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SOME PROPERTIES OF THE EXTREME VALUES OF INFINITY-HARMONIC FUNCTIONS

TILAK BHATTACHARYA

Abstract. We study local behavior of infinity-harmonic functions, in particular, the extreme values of such functions on a ball. We show that the extreme values obey certain relationships, and use this to derive an interior growth estimate.

1. Introduction

In this work we study some properties of infinity-harmonic functions. The questions of local behavior and regularity form the main motivation for our present work, and our hope is that the results in this work will provide some insight into these matters.

We introduce notation for our discussion. We take $\Omega \subset \mathbb{R}^n$, $n \geq 2$ to stand for a domain. Our results, being local in nature, would apply even when Ω is unbounded. We will often use x, y, z to denote points on $\mathbb{R}^n$, and o will stand for the origin. We will occasionally write a point as $x = (x_1, x_2, \dots, x_n)$. A ball of radius r and center x will be denoted by $B_r(x)$.

We define $u : \Omega \to \mathbb{R}$, to be infinity-harmonic in Ω if it satisfies, in the sense of viscosity,

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad \text{in } \Omega. \tag{1.1}$$

For motivation and a detailed discussion of various properties of such solutions, see [2, 5, 9, 10, 12, 17]. An important aspect of these functions is that they are completely characterized by the "cone comparison property", a fact first observed in [12]. Like all other results, our work too exploits this property to achieve its ends. We refer to [2, 10, 12] for a detailed discussion of this issue.

Our motivation for this work is related to the question of local regularity. Despite great recent progress the matter of local regularity remains somewhat unresolved. It is well-known that u is locally Lipschitz continuous [5, 10, 12, 15], and it has been shown that u is $C^{1,\alpha}$ when $n = 2$, see [14, 18]. However, differentiability is still open when $n \geq 3$. The work [14] proves deep results regarding this matter and states a conjecture, whose proof would lead to $C^{1,\alpha}$ regularity of such functions.

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Our goal in this work is to derive new properties of the extreme values of such functions which we hope may lead to some additional insight into this problem.

In what follows, we will limit our discussion to a ball $B_r(x)$ in Ω. Let $\overline{A}$ denote the closure of a set A. Define $M_x(r) = \sup_{\bar{B}_r(x)} u$ and $m_x(r) = \inf_{\bar{B}_r(x)} u$. We will drop the subscript if $x = o$. It is well-known that u satisfies the strong maximum principle, and these values are attained on the boundary of the ball $B_r(x)$ [8, 12]. It is also well-known that $M_x(r) - u(x)$ and $u(x) - m_x(r)$ are both convex in r, and the following limit exists:

$$\lim_{r \downarrow 0} \frac{M_x(r) - u(x)}{r} = \lim_{r \downarrow 0} \frac{u(x) - m_x(r)}{r} := \Lambda(x). \tag{1.2}$$

Moreover, the limit $\Lambda(x)$ would equal $|Du(x)|$, if u is differentiable at x, see [2, 5, 10, 11, 12]. In addition, it was shown in [11] that if $\omega \in S^{n-1}$ is such that $u(x + r\omega) = M_x(r)$ and the set of ω's has a unique limit point as $r \downarrow 0$, then u is differentiable at x. It is also known that if $B_r(x) \subset\subset \Omega$, then u is differentiable at $x + r\omega$ [5]. Moreover, the limit $\Lambda(x)$ is upper semi-continuous [2, 5, 10, 12] and satisfies a maximum principle, [4, 12]. An important question in this context is whether or not infinity-harmonic functions, defined on all of $\mathbb{R}^n$ with bounded Λ, are affine. This was proven in $n = 2$ in [18], as a consequence of differentiability. Some of the facts mentioned above will play a role in this work.

From hereon we assume that $o \in \Omega$ and work in a ball $B_R(o) \subset \Omega$. Additionally, we always take $u(o) = 0$. To make our presentation clearer, we redefine $m(r) = -\inf_{\bar{B}_r(o)} u$, and unless otherwise mentioned, from hereon we will always take this to be the definition of $m(r)$. We now state the first of our main results.

Theorem 1.1. *Let u be infinity-harmonic in $B_R(o)$. Suppose that $u(o) = 0$; define for $r \in [0, R)$, $m(r) = -\inf_{\bar{B}_r(o)} u$ and $M(r) = \sup_{\bar{B}_r(o)} u$. The following two inequalities then hold:*

(a) $m(r) \geq \left(\sqrt{rM'(r-)} - \sqrt{rM'(r-) - M(r)}\right)^2$ *for* $r \in [0, R)$;

(b) $M(r) \geq \left(\sqrt{rm'(r-)} - \sqrt{rm'(r-) - m(r)}\right)^2$ *for* $r \in [0, R)$.

Moreover, if u is infinity-harmonic in $\mathbb{R}^n$ and equality holds in both the inequalities for every $r > 0$, then u is affine.

A proof appears in Section 2. An easy consequence of Theorem 1.1 are the inequalities,

$$m(r)M'(r-) \geq \frac{M(r)^2}{4r} \quad \text{and} \quad M(r)m'(r-) \geq \frac{m(r)^2}{4r},$$

see Remark 2.1. The inequalities in Theorem 1.1 place no restrictions on how small, for instance, $m(r)$ could be. As a matter of fact we construct an example in Lemma 2.2 in Section 2 which supports this observation. Using the inequalities in Theorem 1.1, we also prove the following growth rate in Theorem 1.2.

Theorem 1.2. *Suppose that u is infinity-harmonic in $B_1(o)$ and $u(o) = 0$. Define for $r \in [0, 1)$ $m(r) = -\inf_{\bar{B}_r(o)} u$ and $M(r) = \inf_{\bar{B}_1(o)} u$.*
(i) Suppose that $M(r) \leq r$ for every $r \in [0, 1)$. Then either $m(r) \leq r$ for every $r \in [0, 1)$, or there is an $a \in (0, 1)$ such that $m(a) > a$ and

$$m(r) \geq r(1 + k \log(r/a)), \ \forall\, a < r < 1,$$

where $k = (m(a) - a)^2/4a^2$,

(ii) Analogously, if $m(r) \le r$ for every $r \in [0,1)$, then either $M(r) \le r$ for every $r \in [0,1)$, or $M(a) > a$ for some $a \in (0,1)$ and

$$M(r) \ge r(1 + k\log(r/a)),\ \forall\, a < r < 1,$$

where $k = (M(a) - a)^2/4a^2$.

We provide a proof in Section 2. It is to be noted that the hypothesis $M(r) \le r$, or for that matter $m(r) \le r$, is not restrictive. One can scale u to obtain this inequality, see Remark 2.3 for a general version.

We now bring up a related matter. If u is infinity-harmonic in $B_1(o)$, by convexity, $M'(r+)$ and $M'(r-)$ exist everywhere and $M'(r)$ exists for almost every r. As pointed out before (see [5]), if $p \in \partial B_r(o)$ for $r < 1$, is a point of maximum then u is differentiable at p and

$$M'(r-) \le |Du(p)| \le M'(r+).$$

Moreover, as Lemma 3.1 shows there are points p on $\partial B_r(o)$ where the values $M'(r+)$ and $M'(r-)$ are attained by $|Du(p)|$. Our question is: does $M'(r)$ exist at every $r \in [0,1)$, or equivalently, is $M'(r-) = M'(r+)$? Clearly, this equality holds at $r = 0$. We have been unable to settle the matter, however, the example in Lemma 3.3 shows that these may disagree on the boundary $\partial B_1(o)$ in the event $u \in C(\bar{B}_1(o))$. In particular, we show that $M'(r)$ exists for every $r \in [0,1)$ but there are points p on $\partial B_1(o)$ such that $u(p) = M(1)$, and the one sided gradient $\Lambda(p) > M'(1-)$. As our calculations will show this difference can be made arbitrarily large.

We have divided our work as follows. Proofs of Theorems 1.1 and 1.2, and Lemma 2.2 appear in Section 2. Section 3 contains proofs of Lemmas 3.1, 3.2 and 3.3. In this work, all sets are subsets of $\mathbb{R}^n$, $n \ge 2$, unless otherwise mentioned.

2. Proofs of Theorems 1.1 and 1.2

We first state a few properties of infinity-harmonic functions that could be thought of as "monotonicity" along radial segments, see [7, 10]. For completeness, we provide short proofs of these, also see Exercise 16 in [10]. The proofs will use the comparison principle [2, 3, 10, 13, 15].

Let v be a positive infinity-harmonic function in a domain Ω and $B_\rho(o) \subset \Omega$. Take $y \in B_\rho(o)$ and let $d > 0$ be chosen to be any value that does not exceed the distance of y from $\partial\Omega$. By comparing $v(x)$ to the infinity-harmonic function $v(y)(1 - |x-y|/d)$, we see

$$v(x) \ge v(y)\Big(1 - \frac{|x-y|}{d}\Big), \quad \text{for } x \in B_d(y),$$

In this discussion, we refer to the above inequality as *cone comparison in $B_d(y)$*. Taking $d = \rho - |y|$, a rearrangement leads to the inequality

$$\frac{v(y) - v(x)}{|x-y|} \le \frac{v(x)}{d - |x-y|}.$$

It is clear by using (1.2) that $\Lambda(y) \le v(y)/d$, for every $y \in B_\rho(o)$. Moreover, if $\omega \in S^{n-1}$, by selecting the points x and y on the radial ray, in $B_\rho(o)$ and along ω, it is clear that $v(\theta\omega)/(\rho - \theta)$ is an increasing function of θ in $[0,\rho)$.

Next we show that $v(\theta\omega)(\rho - \theta)$ is decreasing. To see this, take $x = \theta\omega$ with $\theta > 0$, small, so that $B_\rho(x) \subset \Omega$ (one may need the assumption $\bar{B}_\rho(o) \subset \Omega$, but this is not restrictive). Using cone comparison in $B_\rho(x)$, we have $v(x)(\rho - |x|) \le v(o)\rho$.

Taking $y = \bar{\theta}\omega$ with $\bar{\theta} > \theta$ and $\bar{\theta}$ close to θ, cone comparison in $B_{\rho-|x|}(y)$, leads to $v(x)(\rho - |x|) \geq v(y)(\rho - |y|)$. Proceeding in this way we obtain our claim. An alternative is to use the Harnack inequality [7]. Collecting our conclusions, for any $\omega \in S^{n-1}$ and $\theta \in [0, \rho)$,

$$\begin{aligned}
&\text{(i) } v(\theta\omega)/(\rho - \theta) \text{ is increasing in } \theta,\\
&\text{(ii) } v(\theta\omega)(\rho - \theta) \text{ is decreasing in } \theta, \text{ and}\\
&\text{(iii) } \Lambda(y) \leq v(y)/(\rho - |y|), \text{ for all } y \in B_\rho(o).
\end{aligned} \tag{2.1}$$

For applications, we will often use these properties with $v = M - u$ or $v = u - l$, where M and l are any numbers with $M \geq \sup_{\bar{B}_\rho(o)} u$ and $l \leq \inf_{\bar{B}_\rho(o)} u$. We also note another property. For $r \leq \rho$, if $\gamma, \bar{\gamma} \in S^{n-1}$ are such that $v(r\gamma) = \sup_{\bar{B}_r(o)} v$ and $v(r\bar{\gamma}) = \inf_{\bar{B}_r(o)} v$, then u is differentiable at $r\gamma$ and $r\bar{\gamma}$ and

$$Du(r\gamma) = \alpha\gamma, \quad Du(\bar{\gamma}r) = \beta\bar{\gamma}. \tag{2.2}$$

for some $M'(r-) \leq \alpha \leq M'(r+)$ and $m'(r-) \leq \beta \leq m'(r+)$. For a proof see [5], also see Lemma 3.1 for a related statement.

We thank the referee for his/her suggestions that have simplified the following proof.

Proof of Theorem 1.1. We prove inequality (a) of the theorem, as inequality (b) follows by symmetry. First observe that $(r - t)(M(t) + m(r))$ is non-increasing in t. This may be argued by using 3(ii) with $v = u + m(r)$. Alternatively, one could use [10, Lemma 4.6] or (2.1)(iii) to first derive the following differential inequality

$$M'(t+) \leq \frac{M(t) + m(r)}{r - t},$$

and then conclude the same. In any case, since $u(o) = 0$,

$$M(t) \leq \Big(\frac{t}{r - t}\Big) m(r), \quad 0 \leq t < r.$$

Combining this with the convexity property $M(t) \geq M(r) - (r - t)M'(r-)$, we have

$$\Big(\frac{r - t}{t}\Big)\Big(M(r) - (r - t)M'(r-)\Big) \leq m(r), \quad 0 < t \leq r. \tag{2.3}$$

Our idea is to select a value of t that optimizes the above inequality in $0 < t \leq r$. To this end define

$$f(t) = \Big(\frac{r - t}{t}\Big)\Big(M(r) - (r - t)M'(r-)\Big) \quad 0 < t \leq r,$$

and observe that $f(r) = 0$, $f(t) \geq 0$, for t near r, and $\lim_{t\downarrow 0} f(t) < 0$ unless $M(r) = rM'(r-)$. It is easy to show that f is optimized when

$$t_0 = \Big(\frac{r[rM'(r-) - M(r)]}{M'(r-)}\Big)^{1/2}.$$

As $rM'(r-) \geq M(r)$ by convexity, we have $0 \leq t_0 \leq r$. Inserting the value of t_0 in (2.3) the first inequality of the theorem follows,

$$m(r) \geq \Big(\sqrt{rM'(r-)} - \sqrt{rM'(r-) - M(r)}\Big)^2. \tag{2.4}$$

We discuss briefly the cases when $t_0 = 0$ and $t_0 = r$. If $t_0 = 0$ then $rM'(r-) = M(r)$, and convexity leads to $M(t) = tM(r)/r$. If $t_0 = r$, then $M(r) = 0$ leading to $u = 0$ in $B_r(o)$. Also see [4, Lemma 3.1].

Let us now assume that u is infinity-harmonic in $\mathbb{R}^n$ and equality holds every where, in the two inequalities of the theorem. We use an equivalent form of (2.4),

$$M(r)^2 \le m(r)\left(\sqrt{rM'(r-)}+\sqrt{rM'(r-)-M(r)}\right)^2. \tag{2.5}$$

With equality in place in (2.5) and replacing $rM'(r-)$ by the smaller quantity $M(r)$ and disregarding the second term on the right hand side, we deduce that $M(r) \ge m(r)$. Analogously, by exploiting inequality (b) of the theorem, $m(r) \ge M(r)$, which leads to $M(r) = m(r)$. Next, we use this in (2.4) and (2.5), with equality in place, and compare the two right hand sides to conclude that $rM'(r-) = M(r)$. This leads to $M(r) = m(r) = kr$, for some $k > 0$. Now applying the "tight on a line" result in Section 7.2 in [10], it follows that u is affine. $\square$

Remark 2.1. We discuss briefly the inequalities in Theorem 1.1. From (2.5), it follows easily that $m(r)M'(r-) \ge M(r)^2/4r$, and, similarly, $M(r)m'(r-) \ge m(r)^2/4r$. Next in (2.5), we set $x = rM'(r-)$ and $y = M(r)$, and use an expansion for $\sqrt{x-y}$. Working with

$$\sqrt{x-y} \le \sqrt{x} - \frac{y}{2x^{1/2}} - \frac{y^2}{8x^{3/2}} - \frac{y^3}{16x^{5/2}}, \quad 0 < y \le x,$$

and, for instance, using the first two terms and convexity,

$$rM'(r-) \ge \frac{M(r)^{3/2}}{2m(r)^{1/2}} + \frac{M(r)}{4}.$$

An analogue, from the second inequality in Theorem 1.1, can be easily worked out.

Next we construct an example to show that the inequalities of Theorem 1.1 place no restrictions on the lower bounds of $m(r)$ and $M(r)$. First we discuss some preliminaries.

Our construction will involve the Aronsson singular example, and we recall below some of its properties that will be used in Lemma 2.2. For a more detailed discussion, see [1], [6, Lemmas 2.5, 2.6, 2.9], and (b) in part II of the appendix in [7]. We use $x = (x_1, \dots, x_n) = (\bar{x}, x_n)$ to denote a point in $\mathbb{R}^n$. Let e_n be the unit vector along the positive x_n-axis and $\theta = \cos^{-1}(\langle x, e_n\rangle/|x|)$, the angle made with the x_n-axis. The Aronsson example is given by $v(x) = \psi(\theta)/|x|^{1/3}$, where $v(x)$ is infinity-harmonic in $\mathbb{R}^n \setminus \{o\}$. We list only what we need. For $x \ne o$,

$$\begin{aligned}
&\text{(i) } \psi(\theta) > 0 \text{ if } x_n > 0, \text{ and } \psi(\pm\pi/2) = v(\bar{x}, 0) = 0;\\
&\text{(ii) } \sup_{\partial B_r(o)} v = \psi(0)/r^{1/3};\\
&\text{(iii) } \psi(\theta) = \psi(-\theta), \text{ and } \psi(\theta) \text{ is decreasing in } [0, \pi/2].
\end{aligned} \tag{2.6}$$

Lemma 2.2. *Let $\varepsilon > 0$ and $B_1(o) \subset \mathbb{R}^n$, $n \ge 2$. There exists a function $u_\varepsilon \in C(\bar{B}_1(o))$, infinity-harmonic in $B_1(o)$ with $u_\varepsilon(o) = 0$ and $\sup_{\bar{B}_1(o)} u_\varepsilon = 1$, such that $\inf_{\bar{B}_1(o)} u_\varepsilon \ge -\varepsilon$.*

Proof. To construct our example u_ε on $B_1(o)$, we employ a translate of the Aronsson singular function v and use the properties stated in (2.6). Let $\delta > 0$ and let $p_\delta = (0, 0, 0, \dots, -(1+\delta))$. Define

$$v_\delta(x) = \frac{\psi(\theta)}{|x - p_\delta|^{1/3}} = \frac{\psi(\theta)}{|x + (1+\delta)e_n|^{1/3}}, \quad \theta = \theta(x) = \frac{\langle x - p_\delta,\ e_n\rangle}{|x - p_\delta|}.$$

We scale $\psi(0) = 1$. By (2.6)(i) $v > 0$ on $B_1(o)$, and by (2.6)(ii) and (iii), $\sup_{\partial B_\delta(p_\delta)} v_\delta = \sup_{B_1(o)} v = 1/\delta^{1/3}$. Next for $x \in B_1(o)$, define

$$\begin{aligned} w_\delta(x) &= \frac{\delta^{1/3}(1+\delta)^{1/3}(v_\delta - v(o))}{(1+\delta)^{1/3} - \delta^{1/3}} \\ &= \frac{\delta^{1/3}(1+\delta)^{1/3}}{(1+\delta)^{1/3} - \delta^{1/3}}\Big(\frac{\psi(\theta)}{|x+(1+\delta)e_n|^{1/3}} - \frac{1}{(1+\delta)^{1/3}}\Big). \end{aligned}$$

Then w_δ is infinity-harmonic in $x_n > -(1+\delta)$ and clearly so in $B_1(o)$. Also $\sup_{B_1(o)} w_\delta = 1$, $w_\delta(o) = 0$ and $w_\delta \in C(\bar{B}_1(o))$. Noting that $\psi(\theta) > 0$ when $x \in B_1(o)$ (see (2.6)(i)), we see that

$$0 \ge \inf_{\bar{B}_1(o)} w_\delta \ge \frac{-\delta^{1/3}}{(1+\delta)^{1/3} - \delta^{1/3}}.$$

Choosing δ small enough, we obtain our desired infinity-harmonic function u_ε. $\square$

Next we present a proof of the growth estimate in Theorem 1.2. We utilize the inequalities proven in Theorem 1.1.

Proof of Theorem 1.2. We will only prove part (i), part (ii) will follow analogously. To achieve our goal we use inequality (b) of Theorem 1.1. Using the hypothesis, we note

$$\Big(\sqrt{rm'(r-)} - \sqrt{rm'(r-) - m(r)}\Big)^2 \le r, \quad 0 \le r < 1. \tag{2.7}$$

Let us assume that $m(a) > a$, for some $0 < a < 1$. By the convexity of $m(r)$,

$$m'(r-) \ge m'(a-) > 1, \quad m(r) > r, \quad \forall\, a < r < 1.$$

Observing that $2rm'(r-) \ge r + m(r)$, squaring (2.7), rearranging terms and squaring again we obtain for $r > a$,

$$m'(r-) \ge \frac{(m(r)+r)^2}{4r^2}. \tag{2.8}$$

Setting $w = m(r)/r$ in the second inequality in (2.8), we obtain the differential inequality $4rw' \ge (w-1)^2$. An integration from c to r, for any $a \le c < 1$ yields

$$\frac{c}{m(c)-c} - \frac{r}{m(r)-r} \ge \frac{\log(r/c)}{4}, \quad c \le r < 1.$$

Noting that $m(r) - r \ge m(c) - c$, a further rearrangement yields

$$\frac{m(r)}{r} \ge \frac{m(c)}{c} + \Big(\frac{m(c)-c}{c}\Big)^2 \frac{\log(r/c)}{4}, \quad c \le r < 1.$$

Selecting $k = (m(a)-a)^2/4a^2$ we have

$$m(r) \ge \frac{r}{a} m(a) + kr\log(r/a) \ge r + kr\log(r/a), \quad a \le r < 1.$$

The theorem follows. $\square$

Remark 2.3. Firstly, we note that if $M(1) = 1$, by convexity $M(r) \le r$. We now state Theorem 1.2 for the general case. If v is infinity-harmonic in $B_R(o)$ and $\sup_{B_R(o)} v < \infty$, then we scale v and define $u(y) = (v(x) - v(o))/(\sup_{B_R(o)} v - v(o))$, where $y = x/R$. Then $\sup_{B_1(o)} u = 1$ and u satisfies the conditions of Theorem 1.2. Thus either

$$\inf_{B_r(o)} v \ge v(o) - \frac{r}{R}(\sup_{B_R(o)} u - u(o)), \quad \forall\, r \in [0, R),$$

or for some $0 < t < R$,

$$\inf_{B_r(o)} v \le v(o) - \frac{r}{R}\Big(1 + k\log\frac{r}{t}\Big)\big(\sup_{B_R(o)} v - v(o)\big), \quad \forall\, r \in [t, R),$$

where

$$k = \frac{R^2}{4t^2}\Big(\frac{v(o) - \inf_{B_t(o)} v - \frac{t}{R}(\sup_{B_R(o)} v - v(o))}{\sup_{B_R(o)} v - v(o)}\Big)^2.$$

3. Results about $M'(r)$

Our effort in this section is to construct an example in connection with the question raised, in Section 1, about $M'(r)$. To this end, we start with Lemma 3.1. In this connection, recall the definition of Λ in (1.2) and the result in (2.2).

Lemma 3.1. *Let u be infinity-harmonic in $B_R(o)$. For $0 < r < R$, let $p \in \partial B_r(o)$ be a point of maximum of u on $B_r(o)$, then u is differentiable at p and*

$$\Lambda(o) \le M'(r-) \le |Du(p)| \le M'(r+).$$

Moreover, there are points p^+ and p^- on $\partial B_r(o)$ such that $u(p^+) = u(p^-) = M(r)$, with $|Du(p^+)| = M'(r+)$ and $|Du(p^-)| = M'(r-)$. A similar result holds for $m'(r-)$ and $m'(r+)$.

Proof. For the inequality in the lemma, see [5, Remark 2]. Fix $r < R$ and let $p^+ \in \partial B_r(o)$ denote a limit point of a sequence of points of maximum $p \in \partial B_\rho(o)$ with $\rho > r$ and $\rho \downarrow r$. Clearly for each $p \in \partial B_\rho(o)$, $|Du(p)| \ge M'(\rho-)$. Using the convexity of $M(r)$ and the upper semicontinuity of Λ [5, 12], we see that $|Du(p^+)| \ge M'(r+)$. Hence equality follows.

Now let p^- be a limit point of a sequence of such points $p \in \partial B_\rho(o)$ with $\rho < r$ and $\rho \uparrow r$. Applying (2.1)(i) to $M(\rho) - u$ in $B_\rho(o)$, noting (2.2) and the inequality in the lemma, we see that for a fixed $0 < t < 1$,

$$\frac{M(\rho) - u(tp)}{\rho(1-t)} \le |Du(p)| \le M'(\rho+).$$

Letting $\rho \uparrow r$ and selecting a subsequence if needed,

$$\frac{M(r) - u(tp^-)}{r(1-t)} \le M'(r-).$$

Letting $t \uparrow 1$, using (2.2) and the inequality in this lemma, we obtain $|Du(p^-)| = M'(r-)$. □

We introduce additional notations before stating Lemma 3.2. For $\alpha \in (0, \pi)$, let $C_\alpha \subset \partial B_1(o)$ denote the spherical cap, of aperture 2α, centered on the negative x_n-axis. We state an estimate which is based on the Aronsson singular example [17].

Lemma 3.2. *Let $u \in C(\bar{B}_1(o))$ be infinity-harmonic and $0 \le u \le 1$. Suppose that $0 < \alpha < \pi$, and $u(x) = 1$ on C_α. If $\pi - \alpha$ is small then*

$$u(x) \ge 1 - \frac{c(\pi - \alpha)^{1/3}}{|x - (1 + k(\alpha))e_n|^{1/3}}, \quad x \in B_1(o),$$

for some positive constants c, independent of n and α, and $k(\alpha)$, where $k(\alpha) \to 0$ as $\alpha \to \pi$.

For $0 < \alpha < \pi$, let C_α be the spherical cap as described above and $\hat{C}_\alpha \subset \partial B_1(o)$ be the complementary spherical cap of aperture $2\pi - 2\alpha$. We write the point x as $x = (\bar{x}, x_n)$, and define $|\bar{x}| = (\sum_{i=1}^{n-1} x_i^2)^{1/2}$. We will study functions $u \in C(\bar{B}_1(o))$, infinity-harmonic in $B_1(o)$, with

$$u(o) = 0, \quad u < 1 \quad \text{in } B_1(o) \quad \text{and} \quad u = \phi \quad \text{on } \partial B_1(o), \tag{3.1}$$

where $\phi(x) = \phi(|\bar{x}|, x_n)$ is axially symmetric about x_n axis, $\phi = 1$ on C_α, and $\phi(x)$ is decreasing in x_n for $x \in \hat{C}_\alpha$. Such functions u are easily constructed. Our chief interest is their behavior when α is near π. As we will see in Lemma 3.3, when $r \in [0, 1)$, $M'(r)$ exists for $r < 1$. However, this does not extend to the boundary and the disparity between the one-sided gradients at points of maximum on $r = 1$ can be made quite large by making α close to π. Before we state Lemma 3.3, we mention that if ω is such that $u(r\omega) = M(r)$ then the limit

$$\lim_{t \uparrow r} \frac{M(r) - u(t\omega)}{r - t} = L(r\omega),$$

exists (may be unbounded) [5]. This will be referred to as the one sided gradient at $r\omega$ and equals $|Du(r\omega)|$ if $r < 1$.

Lemma 3.3. *For $0 < \alpha < \pi$, let C_α, $\hat{C}_\alpha$ and L be as described above. Suppose that u is an infinity-harmonic function that satisfies* (3.1) *and $\pi - \alpha$ is sufficiently small. The following then hold.*

(i) *For $r \in [0, 1]$, $m(r) = -u(re_n)$, and as $\alpha \to \pi$, $m(1)$ and $m'(1-)$ become unbounded.*
(ii) *For every $0 \le r < 1$, $M(r) = u(-re_n) \ge 1 - (1 - r)|\sec \alpha|$, $M'(r)$ exists and $1 \le M'(1-) \le |\sec \alpha|$.*

Moreover, if p is any point on the boundary of C_α, relative to $\partial B_1(o)$, then the one sided gradient $L(p) \to \infty$ as $\alpha \to \pi$

Proof. Before proving parts (i) and (ii), we state some symmetry-related properties of the the solution u, obtained by utilizing reflections and the comparison principle. Since u satisfies (3.1), we note that $m(1)$ is attained at $x = e_n$. Moreover, by using reflection about any $n - 1$ plane, containing the x_n axis, and the comparison principle it follows that u is axially symmetric about x_n-axis. For a set A, define $-A = \{-x : x \in A\}$.

Let κ denote a unit vector with $\kappa_n \ge 0$ and P_κ be the $n - 1$ dimensional plane passing through o and having κ as its normal. Define the half-space P_κ^+ to be the set of all points $x \in \mathbb{R}^n$ with $\langle x, \kappa \rangle \ge 0$. Define the half-space $P_\kappa^- = -P_\kappa^+$. We now prove a monotonicity property of u to be used later. Now recall that $u = 1$ on C_α and $u < 1$, and consider u in the half ball $B_1(o) \cap P_\kappa^-$. Using its reflection about P_κ and comparing with u, in $B_1(o) \cap P_\kappa^+$, we claim

$$\text{for all } y \in P_\kappa \cap B_1(o) \text{ and } t > 0 \text{ with } y \pm t\kappa \in B_1(o),\ u(y - t\kappa) \ge u(y + t\kappa). \tag{3.2}$$

The assertion in (3.2) follows quite easily if the plane P_κ does not intersect $\hat{C}_\alpha$. In case it does, we recall that ϕ is axially symmetric and decreasing in x_n. These properties allows us to compare boundary data after reflection and (3.2) follows.

Let S be a great semicircle, centered at o, with end points re_n and $-re_n$. Applying (3.2) with suitable planes P_κ one sees that for $x \in S$, $u(x)$ decreasing in x_n.

Thus for every $r \in (0,1)$, $u(re_n) = -m(r)$ and $M(r) = u(-re_n)$. We have thus proven the first parts of (i) and (ii) but for the estimates.

To prove the rest of part (i), we note that the function

$$v(x) = \frac{u(x)+m(1)}{1+m(1)}$$

satisfies the hypothesis of Lemma 3.1. For $\pi-\alpha$ small, $v(o) \geq 1-c(\pi-\alpha)^{1/3}$. Since $u(o)=0$, it is clear that for α close to π, $m(1) \geq \hat{c}(\pi-\alpha)^{-1/3}$, for some $\hat{c}>0$. Next by taking $x = re_n$ and applying the inequality in Lemma 3.2 to v, we see

$$m(1)-m(r) \geq (1+m(1))\Big(1-\frac{c(\pi-\alpha)^{1/3}}{(1+k(\alpha)-r)^{1/3}}\Big).$$

Using the convexity of m the above yields, for any $0<r<1$,

$$m'(1-) \geq \frac{m(1)-m(r)}{1-r} \geq \frac{1}{2}\Big(\frac{1+m(1)}{1-r}\Big),$$

as $\alpha \to \pi$. The conclusion for $m'(1-)$ follows.

Next we show that $u \geq 0$ on a large portion of $B_1(o)$. For $x \in B_1(o)$, $x \neq o$, let us denote by $\Pi(x)$ the $n-1$ dimensional plane passing through x with $x/|x|$ as its normal vector. By the set T, let us denote those points $x \in B_1(o)$ such that $x/|x| \in C_{\alpha-\pi/2}$. For such x's $\Pi(x)$ does not intersect $\hat{C}_\alpha$. Recall that $u=1$ on $C_{\alpha-\pi/2}$. Let $x \in T$, using reflection about $\Pi(x)$ and the comparison principle, one notes that for $t>0$, $u(x+tx/|x|)$ is increasing in t. Since $u(o)=0$, we have $u(x) \geq 0$ whenever $x \in T$.

We now prove the rest of the lemma. From the foregoing, $B_{|\cos\alpha|}(-e_n) \cap B_1(o) \subset T$. If we set $w(x) = (1-|x+e_n||\sec\alpha|)$ then by comparison $u(x) \geq w(x)$, $x \in B_{|\cos\alpha|}(-e_n) \cap B_1(o)$, and $M(r) = u(-re_n) \geq 1-(1-r)|\sec\alpha|$. Working with $1-u$, applying (2.1) (i) and convexity, the one-sided gradient $M'(1-)$ exists and

$$1 \leq M'(1-) = \lim_{r\uparrow 1}\frac{1-u(-re_n)}{1-r} \leq |\sec\alpha| \to 1 \quad \text{as } \alpha \to \pi.$$

This proves part (ii).

To show the last part, recall from (i) that $m(r) = -u(re_n)$. To estimate the one-sided gradient $L(p)$, we apply (2.1)(i) to $1-u$ near p, (2.2) and the Harnack inequality [2, 7, 16] to conclude

$$L(p) \geq \frac{1-u(p-\varepsilon p)}{\varepsilon} \geq \frac{1+m(1-\varepsilon)}{\varepsilon}\exp\Big(\frac{-(1-\varepsilon)(\pi-\alpha)}{\varepsilon}\Big).$$

We may take $\pi-\alpha=\varepsilon$ and the conclusion now follows by taking $\varepsilon \to 0$. $\square$

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Tilak Bhattacharya
Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101, USA
E-mail address: tilak.bhattacharya@wku.edu