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BOUNDEDNESS OF SOLUTIONS FOR A LIÉNARD EQUATION WITH MULTIPLE DEVIATING ARGUMENTS

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Abstract. We consider the Liénard equation

$$x''(t)+f_1(x(t))(x'(t))^2+f_2(x(t))x'(t)+g_0(x(t))+\sum_{j=1}^{m}g_j(x(t-\tau_j(t)))=p(t),$$

where f_1, f_2, g_1 and g_2 are continuous functions, the delays $\tau_j(t)\ge 0$ are bounded continuous, and $p(t)$ is a bounded continuous function. We obtain sufficient conditions for all solutions and their derivatives to be bounded.

1. Introduction

Consider the Liénard type equation with multiple deviating arguments

$$x''(t)+f_1(x(t))(x'(t))^2+f_2(x(t))x'(t)+g_0(x(t))+\sum_{j=1}^{m}g_j(x(t-\tau_j(t)))=p(t), \quad (1.1)$$

where f_1, f_2, g_1 and g_2 are continuous functions on $R=(-\infty,+\infty)$, $\tau_j(t)\ge 0$, $j=1,2,\dots,m$ are bounded continuous functions on R, and $p(t)$ is a bounded continuous function on $R^+=[0,+\infty)$. Define

$$a(x)=\exp\Big(\int_0^x f_1(u)du\Big),\quad \varphi(x)=\int_0^x a(u)[f_2(u)-a(u)]du,\quad y=a(x)\frac{dx}{dt}+\varphi(x),$$

then we transform (1.1) into the system

$$\begin{gathered}\frac{dx(t)}{dt}=\frac{1}{a(x(t))}[-\varphi(x(t))+y(t)],\\ \frac{dy(t)}{dt}=a(x(t))\Big\{-y(t)-[g_0(x(t))-\varphi(x(t))]-\sum_{j=1}^{m}g_j(x(t-\tau_j(t)))+p(t)\Big\}.\end{gathered} \quad (1.2)$$

In applied science some practical problems concerning physics, mechanics and the engineering technique fields associated with Liénard equation can be found in [1, 2, 5, 10]. Hence, it has been the object of intensive analysis by numerous authors.

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In particular, there have been extensive results on boundedness of solutions of Liénard equation with delays in the literature. For example, the authors in [3, 4, 8] establish some sufficient conditions to ensure the boundedness for all solutions of (1.1) without delays; Zhang [11], Liu and Huang [6] consider the boundedness for all solutions of (1.1) with constant delays; We only find that Liu and Huang [7] establish some sufficient conditions to ensure the boundedness for all solutions of (1.1) with a deviating argument. However, to the best of our knowledge, few authors have considered boundedness of solutions of Liénard equation with multiple deviating arguments (See [7]). Thus, it is worth while to continue to investigate the boundedness of solutions of (1.1) in this case.

A primary purpose of this paper is to study the boundedness of solutions of (1.2). We will establish some sufficient conditions for all solutions of (1.2) to be bounded. If applying our results to (1.1), one will find that our results are different from those in the references. An illustrative example is given in the last section.

2. Definitions and assumptions

We assume that $h = \max_{1\le j\le m}\{\sup_{t\in\mathbb{R}} \tau_j(t)\} \ge 0$. Let $C([-h,0],R)$ denote the space of continuous functions $\phi : [-h,0] \to R$ with the supremum norm $\|\cdot\|$. It is known in [1, 2, 5, 10] that for $g_1, g_2, \varphi, \tau_j$ and p continuous, given a continuous initial function $\phi \in C([-h,0],R)$ and a number $y_0 \in \mathbb{R}$, then there exists a solution of (1.2) on an interval $[0,T)$ satisfying the initial condition and satisfying (1.2) on $[0,T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $(x(t), y(t)) = (x(t,\phi,y_0), y(t,\phi,y_0))$.

Definition. Solutions of (1.2) are uniformly bounded (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that

$$(\phi, y_0) \in C([-h,0],R) \times R \quad \text{and} \quad \|\phi\| + |y_0| \le B_1$$

imply that $|x(t,\phi,y_0)| + |y(t,\phi,y_0)| \le B_2$ for all $t \in \mathbb{R}^+$.

In this paper, we will assume that the following conditions:

- (C1) There exists a constant $\underline{d} > 1$ such that $\underline{d}|u| \le \operatorname{sign}(u)\varphi(u)$, for all $u \in \mathbb{R}$.
- (C2) For $j = 0,1,2,\dots,m$, there exist nonnegative constants L_j and q_j such that for all $u \in \mathbb{R}$,

$$\sum_{j=0}^{m} L_j < 1, |(g_0(u) - \varphi(u))| \le L_0|u| + q_0,$$
$$|g_1(u)| \le L_1|u| + q_1, \dots, |g_m(u)| \le L_m|u| + q_m.$$

3. Main result

Theorem 3.1. *Suppose that* (C1), (C2) *hold. Then solutions of* (1.2) *are uniformly bounded.*

Proof. Let $(x(t), y(t)) = (x(t,\phi,y_0), y(t,\phi,y_0))$ be a solution of (1.2) defined on $[0,T)$. We may assume that $T = +\infty$ since the estimates which follow give an a priori bound on $(x(t), y(t))$.

Calculating the upper right derivative of $|x(s)|$ and $|y(s)|$ along (1.2), in view of (C1) and (C2), we have

$$D^+(|x(s)|)|_{s=t} = \operatorname{sign}(x(t))\{\frac{1}{a(x(t))}[-\varphi(x(t)) + y(t)]\} \le \frac{1}{a(x(t))}[-\underline{d}|x(t)| + |y(t)|], \tag{3.1}$$

and

$$\begin{aligned} &D^+(|y(s)|)|_{s=t} \\ &= \operatorname{sign}(y(t))a(x(t))\{-y(t) - [g_0(x(t)) - \varphi(x(t))] - \sum_{j=1}^{m} g_j(x(t-\tau_j(t))) + p(t)\} \\ &\le a(x(t))\{-|y(t)| + L_0|x(t)| + \sum_{j=1}^{m} L_j|x(t-\tau_j(t))| + \sum_{j=0}^{m} q_j + |p(t)|\}. \end{aligned} \tag{3.2}$$

Let

$$M(t) = \max_{-h\le s\le t}\{\max\{|x(s)|,\ |y(s)|\}\}, \tag{3.3}$$

where $y(s) = y(0)$, for all $-h \le s \le 0$. It is obvious that $\max\{|x(t)|,\ |y(t)|\} \le M(t)$, and $M(t)$ is non-decreasing, for $t \ge -h$. Now, we consider two cases.

Case (i):

$$M(t) > \max\{|x(t)|, |y(t)|\} \quad \text{for all } t \ge 0\,. \tag{3.4}$$

We claim that

$$M(t) \equiv M(0), \quad \text{a constant for all } t \ge 0. \tag{3.5}$$

Assume, by way of contradiction, that (3.5) does not hold. Then, there exists $t_1 > 0$ such that $M(t_1) > M(0)$. Since

$$\max\{|x(t)|, |y(t)|\} \le M(0) \quad \text{for all } -h \le t \le 0.$$

There must exist $\beta \in (0,\ t_1)$ such that

$$\max\{|x(\beta)|, |y(\beta)|\} = M(t_1) \ge M(\beta),$$

which contradicts (3.4). This contradiction implies that (3.5) holds. It follows that

$$\max\{|x(t)|,\ |y(t)|\} \le M(t) = M(0) \quad \text{for all } t \ge 0. \tag{3.6}$$

Case (ii): There is a point $t_0 \ge 0$ such that $M(t_0) = \max\{|x(t_0)|,\ |y(t_0)|\}$. Let

$$\eta = \min\{\underline{d} - 1,\, 1 - \sum_{j=0}^{m} L_j\},\ \theta = \sum_{j=0}^{m} q_j + \sup_{t\in\mathbb{R}^+} |p(t)| + 1$$

Then, if $M(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |x(t_0)|$, in view of (3.1), we have

$$\begin{aligned} D^+(|x(s)|)|_{s=t_0} &\le \frac{1}{a(x(t_0))}[-\underline{d}|x(t_0)| + |y(t_0)|] \\ &\le \frac{1}{a(x(t_0))}(-\underline{d} + 1)M(t_0) \\ &< \frac{1}{a(x(t_0))}[-\eta M(t_0) + \theta]. \end{aligned} \tag{3.7}$$

If $M(t_0) = \max\{|x(t_0)|,\ |y(t_0)|\} = |y(t_0)|$, in view of (3.2), we obtain

$$\begin{aligned}
&D^+(|y(s)|)|_{s=t_0}\\
&\le a(x(t_0))\{-|y(t_0)| + L_0|x(t_0)| + \sum_{j=1}^{m} L_j|x(t_0-\tau_j(t_0))| + \sum_{j=0}^{m} q_j + |p(t_0)|\}\\
&< a(x(t_0))[(-1+\sum_{j=0}^{m} L_j)M(t_0)+\theta]\\
&\le a(x(t_0))[-\eta M(t_0)+\theta].
\end{aligned} \tag{3.8}$$

In addition, if $M(t_0) \ge \frac{\theta}{\eta}$, it follows from (3.7) and (3.8) that $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0+\delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$\max\{|x(t_0)|,\ |y(t_0)|\} = M(t_0) < \frac{\theta}{\eta}. \tag{3.9}$$

For $t > t_0$, by the same approach used in the proof of (3.9), we have

$$\max\{|x(t)|,\ |y(t)|\} < \frac{\theta}{\eta}, \quad \text{if } M(t) = \max\{|x(t)|,\ |y(t)|\}. \tag{3.10}$$

On the other hand, if $M(t) > \max\{|x(t)|,\ |y(t)|\}$, $t > t_0$. We can choose $t_0 \le t_2 < t$ such that for all $s \in (t_2, t]$,

$$M(t_2) = \max\{|x(t_2)|,\ |y(t_2)|\} < \frac{\theta}{\eta}, \quad M(s) > \max\{|x(s)|,\ |y(s)|\}.$$

Using a similar argument as in the proof of Case (i), we can show that

$$M(s) \equiv M(t_2) \quad \text{a constant, for all } s \in (t_2, t],$$

which implies

$$\max\{|x(t)|,\ |y(t)|\} < M(t) = M(t_2) = \max\{|x(t_2)|,\ |y(t_2)|\} < \frac{\theta}{\eta}.$$

In summary, the solutions of (1.2) are uniformly bounded. The proof is complete. □

4. An example

Consider the Liénard equation with two deviating arguments

$$\begin{aligned}
&x''(t) + (x'(t))^2 + [e^{-x(t)}(3x^2(t)+2) + e^{x(t)}]x'(t) + \frac{1}{2}\sin x(t) + x^3(t) + 2x(t)\\
&+ \frac{1}{6}|x(t-|\sin t|)| + \frac{1}{6}\arctan x(t - \frac{1}{1+t^2}) = e^{\frac{1}{t^2+1}},
\end{aligned} \tag{4.1}$$

All solutions and their derivatives are bounded. Set

$$a(x) = e^x, \quad \varphi(x) = \int_0^x (3u^2+2)du, \quad y = e^x\frac{dx}{dt} + x^3 + 2x, \tag{4.2}$$

then we can transform (1.1) into the system

$$\begin{gathered}
\frac{dx(t)}{dt} = e^{-x(t)}[-(x^3(t)+2x(t))+y(t)],\\
\frac{dy(t)}{dt} = e^{x(t)}\big[-y(t)-\frac{1}{2}\sin x(t)-\frac{1}{6}|x(t-|\sin t|)| \\
-\frac{1}{6}\arctan x(t-\frac{1}{1+t^2})+e^{\frac{1}{t^2+1}}\big].
\end{gathered}\tag{4.3}$$

It is straight forward to check that all assumptions needed in Theorem 3.1 are satisfied. Therefore, solutions of system (4.3) are uniformly bounded. This implies that all solutions of (4.1) and their derivatives are bounded.

Remark. Equation (4.1) is a very simple Liénard equation with two deviating arguments. Liénard equations with constant delays have been studied in [3, 4, 6, 8, 9, 11], and with one deviating argument in [7]. It is also clear that the results obtained in [3, 4, 6, 7, 8, 9] can not be applied to (4.1). Since we proved boundedness of solutions to Liénard equation by a different method, the results in this article are essentially new.

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