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POSITIVE SOLUTIONS FOR THIRD-ORDER STURM-LIOUVILLE BOUNDARY-VALUE PROBLEMS WITH p-LAPLACIAN

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Abstract. In this article, we consider the third-order Sturm-Liouville boundary value problem, with p-Laplacian,

$$(\phi_p(u''(t)))' + f(t,u(t)) = 0, \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \quad u''(0) = 0,$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. By means of the Leggett-Williams fixed-point theorems, we prove the existence of multiple positive solutions. As an application, we give an example that illustrates our result.

1. Introduction

In this paper, we study the existence of multiple positive solutions for the following third-order Sturm-Liouville boundary value problem with p-Laplacian

$$(\phi_p(u''(t)))' + f(t,u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \quad u''(0) = 0, \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta, \gamma, \delta \geq 0$.

During the past decades, wide attention has been paid to the study equations with p-Laplacian operator, which arises in the modelling of different physical and natural phenomena, non-Newtonian mechanics [3, 9], combustion theory [19], population biology [17, 18], nonlinear flow laws [5, 13, 14], and system of Monge-Kantorovich partial differential equations [4]. There exist a very large number of papers devoted to the existence of solutions of the p-Laplacian operator. The second-order problem,

$$(\phi_p(u'(t)))' + f(t,u(t)) = 0, \quad t \in (0,1),$$

with various boundary conditions has been studied by many authors, see [8, 12, 15, 16, 20, 21, 22, 23] and the references therein. However, to the best of our knowledge, few papers can be found in the literature on the existence of multiple positive solutions for the third-order Sturm-Liouville boundary value problem (1.1), (1.2). The purpose here is to fill this gap in the literature. Motivated by the works

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[1] and [7], we shall establish the existence of at least two or at least three positive solutions to third-order Sturm-Liouville boundary value problem with p-Laplacian (1.1), (1.2) by using fixed point theorems in cones.

By a positive solution of (1.1) and (1.2) we understand a function $u(t) \in C^2[0,1]$ which is positive on $0 < t < 1$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

In this article, we use the following assumptions:

(A1) $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$, $0 < \sigma := \min\big\{\frac{4\delta+\gamma}{4(\delta+\gamma)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\big\} < 1$.

(A2) $G(t,s)$ is the Green's function of the differential equation $u''(t) = 0$, $t \in (0,1)$ with respect to the boundary value condition (1.2), i.e.,

$$G(t,s) = \begin{cases} \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1. \\ \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \le t \le s \le 1. \end{cases}$$

Evidently $G(t,s) \le G(s,s)$, $0 \le t, s \le 1$.

(A3) $f \in C([0,1] \times [0,\infty); [0,\infty))$.

For convenience, we denote

$$\zeta(a) = \max\{f(t,u) : 0 \le t \le 1,\ 0 \le u \le a\},$$
$$\psi(b) = \min\{f(t,u) : \frac{1}{4} \le t \le \frac{3}{4},\ b \le u \le \frac{b}{\sigma^2}\}.$$

Where σ is given as in (A1). Our main results are the following.

Theorem 1.1. *Assume (A1)–(A3), and that there exist constants $0 < a < b$ such that*

$$\zeta(a) < (ma)^{p-1} \tag{1.3}$$
$$\psi(b) \ge (lb)^{p-1}. \tag{1.4}$$

Then the boundary value problem (1.1), (1.2) has at least two positive solutions u_1, u_2 satisfying $\|u_1\| < a$, $\min_{t\in[\frac14,\frac34]} u_2(t) < b$ and $\|u_2\| > a$, where

$$m = \Big(\int_0^1 G(s,s)ds\Big)^{-1} = \frac{6\rho}{\alpha\gamma + 3\alpha\delta + 3\beta\gamma + 6\beta\delta},$$
$$l = \frac{2}{\sigma 4^{1-q}}\Big(\int_{1/4}^{3/4} G(\frac{1}{2}, s)ds\Big)^{-1} = \frac{2}{\sigma 4^{1-q}} \cdot \frac{32\rho}{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta}.$$

Theorem 1.2. *Assume (A1)–(A3) and that there exist constants a, b, c such that $0 < a < b < \sigma^2 c$ implies*

$$\zeta(a) < (ma)^{p-1}, \tag{1.5}$$
$$\psi(b) \ge (lb)^{p-1}, \tag{1.6}$$
$$\zeta(c) \le (mc)^{p-1}. \tag{1.7}$$

Then the boundary value problem (1.1), (1.2) has at least three positive solutions u_1, u_2 and u_3 with $\|u_1\| < a$, $\min_{t\in[\frac14,\frac34]} u_2(t) > b$, $\|u_3\| > a$ and $\min_{t\in[\frac14,\frac34]} u_3(t) < b$, where σ is given as in (A1) and m, l are given as in Theorem 1.1.

The proofs of theorems are based upon the Leggett-Williams fixed-point theorems [10]. These theorems have been useful technique for proving the existence of three or two solutions for boundary value problems of differential and difference equations, see [1, 2, 7].

2. Preliminaries

In this section we summarize some basic concepts and results which are taken from Guo and Lakshmikantham [6], and from Leggett and Williams [10].

Definition 2.1. Let E be a real Banach space and P be a nonempty, convex closed set in E. We say that P is a cone if it satisfies the following properties: (i) $\lambda u \in P$ for $u \in P, \lambda \geq 0$; (ii) $u, -u \in P$ implies $u = \theta$(θ denotes the null element of E).

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$. For $u, v \in P$, we write $u \leq v$ if $v - u \in P$.

Definition 2.2. The map φ is said to be a nonnegative continuous concave functional on P provided that $\varphi : P \to [0, \infty)$ is continuous and $\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y)$ for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3. Let $0 < a < b$ be given and let φ be a nonnegative continuous concave functional on the cone P. Define the convex sets $P_r, \bar{P}_r$ and $P(\varphi, a, b)$ by $P_r = \{y \in P : \|y\| < r\}$, $\bar{P}_r = \{y \in P : \|y\| \leq r\}$, $P(\varphi, a, b) = \{y \in P : a \leq \varphi(y), \|y\| \leq b\}$.

Theorem 2.4 (Leggett-Williams [10])**.** *Let $T : \bar{P}_c \to \bar{P}_c$ be a completely continuous operator and let φ be a nonnegative continuous concave functional on P such that $\varphi(y) \leq \|y\|$ for all $y \in \bar{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(a') *$\{y \in P(\varphi, b, d) : \varphi(y) > b\} \neq \emptyset$ and $\varphi(Ty) > b$ for $y \in P(\varphi, b, d)$;*
(b') *$\|Ty\| < a$ for $\|y\| \leq a$;*
(c') *$\varphi(Ty) > b$ for $y \in P(\varphi, b, c)$ with $\|Ty\| > d$.*

Then T has at least three fixed points y_1, y_2, y_3 in $\bar{P}_c$ satisfying $\|y_1\| < a, \varphi(y_2) > b, \|y_3\| > a$ and $\varphi(y_3) < b$

Theorem 2.5 ([10])**.** *Let $T : \bar{P}_c \to P$ be a completely continuous operator and let φ be a nonnegative continuous concave functional on P such that $\varphi(y) \leq \|y\|$ for all $y \in \bar{P}_c$. Suppose that there exist $0 < a < b < c$ such that*

(a") *$\{y \in P(\varphi, b, c) : \varphi(y) > b\} \neq \emptyset$, and $\varphi(Ty) > b$ for $y \in P(\varphi, b, c)$;*
(b") *$\|Ty\| < a$ for $\|y\| \leq a$;*
(c") *$\varphi(Ty) > \frac{b}{c}\|Ty\|$ for $y \in \bar{P}_c$ with $\|Ty\| > c$.*

Then T has at least two fixed points y_1, y_2 in $\bar{P}_c$ satisfying $\|y_1\| < a$, $\|y_2\| > a$ and $\varphi(y_2) < b$.

In the rest of this section we assume that (A1)-(A3) hold. Let $E = C[0,1]$ and $C^+[0,1] = \{x \in E | x(t) \geq 0,\ t \in [0,1]\}$. Define an operator T by

$$(Tu)(t) = \int_0^1 G(t,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv, \quad \forall u \in C^+[0,1].$$

From (A2) and (A3), we can easily get $(Tu)(t) \geq 0$, $t \in [0,1]$ for $u \in C^+[0,1]$.

Remark 2.6. Suppose that $u \in C^+[0,1]$ satisfies of the operator equation, $Tu = u$. We can obtain

$$\begin{aligned} u'(t) = &-\frac{\gamma}{\rho}\int_0^t (\beta + \alpha v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \\ &+ \frac{\alpha}{\rho}\int_t^1 (\gamma + \delta - \gamma v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv, \end{aligned}$$

$$u''(t) = (Tu)''(t) = -\phi_q\Big(\int_0^t f(s,u(s))ds\Big).$$

So we have

$$\phi_p(u''(t)) = -\int_0^t f(s,u(s))ds,$$

and in consequence, $(\phi_p(u''(t)))' = -f(t,u(t))$. Moreover, it is clear that

$$\begin{aligned}\alpha u(0) - \beta u'(0) &= \alpha\int_0^1 G(0,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv\\ &\quad - \beta\cdot\frac{\alpha}{\rho}\int_0^1(\gamma+\delta-\gamma v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv = 0;\end{aligned}$$

$$\begin{aligned}\gamma u(1) + \delta u'(1) &= \gamma\int_0^1 G(1,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv\\ &\quad + \big(-\frac{\gamma}{\rho}\big)\cdot\delta\int_0^1(\beta+\alpha v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv = 0.\end{aligned}$$

Further, $u''(0) = 0$, that is to say, all the fixed points of operator T are the solutions for the problem (1.1), (1.2).

Lemma 2.7 ([11])**.** *Suppose that $G(t,s)$ is defined as in (A2). Then*

$$\begin{gathered}\frac{G(t,s)}{G(s,s)} \le 1 \quad \text{for } t\in[0,1],\ s\in[0,1],\\ \frac{G(t,s)}{G(s,s)} \ge \sigma \quad \text{for } t\in[\frac14,\frac34],\ s\in[0,1].\end{gathered}$$

Lemma 2.8. *The operator $T : C^+[0,1] \to C^+[0,1]$ is completely continuous; i.e., T is continuous and compact.*

Proof. Firstly, we show that $T : C^+[0,1] \to C^+[0,1]$ is continuous. From Remark 2.6, we know that $T : C^+[0,1] \to C^+[0,1]$. Suppose $\{u_n\} \subset C^+[0,1]$, $u_n \to \bar u (n \to \infty)$. Then $\bar u \in C^+[0,1]$ and there exists a constant $M_0 > 0$ such that $\|u_n\| \le M_0$, $\|\bar u\| \le M_0$. Let $M_1 = \max\{f(t,u)|t\in[0,1],\ u\in[0,M_0]\}$. Then for $t\in[0,1]$ we have

$$\begin{aligned}|Tu_n(t) - T\bar u(t)| &\le \int_0^1 G(t,v)\Big|\phi_q\Big(\int_0^v f(s,u_n(s))ds\Big) - \phi_q\Big(\int_0^v f(s,\bar u(s))ds\Big)\Big|dv\\ &\le \int_0^1 G(v,v)\Big|\phi_q\Big(\int_0^v f(s,u_n(s))ds\Big) - \phi_q\Big(\int_0^v f(s,\bar u(s))ds\Big)\Big|dv\\ &\le \int_0^1 2\phi_q(M_1)G(v,v)dv.\end{aligned}$$

Note that $f(t,u)$ is continuous. We know that $\phi_q(\int_0^v f(s,u)ds)$ is continuous in u on $[0,\infty)$. Then for for each $\varepsilon > 0$, there exists $\delta_1 > 0$, such that $|u_1 - u_2| < \delta_1$ and we

$$\Big|\phi_q\Big(\int_0^v f(s,u_1(s))ds\Big) - \phi_q\Big(\int_0^v f(s,u_2(s))ds\Big)\Big| < \frac{\varepsilon}{G(v,v)}.$$

In view of $u_n(s) \to \bar{u}(s)$, as $n \to \infty$, there exists a natural number $N > 0$, for $n > N$ with $|u_n(s) - \bar{u}(s)| < \delta_1$, we have

$$\big|\phi_q\Big(\int_0^v f(s, u_n(s))ds\Big) - \phi_q\Big(\int_0^v f(s, \bar{u}(s))ds\Big)\big| < \frac{\varepsilon}{G(v,v)}.$$

Thus for $\varepsilon > 0$, there exists $N > 0$, such that when $n > N$,

$$G(v,v)\big|\phi_q\Big(\int_0^v f(s, u_n(s))ds\Big) - \phi_q\Big(\int_0^v f(s, \bar{u}(s))ds\Big)\big| < \varepsilon, \quad \text{a.e. } [0,1].$$

An application of Lebesgue's dominated convergence theorem implies

$$|Tu_n(t) - T\bar{u}(t)| \to 0 (\text{as } n \to \infty),\ t \in [0,1].$$

So operator $T : C^+[0,1] \to C^+[0,1]$ is continuous.

Next we prove that T is compact. Let $\Omega \subset C^+[0,1]$ be a bounded set. Then there exists $R > 0$ such that $\Omega \subset \{u \in C^+[0,1] | \|u\| \le R\}$. Set $M = \max\{f(t,u) | t \in [0,1],\ u \in \Omega\}$. For any $u \in \Omega$, we have

$$|(Tu)(t)| = \big|\int_0^1 G(t,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv\big| \le \int_0^1 G(v,v)\phi_q(M)dv,$$

which implies that $T(\Omega)$ is uniformly bounded.

Furthermore, for any $u \in \Omega$ and $t \in [0,1]$, we have

$$\begin{aligned}
|(Tu)'(t)| &= \Big| -\frac{\gamma}{\rho}\int_0^t (\beta + \alpha v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \\
&\quad + \frac{\alpha}{\rho}\int_t^1 (\gamma + \delta - \gamma v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv\Big| \\
&\le \phi_q(M)\Big[\frac{\gamma}{\rho}\int_0^t (\beta + \alpha v)dv + \frac{\alpha}{\rho}\int_t^1 (\gamma + \delta - \gamma v)dv\Big] \\
&= \phi_q(M)t \le \phi_q(M).
\end{aligned}$$

Hence $\|(Tu)'\| \le \phi_q(M)$. So we can easily prove that $T(\Omega)$ is equicontinuous. The Arzela-Ascoli Theorem guarantee that $T(\Omega)$ is relatively compact and therefore that T is compact. $\square$

3. Proofs of main results

In this section, we prove the existence of multiplicity results. Let $E = C[0,1]$ be endowed with the maximum norm $\|y\| = \max_{t\in[0,1]} |y(t)|$, and the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]$. Define the cone $P \subset E$ by

$$P = \{u \in C^+[0,1] : \min_{t\in[\frac14,\frac34]} u(t) \ge \sigma\|u\|\},$$

where σ is given as in (A1). Next we show that $T(P) \subset P$. For any $u \in P$ and $t \in [0,1]$, from Lemma 2.7 we have

$$Tu(t) = \int_0^1 G(t,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \le \int_0^1 G(v,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv.$$

Consequently,

$$\|Tu\| \le \int_0^1 G(v,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv.$$

Further, for $u \in P$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, from Lemma 2.7 we obtain

$$\begin{aligned}\min_{t\in[\frac{1}{4},\frac{3}{4}]} Tu(t) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 G(t,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \\ &\geq \sigma \int_0^1 G(v,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \geq \sigma\|Tu\|.\end{aligned}$$

From Lemma 2.8, we know that $T : P \to P$ is completely continuous. Let $\varphi : P \to [0,\infty)$ be the nonnegative continuous concave functional defined by

$$\varphi(u) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t), \quad u \in P.$$

Evidently, for each $u \in P$, we have $\varphi(u) \leq \|u\|$. We are now in a position to proving the main results.

Proof of Theorem 1.1. It is easy to see that $T : \bar{P}_{\frac{b}{\sigma^2}} \to P$ is completely continuous and $0 < a < b < \frac{b}{\sigma^2}$. Choose $u(t) = \frac{b}{\sigma^2}$, then

$$u \in P\big(\varphi, b, \frac{b}{\sigma^2}\big), \quad \varphi(u) = \frac{b}{\sigma^2} > b.$$

So $\{u \in P(\varphi, b, \frac{b}{\sigma^2}) : \varphi(u) > b\} \neq \emptyset$. Hence, if $u \in P(\varphi, b, \frac{b}{\sigma^2})$, then $b \leq u(t) \leq \frac{b}{\sigma^2}$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Thus for $t \in [\frac{1}{4}, \frac{3}{4}]$, from assumption (1.4), we have

$$f(t,u(t)) \geq \psi(b) \geq (lb)^{p-1},\ t \in [\frac{1}{4},\frac{3}{4}].$$

Hence

$$\begin{aligned}Tu(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2},v)\phi_q\Big(\int_0^v f(s,u(s)\Big)ds)dv \\ &\geq \int_{1/4}^{3/4} G(\frac{1}{2},v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \\ &\geq \int_{1/4}^{3/4} G(\frac{1}{2},v)lbv^{q-1}dv \\ &\geq (\frac{1}{4})^{q-1}lb\int_{1/4}^{3/4} G(\frac{1}{2},v)dv = \frac{2b}{\sigma} > \frac{b}{\sigma}.\end{aligned}$$

Consequently,

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} Tu(t) \geq \sigma\|Tu\| > \sigma \times \frac{b}{\sigma} = b \ \text{ for } b \leq u(t) \leq \frac{b}{\sigma^2},\ t \in [\frac{1}{4},\frac{3}{4}].$$

That is,

$$\varphi(Tu) > b, \forall\ u \in P\big(\varphi, b, \frac{b}{\sigma^2}\big).$$

Therefore, condition (a") of Theorem 2.5 is satisfied. Now if $u \in \bar{P}_a$, then $\|u\| \leq a$. By assumption (1.3), we have $f(t,u(t)) \leq \zeta(a) < (ma)^{p-1}$, $t \in [0,1]$. Consequently,

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]} |Tu(t)| = \max_{t\in[0,1]} \int_0^1 G(t,v)\phi_q\Big(\int_0^v f(s,u(s))ds\Big)dv \\ &< ma\max_{t\in[0,1]} \int_0^1 G(t,v)dv \leq ma\int_0^1 G(v,v)dv = a.\end{aligned}$$

This shows that $T:\bar{P}_a\to P_a$. That is, $\|Tu\|<a$ for $u\in\bar{P}_a$. This shows that condition (b") of Theorem 2.5 is satisfied. Finally, we show that (c") of Theorem 2.5 also holds. Assume that $u\in \bar{P_{\frac{b}{\sigma^2}}}$ with $\|Tu\|>\frac{b}{\sigma^2}$, then by the definition of cone P, we have

$$\varphi(Tu)=\min_{t\in[\frac14,\frac34]}Tu(t)\ge\sigma\|Tu\|>\sigma^2\|Tu\|=b/\frac{b}{\sigma^2}\|Tu\|.$$

So condition (c") of Theorem 2.5 is satisfied. Thus using Theorem 2.5, T has at least two fixed points. That is to say, problem (1.1),(1.2) has at least two positive solutions u_1,u_2 in $\bar{P_{\frac{b}{\sigma^2}}}$ satisfying $\|u_1\|<a$, $\min_{t\in[\frac14,\frac34]}u_2(t)<b$ and $\|u_2\|>a$. □

Proof of Theorem 1.2. It follows from the conditions (1.5)-(1.7) in Theorem 1.2 that $a<b<\frac{b}{\sigma^2}<c$. Using the same arguments as in the proof of Theorem 1.1, we have: $T:\bar{P}_c\to\bar{P}_c$ is a completely continuous operator and $T:\bar{P}_a\to P_a$. Also

$$\{u\in P\big(\varphi,b,\frac{b}{\sigma^2}\big):\varphi(u)>b\}\ne\emptyset,\quad \varphi(Tu)>b\ \forall u\in P\big(\varphi,b,\frac{b}{\sigma^2}\big).$$

Moreover, for $u\in P(\varphi,b,c)$ and $\|Tu\|>\frac{b}{\sigma^2}$, we have

$$\varphi(Tu)=\min_{t\in[\frac14,\frac34]}Tu(t)\ge\sigma\|Tu\|>\frac{b}{\sigma}>b.$$

So all the conditions of Theorem 2.4 are satisfied. Thus using Theorem 2.4, T has at least three fixed points. That is to say, the boundary value problem (1.1),(1.2) has at least three positive solutions $u_1,u_2\,u_3$ with $\|u_1\|<a$, $\min_{t\in[\frac14,\frac34]}u_2(t)>b$, $\|u_3\|>a$ and $\min_{t\in[\frac14,\frac34]}u_3(t)<b$. □

Corollary 3.1. *Assume* (A1)–(A3) *and that there exist constants* $0<a_j<b_j<\sigma^2a_{j+1}$ $(j=1,2,\dots,n-1)$, $n\in N$ *such that*

- (B1) $\zeta(a_j)<(ma_j)^{p-1}$, $1\le j\le n$.
- (B2) $\psi(b_j)\ge(lb_j)^{p-1}$, $1\le j\le n-1$, *where* σ *is given as in (A1) and* m,l *are given as in Theorem 1.1.*

Then the boundary value problem (1.1),(1.2) *has at least* $2n-1$ *positive solutions.*

The proof of the above corollary is an immediate consequence of Theorem 1.2.

Remark 3.2. When $p=2$, problem (1.1), (1.2) is the usual form of third-order Sturm-Liouville boundary value problem

$$u'''(t)+f(t,u(t))=0,\ t\in(0,1),$$
$$\alpha u(0)-\beta u'(0)=0,\quad \gamma u(1)+\delta u'(1)=0,\quad u''(0)=0.$$

Using the same method, we can present some sufficient conditions that guarantee the existence of at least two or three positive solutions for the above boundary value problem. These results are also new and different from previous results.

4. An example

Now we consider an example to illustrate our results. Consider the third-order Sturm-Liouville boundary value problem, with p-Laplacian,

$$(\phi_3(u''(t)))'+[\varphi(t)h(u(t))]^2=0,\quad t\in(0,1),\tag{4.1}$$
$$u(0)-u'(0)=0,\quad u(1)=0,\quad u''(0)=0,\tag{4.2}$$

where $\varphi(t) = 4t$, $t \in [0,1]$ and

$$h(u) = \begin{cases} u/2, & 0 \le u \le 3/512; \\ \frac{1021}{4}u - \frac{3057}{2048}, & \frac{3}{512} \le u \le \frac{5}{512}; \\ 1, & \frac{5}{512} \le u \le \frac{5}{32}; \\ \frac{8}{59}u + \frac{231}{236}, & \frac{5}{32} \le u \le 2; \\ 5u/8, & u \ge 2. \end{cases}$$

In this example, we note that $p = 3$, $\alpha = \beta = \gamma = 1$, $\delta = 0$. After a simple calculation, we get $q = 3/2$, $\rho = 2$, $\sigma = \frac{1}{4} < 1$, $G(s,s) = \frac{1}{2}(1 - s^2)$ and

$$m = \frac{6\rho}{\alpha\gamma + 3\alpha\delta + 3\beta\gamma + 6\beta\delta} = 3, \quad l = \frac{2}{\sigma 4^{1-q}} \cdot \frac{32\rho}{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta} = \frac{512}{5}.$$

We choose $a = \frac{3}{512}$, $b = \frac{5}{512}$, $c = 2$. Evidently, $a < b < \sigma^2 c$ and

(i) for $t \in [0,1]$, $0 \le u \le \frac{3}{512}$, we have

$$f(t,u) = [\varphi(t)h(u)]^2 \le \big[4 \times \frac{1}{2} \times \frac{3}{512}\big]^2 < (ma)^2.$$

(ii) for $t \in [\frac{1}{4}, \frac{3}{4}]$, $\frac{5}{512} \le u \le \frac{b}{\sigma^2} = \frac{5}{32}$, we have

$$f(t,u) = [\varphi(t)h(u)]^2 \ge \big[4 \times \frac{1}{4} \times 1\big]^2 = (lb)^2.$$

(iii) for $t \in [0,1]$, $0 \le u \le 2$, we have

$$f(t,u) = [\varphi(t)h(u)]^2 \le \big[4 \times 1 \times \big(\frac{8}{59} \times 2 + \frac{231}{236}\big)\big]^2 \le (mc)^2.$$

Thus, $\zeta(a) < (ma)^2$, $\psi(b) \ge (lb)^2$, $\zeta(c) \le (mc)^2$.

Hence, all the conditions of Theorem 1.2 are satisfied. An application of Theorem 1.2 implies that (4.1), (4.2) has at least three positive solutions u_1, u_2 and u_3 with $\|u_1\| < \frac{3}{512}$, $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) > \frac{5}{512}$, $\|u_3\| > \frac{3}{512}$ and $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) < \frac{5}{512}$.

References

[1] R. P. Agarwal, D. O'Regan; A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem, *Appl. Math. Comput.*, **161** (2005), 433-439.

[2] D. Anderson, R. I. Avery and A. C. Peterson; Three positive solutions to a discrete focal boundary value problem, *J. Comput. Appl. Math.*, **88**(1998), 103-118.

[3] J. Diaz, F. de Thelin; On a nonlinear parabolic problem arising in some models related to turbulent flows, *SIAM J. Math. Anal.*, **25** (1994), (4), 1085-1111.

[4] L. Evans, W. Gangbo; Differential equations methods for the Monge-Kantorovich mass transfer problem, *Mem. Amer. Math. Soc.*, **137** (653), (1999).

[5] R. Glowinski, J. Rappaz; Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology, *Math. Model. Numer. Anal.*, **37** (2003) (1), 175-186.

[6] D. J. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, New York: Academic Press, 1988.

[7] J. Henderson and H. B. Thompson; Multiple symmetric positive solutions for a second order boundary value problem, *Proc. Amer. Math. Soc.*, **128** (2000), 2373-2379.

[8] X. He, W. Ge; Existence of positive solutions for the one-dimension p-Laplacian equation, *Acta Math.Sinica.*, **46** (2003) (4), 805-810(in Chinese).

[9] U. Janfalk; *On Certain Problem Concerning the p-Laplace Operator, in: Likping Studies in Sciences and Technology*, Dissertations, vol. 326, 1993.

[10] R. W. Leggett and L. R. Williams; Multiple positive fixed points of nonlinear operaters on ordered Banach space, *Indiana. univ. Math. J.*, **28** (1979), 673-688.

[11] W. C. Lian, F. H. Wong, C. C. Yeh; On the existence of positive solutions of nonlinear second order differential equations, *Proc. Amer. Math. Soc.*, **124** (1996), 1117-1126.

[12] B. Liu; Positive solutions of singular three-point boundary value problems for the one-dimensional p-Laplacian, *Comput. Math. Appl.*, **48** (2004), 913-925.

[13] C. Liu; Weak solutions for a viscous p-Laplacian equation, *Electron. J. Differential Equations*, **2003** (2003), no. 63, 1-11.

[14] I. Ly, D. Seck; Isoperimetric inequality for an interior free boundary problem with p-Laplacian operator, *Electron. J. Differential Equations*, **2004** (2004), Nov. 109, 1-12.

[15] D. Ma, Z. Du, W. Ge; Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator, *Comput. Math. Appl.*, **50** (2005), 729-739.

[16] D. O'Regan; Some general existence principles results for $(\psi_p(y'))' = q(t)f(t, y, y'), 0 < t < 1$, *SIAM J. Math. Anal.*, **24** (1993) (3) , 648-668.

[17] S. Oruganti, J. Shi, R. Shivaji; Diffusive logistic equation with constant yield harvesting, I:Steady-states, *Trans. Amer. Math. Soc.*, **354** (2002) (9) , 3601-3619.

[18] S. Oruganti, J. Shi, R. Shivaji; Logistic equation with p-Laplacian and constant yield harvesting, *Abstr. Appl. Anal.*, **9** (2004), 723-727.

[19] M. Ramaswamy, R. Shivaji; Multiple positive solutions for classes of p-Laplacian equations, *Differential Integral Equations*, **17**(2004) (11-12) , 1255-1261.

[20] Y. Wang, C. Hou; Existence of multiple positive solutions for one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **315** (2006), 144-153.

[21] Y. Wang, W. Ge; Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian, *Nonlinear Anal.*, **66** (2007), 1246-1256.

[22] C. B.Zhai; The existence of positive solutions for mixed boundary value problems of p-Laplace equations, *(English) Acta. Anal. Funct. Appl.*, **5(2)** (2003), 170-173.

[23] C.B.Zhai; Existence of positive solutions for mixed boundary value problems of p-Laplacian equations, *Chinese J. Engineering Mathematics* (in Chinese), **23**(2006), (6) 1053-1057 .

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