eqns., Vol. 2004(2004), No. 132, pp. 115.
- [4] C. C. Aranda and E. Lami Dozo. *Multiple solutions to a singular Lane-Emden-Fowler equation with convection term* Electron. J. of Diff. Eqns., Vol. 2007(2007), No. 05, pp. 121.
- [5] R. Agarwal and D. O'Reagan. *Existence theory for single and multiple solutions to singular positone boundary value problems*, J. Differential Equations 175 (2001) 393-414.
- [6] H. Brezis and X. Cabre. *Some simple nonlinear PDE's without solutions*. Bolletino della Unione Matematica Italiana, 1-B (1998) 223-262.
- [7] T. Carelman. *Problèmes mathématiques dans la théorie cinétique des gaz*. Almquist-Wiksell, Uppsala, 1957.
- [8] J. I. Diaz, J. M. Morel and L. Oswald. *An elliptic equation with singular nonlinearity* Comm. Part. Diff. Eq., 12, 12 (1987) 1333-1334.
- [9] V. R. Emden. *Gaskugeln*. Teubner, Leipzig, 1907.

- [10] P. Esposito, N. Ghossoub and Y. Guo. *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*. Comm. on Pure and Appl. Math. Volume 60 Issue 12, (2007) 1731 - 1768.
- [11] R. H. Fowler. *Further studies of Emden's and similar differential equations*. Q. J. Math. (Oxford Series) 2 (1931), 259-288.
- [12] W. Fulks and J. S. Maybe. *A singular nonlinear equation*. Osaka J. Math. 12 (1960), 1-19.
- [13] S. Gaucel and M. Langlais. *Some mathematical problems arising in heterogeneous insular ecological modes*. RACSAM 96 (2002), 389-400.
- [14] M. Ghergu, C. Niculescu, and V. Rădulescu. *Explosive solutions of elliptic equations with absorption and nonlinear gradient term*, Proc. Indian Acad. Sci. (Math. Sci.) 112 (2002), 441-451.
- [15] M. Ghergu and V. Rădulescu. *Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term*, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005), no. 1, 61-83.
- [16] M. Ghergu and V. Rădulescu. *On a class of sublinear singular elliptic problems with convection term*. J. Math. Anal. Appl. 311 (2005), 635-646.
- [17] M. Ghergu and V. Rădulescu. *Singular Elliptic Problems: Bifurcation & Asymptotic Analysis*. Oxford Lecture Series in Mathematics and Its Applications, 2008.
- [18] D. Gilbart, N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Second edition 1983.
- [19] J. Homer Lane. *On the Theoretical Temperature of the Sun under the Hypothesis of a gaseous mass Maintaining Its Volume by Its Internal Heat and Depending on the Law of Gases Known to Terrestrial Experiment*. Amer. J. Sci, 2d ser. 50 (1869), 57-74.
- [20] Y. Haitao, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*. J. Differential Equations 189 (2003), 487-512.
- [21] J. Hernández and F. J. Mancebo. *Singular elliptic and parabolic equations*. In *Handbook of differential equations* (ed. M. Chipot and P. Quittner), vol 3 (Elsevier 2006).
- [22] J. Hernández, F. J. Mancebo and J. M. Vega. *Positive solutions for a singular nonlinear elliptic equations*. Proc. Roy. Soc. Edinburgh, 137 A, 41-62, 2007.
- [23] N. Hirano, C. Saccon and N. Shioji. *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*. Adv. in Differential Equations 9 (2004) 197-220.
- [24] H. Kielhöfer, S. Maier. *Infinitely many positive solutions of semilinear elliptic problems via sub- and supersolutions*. Commun. in Partial Differential Equations, 18 (7&8), 1219-1229, 1993.
- [25] M. Struwe. *Variational Methods*. Second edition, 1996.
- [26] G. B. Whitman. *Linear and Nonlinear Waves*. WileyInterscience, New York, 1973.
- [27] S. Yijing, W. Shaoping and L. Yiming. *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*. J. of Differential Equations 176 (2001), 511-531.
- [28] Z. Zhang. *Critical points and positive solutions of singular elliptic boundary value problems*. J. Math. Anal. Appl. 302 (2005), 476-483.
- [29] Z. Zhang. *Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problems with a convection term*. Nonlinear Anal. 27 (1996), 957-961.
- [30] Z. Zhang and J. Yu. *On a singular nonlinear Dirichlet problem with a convection term*. SIAM J. Math. Anal. 4 (2000), 916-927.

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