

STABILITY OF NEGATIVE SOLITARY WAVES

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ABSTRACT. The generalized regularized long-wave equation admits a family of negative solitary waves. We show that there is a critical wave speed dividing the range of stable and unstable negative solitary waves. Our proofs of stability and instability are based on a variant of the general theory by Grillakis, Shatah and Strauss.

1. INTRODUCTION

In this article, we consider the dynamic stability of negative solitary-wave solutions of the generalized regularized long-wave equation

$$u_t + u_x + (u^p)_x - u_{xxt} = 0, \quad (1.1)$$

where $p \geq 2$ is a positive integer. For $p = 2$, this equation is used to model the propagation of small-amplitude waves on the surface of a fluid contained in a long narrow channel [5, 18, 21].

It is well known that (1.1) admits solitary-wave solutions of the form $u(x, t) = \Phi_c(x - ct)$. Indeed, when this ansatz is substituted into (1.1), there appears the ordinary differential equation

$$-c\Phi_c + \Phi_c + c\Phi_c'' + \Phi_c^p = 0, \quad (1.2)$$

where $\Phi_c' = \frac{d\Phi_c}{d\xi}$, for $\xi = x - ct$. It is elementary to check that a solution of this equation is given by

$$\Phi_c(\xi) = A \operatorname{sech}^\sigma(K\xi), \quad (1.3)$$

where $\sigma = \frac{2}{p-1}$, $K = \frac{p-1}{2} \sqrt{\frac{c-1}{c}}$, and $A = \left[\frac{(p+1)(c-1)}{2}\right]^{1/(p-1)}$. For $c > 1$, these solutions are strictly positive progressive waves which propagate to the right (in the direction of increasing values of x) without changing their profile over time. Naturally, the question arises what happens for values of c less than one. Upon contemplating the formula (1.3), it appears that it gives a valid representation also for negative values of c , as long as p is even. The expression (1.3) then defines a strictly negative solitary wave propagating to the left (in the direction of decreasing values of x). As will be shown in Section 3, there are no solitary-wave solutions of (1.1) with $0 < c < 1$ for any p , and there are no solitary waves with $c < 0$ if p is odd. Negative solitary waves are possible if p is odd, but they are given by $-\Phi_c$ for

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$c > 1$. Thus it turns out that all solitary-wave solutions of (1.1) are given by the formula (1.3).

The main goal of this paper is to provide a sharp criterion for the stability and instability of solitary waves with negative propagation speed. Since the stability properties of solitary waves with positive propagation speeds are already well understood, a complete classification of the existence of positive and negative solitary waves and their stability properties is achieved. The proof of stability and instability given here is based on the general theory of Albert, Bona, Grillakis, Henry, Souganidis, Shatah and Strauss laid down in [1, 3, 8, 12], and pioneered by Bousinesq, Benjamin and others [4, 6, 9, 19]. However, the negativity of the solitary waves under study here necessitates an extension of the theory presented in the papers mentioned above.

We begin by recalling the relevant well-posedness theorems for (1.1) in Section 2. Then, in Section 3, the existence of solitary waves is considered, and the precise notion of stability to be shown is explained. Section 4 gives the relevant proof of instability, and Section 5 provides the proof of stability.

Before we leave the Introduction, some notation is established. For $1 \leq p < \infty$, the space $L^p = L^p(\mathbb{R})$ is the set of measurable real-valued functions of a real variable whose p^{th} powers are integrable over \mathbb{R} . If $f \in L^p$, its norm is denoted $\|f\|_{L^p}$. For $s \geq 0$, the space $H^s = H^s(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of functions such that

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + |\eta|^2)^s |\hat{f}(\eta)|^2 d\eta < +\infty,$$

where \hat{f} denotes the Fourier transform of f . The principal space to be used for the well-posedness theory will be $\mathcal{C}([0, T]; H^s)$ which consists of all functions $v(x, t)$, such that $v(\cdot, t)$ is a continuous function $t \mapsto H^s$ for $t \in [0, T]$. The norm is defined by

$$\|v\|_{C_T^s} = \sup_{0 \leq t \leq T} \|v(\cdot, t)\|_{H^s}.$$

In the same way, we define

$$\mathcal{C}^n([0, T]; H^s) = \{v(x, t) : \partial_t^k v(\cdot, t) \in \mathcal{C}([0, T]; H^s) \text{ for } 0 \leq k \leq n\},$$

and the corresponding norms $\|v\|_{C_T^{n,s}} = \sum_{k \leq n} \|\partial_t^k v\|_{C_T^s}$. Finally, we define the space $\mathcal{C}^\infty([0, T]; H^s) = \cap_{n \geq 0} \mathcal{C}^n([0, T]; H^s)$. Since all functions considered here are real-valued, we take the L^2 -inner product to be $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x) dx$. The convolution of two functions is defined as usual by $g * f(x) = \int_{-\infty}^{\infty} g(y)f(x-y)dy$.

2. WELL POSEDNESS AND INVARIANT INTEGRALS

To set the stage for the proof of stability and instability of the solitary-wave solutions, we will recall the well posedness theory for (1.1).

Theorem 2.1. *For each $u_0 \in H^1(\mathbb{R})$, there exists a unique global solution $u(x, t)$ of (1.1) with $u(\cdot, 0) = u_0$. Moreover, the solution depends continuously on the initial data in $\mathcal{C}([0, T]; H^1)$, for any $T > 0$.*

Remark 2.2. The solution is global in the sense that T can be chosen arbitrarily, and $\|u(\cdot, t)\|_{H^1}$ is bounded as a function of t . Thus we can conclude that $u \in \mathcal{C}([0, \infty); H^1)$. However, continuous dependence on the initial data holds only for a given finite T . Note also that u can be differentiated any number of times with respect to t , and therefore $u \in \mathcal{C}^\infty([0, \infty); H^1)$.

Proof of Theorem 2.1. The proof of this theorem is based on the works of Benjamin, Bona and Mahoney [5], and Albert and Bona [2]. While nothing new is presented here, we provide a short outline of the proof for the interested reader. First, local existence of a solution is established. Using the differential operator $1 - \partial_x^2$, the equation (1.1) can be rewritten in the form

$$(1 - \partial_x^2)u_t = -\partial_x(u + u^p). \quad (2.1)$$

It is elementary to check directly that $1 - \partial_x^2 : H^2 \subset L^2 \rightarrow L^2$ is self-adjoint with respect to the L^2 -inner product. Because the Green's function for $1 - \partial_x^2$ is $G(x) = \frac{1}{2}e^{-|x|}$, this equation is equivalent (at least in the sense of distributions) to

$$u_t = g * (u + u^p) = \mathcal{G}(u + u^p), \quad (2.2)$$

where $g(x) = -G'(x) = \frac{1}{2} \text{sign}(x)e^{-|x|}$, and the operator \mathcal{G} is defined by convolution with g . Recalling that the Fourier transform of g is given by $\hat{g}(\eta) = \frac{-1}{\sqrt{2\pi}} \frac{i\eta}{1+\eta^2}$, it is immediate that \mathcal{G} is a bounded operator on any Sobolev class $H^s(\mathbb{R})$. Integrating (2.2) in t , the following integral equations appears.

$$u(x, t) = u_0(x) + \int_0^t [\mathcal{G}(u(\cdot, \tau) + u^p(\cdot, \tau))](x) d\tau. \quad (2.3)$$

Thus, the first step of solving (1.1) will be to find a fixed-point for the map

$$\Gamma(v) = u_0 + \int_0^t [\mathcal{G}(v + v^p)] d\tau.$$

To this end, it will be shown that for sufficiently small t_0 , the map Γ is a contraction in a ball $B \subset \mathcal{C}([0, t_0]; H^1)$, where the radius of B is $2\|u_0\|_{H^1}$. Consider the estimate

$$\begin{aligned} \|\Gamma v(t)\|_{H^1} &\leq \|u_0\|_{H^1} + \int_0^t \|v(\cdot, \tau) + v^p(\cdot, \tau)\|_{H^1} d\tau \\ &\leq \|u_0\|_{H^1} + t(\|v\|_{C_{t_0}^1} + \|v\|_{C_{t_0}^p}). \end{aligned} \quad (2.4)$$

Taking the supremum over $t \in [0, t_0]$, it appears that Γ is a mapping on B if t_0 is chosen small enough. Now for the contractive property, consider

$$\begin{aligned} \|\Gamma v_1(t) - \Gamma v_2(t)\|_{H^1} &\leq \int_0^t \|(v_1 + v_1^p) - (v_2 + v_2^p)\|_{H^1} d\tau \\ &\leq t\|(v_1 - v_2)\|_{C_{t_0}^1} \|1 + v_1^{p-1} + v_1^{p-2}v_2 + \cdots + v_2^{p-1}\|_{C_{t_0}^1}. \end{aligned}$$

Taking the supremum over $t \in [0, t_0]$, yields

$$\begin{aligned} &\|\Gamma v_1 - \Gamma v_2\|_{C_{t_0}^1} \\ &\leq t_0 \{1 + \|v_1^{p-1}\|_{C_{t_0}^1} + \|v_1^{p-2}v_2\|_{C_{t_0}^1} + \cdots + \|v_2^{p-1}\|_{C_{t_0}^1}\} \|v_1 - v_2\|_{C_{t_0}^1}. \end{aligned} \quad (2.5)$$

It follows that the map Γ is contractive if v is restricted to lie in B , and t_0 is chosen such that

$$t_0 \leq \frac{1/2}{1 + p 2^{p-1} \|u_0\|_{H^1}^{p-1}}. \quad (2.6)$$

Indeed, consulting (2.4), it appears that this choice of t_0 will be also sufficient to ensure that Γ is a mapping on B . Therefore, according to the contraction-mapping principle Γ has a unique fixed-point u in the ball B .

Having in hand a solution u of (2.3) on a time interval $[0, t_0]$, we turn to the regularity and global existence of u . From the formulation of (2.3), it appears immediately that $u_t \in \mathcal{C}([0, t_0]; H^1)$. Since $u \in H^1(\mathbb{R})$, and therefore $u^p \in H^1(\mathbb{R})$, rearranging (2.1) as

$$u_{xxt} = u_t + u_x + (u^p)_x$$

shows that $u_{xxt}(\cdot, t) \in L^2(\mathbb{R})$ for any $t \in [0, t_0]$. Multiplying each term in (2.1) by u and integrating yields

$$\int_{-R}^R uu_t dx - \int_{-R}^R uu_{xxt} dx = - \int_{-R}^R uu_x dx - \int_{-R}^R u(u^p)_x dx.$$

The previous considerations show that $u, u_{xt} \in H^1(\mathbb{R})$ for any $t \in [0, t_0]$, so that an integration by parts is justified in the second integral (cf. Brezis [10]). Hence there appears

$$\int_{-R}^R uu_t dx + \int_{-R}^R u_x u_{xt} dx - uu_{xt} \Big|_{-R}^R = -\frac{1}{2}u^2 \Big|_{-R}^R - \frac{p}{p+1}u^{p+1} \Big|_{-R}^R.$$

Letting $R \rightarrow \infty$, we see that

$$\int_{-\infty}^{\infty} uu_t dx + \int_{-\infty}^{\infty} u_x u_{xt} dx = 0.$$

Here, use was made of the fact that functions in $H^1(\mathbb{R})$ must vanish at infinity. A proof of this fact can be given for instance with help of the Riemann-Lebesgue lemma. Since each of the terms u , u_t , u_x , and u_{xt} is in $\mathcal{C}([0, t_0]; L^2)$, the dominated convergence theorem establishes that

$$\frac{d}{dt} \int_{-\infty}^{\infty} (u^2 + u_x^2) dx = 0.$$

In conclusion, the solution of the integral equation is regular enough to satisfy (1.1) in the L^2 -sense, and moreover the H^1 -norm is constant on the time interval $[0, t_0]$. Consequently, the solution may be continued to any interval $[0, T]$ by repeating the contraction argument a sufficient number of times.

It remains to establish continuous dependence on the initial data. Suppose we have solutions u and v , corresponding to initial data u_0 and v_0 , respectively. Then (2.5) shows that

$$\|u - v\|_{C_{t_0}^1} = \|\Gamma u - \Gamma v\|_{C_{t_0}^1} \leq \|u_0 - v_0\|_{H^1} + \frac{1}{2} \|u - v\|_{C_{t_0}^1},$$

showing continuous dependence in $\mathcal{C}([0, t_0]; H^1)$. Continuous dependence can be extended to $\mathcal{C}([0, T]; H^1)$ by an obvious bootstrapping argument.

Since the existence of u was provided by the contraction mapping principle, the solution is automatically unique in the ball B . The uniqueness can also be extended to $\mathcal{C}([0, T]; H^1)$. A detailed description can be found in [2]. \square

Since the invariance of the H^1 -norm and two other integral quantities is of major importance in the proof of stability, we state these as a separate proposition.

Proposition 2.3. *Suppose u is a solution of (1.1) in $C^1([0, \infty); H^1)$, then the functionals*

$$\begin{aligned} I(u) &= \int_{-\infty}^{\infty} u(x, t) \, dx, \\ V(u) &= \int_{-\infty}^{\infty} \left(\frac{1}{2} u^2 + \frac{1}{2} u_x^2 \right) \, dx, \\ E(u) &= \int_{-\infty}^{\infty} \left(\frac{1}{2} u^2 + \frac{1}{1+p} u^{1+p} \right) \, dx \end{aligned} \quad (2.7)$$

are constant as functions of t . Moreover, I , V and E are invariant with respect to spatial translations and continuous with respect to the $H^1(\mathbb{R})$ -norm.

The invariance of $I(u)$, and $E(u)$ as functions of t can be proved in the same way as it was done above for $V(u)$. Spatial invariance, and continuity with respect to the $H^1(\mathbb{R})$ -norm are also straightforward.

While the functional $I(u)$ is constant as a function of t , the proof of instability in Section 4 requires another related expression which will be defined presently.

Theorem 2.4. *Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, and let $u(x, t)$ be the solution of (1.1) with initial data u_0 . Then there exists $0 < \zeta < 1$ such that*

$$\sup_{-\infty < z < \infty} \left| \int_z^{\infty} u(x, t) \, dx \right| \leq C(1 + t^\zeta),$$

for $t \geq 0$, where the constant C depends on u_0 .

A proof of this theorem can be found in [20].

3. SOLITARY WAVES AND ORBITAL STABILITY

In this section, it is shown that all solitary-wave solutions of (1.1) are given by the expression (1.3).

Proposition 3.1. *Let p be even. Then the positive solitary-wave solutions of (1.1) are given by (1.3) for $c > 1$. The negative solitary-wave solutions are given by (1.3) with negative wavespeed $c < 0$. For $0 < c < 1$, there are no nontrivial solitary waves.*

Proof. For $c > 1$, the formula (1.3) is valid, and it is easily verified that (1.3) is valid also for $c < 0$, resulting in negative solitary waves which propagate to the left [14, 16]. These are the unique homoclinic solutions of equation (1.3) as shown by a standard phase-plane argument.

Now for $0 < c < 1$, suppose there exists a solution Φ_c . Multiply equation (1.2) by Φ_c' to obtain

$$\frac{c}{2} \Phi_c'^2 = \Phi_c^2 \left(\frac{c-1}{2} - \frac{1}{1+p} \Phi_c^{p-1} \right). \quad (3.1)$$

Observe that the left-hand side of this equality is positive for $0 < c < 1$, and hence the right hand side must also be positive. This means that we must have $\Phi_c^{p-1} < \frac{1+p}{2}(c-1)$. Hence Φ_c^{p-1} , and therefore also Φ_c is negative, and bounded above by the negative constant $\frac{1+p}{2}(c-1)$. But this is not possible if Φ_c is required to vanish at infinity. \square

Proposition 3.2. *Let p be odd. Then the positive solitary-wave solutions of (1.1) are given by (1.3) for $c > 1$. The negative solitary-wave solutions are given by $u(x, t) = -\Phi_c$, with positive wavespeed $c > 1$. For $c < 0$ and $0 < c < 1$, there are no nontrivial solitary waves.*

Proof. As also observed in [7], for $p \geq 2$ odd, $-\Phi_c(x - ct)$ is also a solution of (1.1). This follows immediately from the fact that $-u$ satisfies equation (1.1) if u does.

In the case $0 < c < 1$, consider the argument in the proof of Proposition 3.1. As above, the inequality $\Phi_c^{p-1} < \frac{1+p}{2}(c-1)$ shows that Φ_c^{p-1} is negative. But this is not possible when p is odd.

Next consider the case $c < 0$. If a solitary wave existed, it would clearly be differentiable, and a phase plane analysis using (1.2) shows in general that a solitary wave is symmetric about its crest, and has a single maximum or minimum. Consider equation (3.1) at the critical point ξ^* , i.e. where $\Phi'_c(\xi^*) = 0$. Because $\Phi_c(\xi^*) \neq 0$, this leads to $\Phi_c^{p-1}(\xi^*) = \frac{c-1}{2}(1+p)$. However, this equation cannot be satisfied because the left-hand side is positive when p is odd, while the right-hand side is negative. \square

Figure 1 summarizes the existence of negative and positive solitary-wave solutions of (1.1) with both negative and positive propagation velocities.

Next we turn to the discussion relating to dynamic stability of the solitary waves. As already observed by Benjamin and others [4, 5], a solitary wave cannot be stable in the strictest sense of the word. To understand this, consider two solitary waves of different heights, centered initially at the same point. Since the two waves have different amplitudes they have different velocities according to the formula (1.3). As time passes the two waves will drift apart, no matter how small the initial difference was.

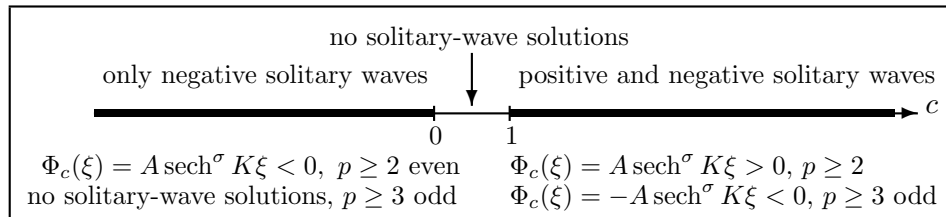


FIGURE 1. Solitary-wave solutions of (1.1).

However, in the situation just described, it is evident that two solitary waves with slightly differing height will stay similar in shape during the time evolution. Measuring the difference in shape will therefore give an acceptable notion of stability. Thus, we say the solitary wave is orbitally stable, if a solution u of the equation (1.1) that is initially sufficiently close to a solitary-wave will always stay close to a translation of the solitary-wave during the time evolution. A more mathematically precise definition is as follows. For any $\varepsilon > 0$, consider the tube

$$U_\varepsilon = \{u \in H^1 : \inf_s \|u - \tau_s \Phi_c\|_{H^1} < \varepsilon\},$$

where $\tau_s \Phi_c(x) = \Phi_c(x - s)$ is a translation of Φ_c . The set U_ε is an ε -neighborhood of the collection of all translates of Φ_c .

Definition 3.3. The solitary wave is **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 = u(\cdot, 0) \in U_\delta$, then $u(\cdot, t) \in U_\varepsilon$ for all $t \geq 0$. The solitary wave Φ_c is **unstable** if Φ_c is not stable.

Solitary waves (both negative and positive waves) with positive propagation velocity are always stable if $p \leq 5$. However, if $p > 5$, there exists a critical speed $c_p^+ = \frac{1+\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, where $\sigma = \frac{2}{p-1}$ such that the positive solitary waves are stable for $c > c_p^+$, and they are unstable for $1 < c < c_p^+$. This result was proved by Souganidis and Strauss in [20] using the general theory of Grillakis, Shatah and Strauss [12] as mentioned in the Introduction. For a thorough review of the results, and a numerical study of the stability of positive solitary waves, the reader may consult the work of Bona, McKinney and Restrepo [7].

Now contrary to what one might expect, negative solitary waves with negative propagation velocity can be unstable even if $p \leq 5$. The main contribution of this paper is a proof of the stability and instability of these negative solitary waves for both subcritical and supercritical p . Note that instability in the case $p = 2$ was already treated by one of the authors in [14]. Furthermore, in [16], the critical speed was computed as $c_p^- = \frac{1-\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, where $\sigma = \frac{2}{p-1}$. In the present paper, we will give a complete proof of the following theorem.

Theorem 3.4. *Let $p \geq 2$ be even, and define*

$$c_p^- = \frac{1 - \sqrt{2 + \sigma^{-1}}}{2(\sigma + 1)},$$

where $\sigma = \frac{2}{p-1}$. *Solitary-wave solutions of (1.1) are stable for $c < c_p^-$, and unstable for $c_p^- < c < 0$.*

Figure 2 summarizes the stable and unstable regimes for both negative and positive speeds of these solitary waves. The reader may consult [16] for an illustration of Theorem 3.4 by numerical simulation.

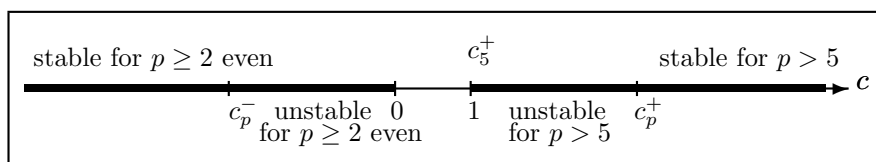


FIGURE 2. The stable and unstable regimes of the solitary waves for both negative and positive speed c . Here, $c_p^+ = \frac{1+\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, and $c_p^- = \frac{1-\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, where $\sigma = \frac{2}{p-1}$. Note that $c_5^+ = 1$.

The criterion for stability and instability follows from close examination of the convexity properties of the function

$$d(c) = E(\Phi_c) - cV(\Phi_c). \tag{3.2}$$

As it was shown in [16] that $d(c)$ is strictly convex (upwards) for $c < c_p^-$, and strictly concave (downwards) for $c_p^- < c < 0$, the proof of Theorem 3.4 will be accomplished by proving that a solitary wave with wave speed c_0 is stable if $d(c)$ is strictly convex at $c = c_0$, and it is unstable if $d(c)$ is strictly concave at $c = c_0$. A

more elementary proof of stability not relying on the convexity properties of $d(c)$ has been provided for the case $p = 2$ in [17] for a restricted range of wave speeds c .

In the remaining part of this section, we establish a few general facts which are important for the proof of instability and stability which will be taken up in the next two sections

Lemma 3.5. *There exists $\varepsilon > 0$ and a unique C^1 map $\alpha : U_\varepsilon \rightarrow \mathbb{R}$, such that for every $u \in U_\varepsilon$,*

$$\langle u(\cdot + \alpha(u)), \Phi'_c \rangle = 0.$$

The proof of this lemma can be found in [8].

Now, observe that equation (1.2) can be written in variational form as

$$E'(\Phi_c) - cV'(\Phi_c) = 0, \quad (3.3)$$

where $E'(\Phi_c) = \Phi_c + \Phi_c^p$ and $V'(\Phi_c) = \Phi_c - \Phi_c''$ are the Fréchet derivatives at Φ_c of E and V , respectively. The functional derivative of $E'(\Phi_c) - cV'(\Phi_c)$ is given by the linear operator

$$\mathcal{L}_c = E''(\Phi_c) - cV''(\Phi_c) = c\partial_x^2 - c + 1 + p\Phi_c^{p-1}. \quad (3.4)$$

Note that since $c < 0$, $c\partial_x^2 - c + 1$ is a positive operator. Moreover, we have the following relation involving the derivative of Φ_c with respect to c .

Lemma 3.6. *In the notation established above, the following relation holds.*

$$\mathcal{L}_c(d\Phi_c/dc) = V'(\Phi_c). \quad (3.5)$$

Proof. The relation (3.5) follows from (3.3) after the following computation.

$$\begin{aligned} 0 &= \partial_c [E'(\Phi_c) - cV'(\Phi_c)] \\ &= [E''(\Phi_c) - cV''(\Phi_c)]d\Phi_c/dc - V'(\Phi_c) \\ &= \mathcal{L}_c(d\Phi_c/dc) - V'(\Phi_c). \end{aligned}$$

□

For the proofs of stability and instability, it will be convenient to have some spectral information about \mathcal{L}_c at our disposal. First of all, it is elementary to check that $\mathcal{L}_c : H^2 \subset L^2 \rightarrow L^2$ is self-adjoint with respect to the L^2 -inner product. Furthermore, a simple scaling transforms \mathcal{L}_c to an operator for which the exact spectral representation is known. Consulting [15] page 768-769 yields the following.

Proposition 3.7. *\mathcal{L}_c has positive continuous spectrum bounded away from zero by $\rho_0 > 0$, a simple zero eigenvalue with eigenfunction Φ'_c , and one negative simple eigenvalue $-\lambda^2 = [\frac{1}{4}(p+1)^2 - 1](c-1)$ with corresponding eigenfunction*

$$\chi_c(\xi) = \kappa \left[\operatorname{sech} \left(\frac{p-1}{2} \sqrt{\frac{c-1}{c}} \xi \right) \right]^{\frac{p+1}{p-1}}, \quad (3.6)$$

where the constant κ is chosen such that $\|\chi_c\|_{L^2} = 1$.

4. PROOF OF INSTABILITY

As mentioned in the Introduction, the existing literature focuses almost entirely on positive solitary waves, and the strict positivity of these waves is used in an important part of the proof of instability, namely in establishing that the functional E has a constrained maximum near the critical point Φ_c . For the proof of instability of negative solitary waves, we need to provide a new tool which allows us to dispense with the assumption of positivity. This is done in Lemma 4.1. Then, after stating a few more preliminary lemmas, we present the proof of instability.

Lemma 4.1. *Let c be fixed. If $d''(c) < 0$, then there exists a curve $\omega \mapsto \Psi_\omega$ in a neighborhood of c , such that $\Psi_c = \Phi_c$, $V(\Psi_\omega) = V(\Phi_c)$ for all ω , and $E(\Psi_\omega) < E(\Phi_c)$ for $\omega \neq c$.*

Proof. Consider a mapping $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $(\omega, s) \mapsto V(\Phi_\omega + s\chi_c)$, where χ_c is the eigenfunction corresponding to the negative eigenvalue of the operator \mathcal{L}_c , as defined in (3.6). Note that $(c, 0)$ maps to $V(\Phi_c)$. To obtain the curve $\omega \mapsto \Psi_\omega$, we first apply the implicit function theorem to find a mapping $\omega \rightarrow s(\omega)$, such that $V(\Phi_\omega + s(\omega)\chi_c)$ is constant. To this end, it has to be shown that

$$\left. \frac{\partial}{\partial s} \{V(\Phi_\omega + s\chi_c)\} \right|_{\omega=c, s=0} = \int_{-\infty}^{\infty} V'(\Phi_c) \chi_c dx$$

is nonzero. Using (3.3) and (3.6), it can be seen that

$$\begin{aligned} & \left. \frac{\partial}{\partial s} \{V(\Phi_\omega + s\chi_c)\} \right|_{\omega=c, s=0} \\ &= \frac{\kappa}{c} \int_{-\infty}^{\infty} (\Phi_c + \Phi_c^p) \operatorname{sech}^{\frac{p+1}{p-1}}(Kx) dx \\ &= \frac{\kappa}{c} \int_{-\infty}^{\infty} [A \operatorname{sech}^\sigma(Kx) + A^p \operatorname{sech}^{\sigma p}(Kx)] \operatorname{sech}^{\frac{p+1}{p-1}}(Kx) dx \\ &= \frac{\kappa A}{cK} \int_{-\infty}^{\infty} \left[\operatorname{sech}^{\frac{p+3}{p-1}} z + \frac{(p+1)(c-1)}{2} \operatorname{sech}^{\frac{3p+1}{p-1}} z \right] dz, \end{aligned} \tag{4.1}$$

where κ is defined in Proposition 3.7, and A and K are defined in the Introduction. We claim that this integral is negative. To verify the claim, we integrate by parts to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \operatorname{sech}^{\frac{3p+1}{p-1}} z dz &= \int_{-\infty}^{\infty} \operatorname{sech}^{\frac{p+3}{p-1}} z \operatorname{sech}^2 z dz \\ &= \frac{p+3}{p-1} \int_{-\infty}^{\infty} \tanh^2 z \operatorname{sech}^{\frac{p+3}{p-1}} z dz \\ &= \frac{p+3}{p-1} \int_{-\infty}^{\infty} \left(\operatorname{sech}^{\frac{p+3}{p-1}} z - \operatorname{sech}^{\frac{3p+1}{p-1}} z \right) dz. \end{aligned}$$

After rearranging, it appears that

$$\int_{-\infty}^{\infty} \operatorname{sech}^{\frac{3p+1}{p-1}} z dz = \frac{p+3}{2(p+1)} \int_{-\infty}^{\infty} \operatorname{sech}^{\frac{p+3}{p-1}} z dz.$$

Consequently, the integral in (4.1) has the simpler expression

$$\left[1 + \frac{(c-1)(p+3)}{4} \right] \int_{-\infty}^{\infty} \operatorname{sech}^{\frac{p+3}{p-1}} z dz,$$

and this is negative because $1 + \frac{(c-1)(p+3)}{4} < 0$, since $p \geq 2$ and $c < 0$. Thus the implicit function theorem may be used to find the mapping $\omega \rightarrow s(\omega)$, and Ψ_ω is defined by $\Psi_\omega = \Phi_\omega + s(\omega)\chi_c$.

Next, we show that c is a critical point of $\omega \rightarrow E(\Psi_\omega)$. Since $V(\Psi_\omega)$ is constant near c , we have

$$\frac{d}{d\omega}E(\Psi_\omega) = \frac{d}{d\omega}\{E(\Psi_\omega) - cV(\Psi_\omega)\}, \quad (4.2)$$

and in light of (3.3), the above expression is zero when evaluated at $\omega = c$. Furthermore, as will be shown next, at this critical point, the curve $\omega \rightarrow E(\Psi_\omega)$ is strictly concave, i.e., $\frac{d^2}{d\omega^2}E(\Psi_\omega)|_{\omega=c} < 0$, and hence has a local maximum. Differentiating equation (4.2) and using (3.3) gives

$$\frac{d^2}{d\omega^2}E(\Psi_\omega)|_{\omega=c} = \left\langle [E''(\Phi_c) - cV''(\Phi_c)] \frac{d\Psi_\omega}{d\omega}|_{\omega=c}, \frac{d\Psi_\omega}{d\omega}|_{\omega=c} \right\rangle.$$

Recall now that $\mathcal{L}_c = E''(\Phi_c) - cV''(\Phi_c)$, and χ_c is an eigenfunction corresponding to the negative eigenvalue $-\lambda^2$. Therefore, if we define

$$y = \frac{d\Psi_\omega}{d\omega}|_{\omega=c} = \frac{d\Phi_c}{dc} + s'(c)\chi_c, \quad (4.3)$$

then

$$\frac{d^2}{d\omega^2}E(\Psi_\omega)|_{\omega=c} = \langle \mathcal{L}_c y, y \rangle.$$

Thus, the proof of Lemma 4.1 will be completed if it can be shown that $\langle \mathcal{L}_c y, y \rangle < 0$. First observe that

$$\langle V'(\Phi_c), y \rangle = 0. \quad (4.4)$$

This can be seen from differentiating $\omega \rightarrow V(\Psi_\omega)$ as follows.

$$0 = \frac{d}{d\omega}V(\Psi_\omega)|_{\omega=c} = \left\langle V'(\Phi_c), \frac{d\Psi_\omega}{d\omega}|_{\omega=c} \right\rangle = \langle V'(\Phi_c), y \rangle.$$

Combining (4.4) and Lemma 3.6, we obtain

$$\begin{aligned} \langle \mathcal{L}_c y, y \rangle &= \left\langle \mathcal{L}_c \left(\frac{d\Phi_c}{dc} + s'(c)\chi_c \right), y \right\rangle \\ &= \left\langle V'(\Phi_c) + s'(c)\mathcal{L}_c\chi_c, y \right\rangle \\ &= s'(c)\langle \mathcal{L}_c\chi_c, y \rangle. \end{aligned}$$

Since \mathcal{L}_c is self-adjoint, we obtain further

$$\begin{aligned} \langle \mathcal{L}_c y, y \rangle &= s'(c)\langle \chi_c, \mathcal{L}_c y \rangle \\ &= s'(c)\left\langle \chi_c, \mathcal{L}_c \left(\frac{d\Phi_c}{dc} + s'(c)\chi_c \right) \right\rangle \\ &= s'(c)\langle \chi_c, V'(\Phi_c) + s'(c)\mathcal{L}_c\chi_c \rangle \\ &= s'(c)\langle \chi_c, V'(\Phi_c) \rangle + [s'(c)]^2\langle \chi_c, \mathcal{L}_c\chi_c \rangle. \end{aligned}$$

Observe that the first term on the right of this equation is exactly $d'(c)$. Indeed, since $d(c) = E(\Phi_c) - cV(\Phi_c)$, we have

$$d'(c) = \langle E'(\Phi_c) - cV'(\Phi_c), d\Phi_c/dc \rangle - V(\Phi_c) = -V(\Phi_c),$$

and hence,

$$d''(c) = -\langle V'(\Phi_c), d\Phi_c/dc \rangle = s'(c)\langle V'(\Phi_c), \chi_c \rangle, \quad (4.5)$$

in light of (4.3) and equation (4.4). Therefore,

$$\langle \mathcal{L}_c y, y \rangle = d''(c) + [s'(c)]^2 \langle \chi_c, \mathcal{L}_c \chi_c \rangle = d''(c) - \lambda^2 [s'(c)]^2 \|\chi_c\|_{L^2}^2 < 0,$$

since $d''(c)$ is negative. Therefore, we have shown that $\frac{d^2}{d\omega^2} E(\Psi_\omega)|_{\omega=c} = \langle \mathcal{L}_c y, y \rangle < 0$, and thus $\omega \mapsto E(\Psi_\omega)$ has a local maximum at $\omega = c$. \square

Next, an auxiliary operator B is defined which will play a critical role in the proof of instability. For $u \in U_\varepsilon$, define $B(u)$ by the formula

$$B(u) = y(\cdot - \alpha(u)) - \langle (1 - \partial_x^2)u, y(\cdot - \alpha(u)) \rangle (1 - \partial_x^2)^{-1} \partial_x \alpha'(u), \tag{4.6}$$

where $(1 - \partial_x^2)^{-1}$ denotes the inverse of the operator $1 - \partial_x^2$. The operator $(1 - \partial_x^2)^{-1}$ is defined by convolution with the Green's function $G(x)$, as explained in Section 2. Because the Fourier transform of $G(x)$ is given by $\widehat{G}(\eta) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + \eta^2}$, it is immediate that $(1 - \partial_x^2)^{-1}$ is a bounded operator on any Sobolev class $H^s(\mathbb{R})$, and is self-adjoint with respect to the L^2 -inner product. With the help of Lemma 4.1, the next lemma can be proved exactly as in the analogous case of [8], and we therefore state it without proof.

Lemma 4.2. *Let c be fixed. If $d''(c) < 0$, there is a C^1 -functional $\Lambda : D_\varepsilon \rightarrow \mathbb{R}$, where $D_\varepsilon = \{v \in U_\varepsilon : V(v) = V(\Phi_c)\}$, such that $\Lambda(\Phi_c) = 0$, and if $v \in D_\varepsilon$ and v is not a translate of Φ_c , then*

$$E(\Phi_c) < E(v) + \Lambda(v) \langle E'(v), B(v) \rangle.$$

Furthermore, $\langle E'(\Psi_\omega), B(\Psi_\omega) \rangle$ changes sign as ω passes through c , where $\omega \mapsto \Psi_\omega$ is the curve constructed in Lemma 4.1.

Proof of Instability. As was shown in [16], the function $d(c)$ is strictly concave if $c_p^- < c < 0$. Thus to prove the instability part of Theorem 3.4, it is enough to prove the following.

Theorem 4.3. *If $d''(c) < 0$, the solitary wave is unstable.*

Proof. The proof is based on the techniques in [8], [12] and [20]. Let $\varepsilon > 0$ sufficiently small be given. By Lemma 4.1 and Lemma 4.2, we can choose $u_0 \in H^1 \cap L^1$ arbitrary close to Φ_c , such that $u_0 \in U_\varepsilon$, $V(u_0) = V(\Phi_c)$, $E(u_0) < E(\Phi_c)$, and $|\langle E'(u_0), B(u_0) \rangle| > 0$. Note that the last condition guarantees that u_0 is not a translate of Φ_c . For example, let $u_0 = \Phi_\omega + s(\omega)\chi_c$, for an arbitrary ω close to c , but not exactly equal to c . \square

Now, if $u(x, t)$ is the solution of equation (1.1) with initial condition u_0 , let $[0, t_1)$ denote the maximal time interval for which $u(\cdot, t) \in U_\varepsilon$. By Theorem 2.1, $t_1 > 0$. Instability of the solitary-wave will be demonstrated by showing that $t_1 < \infty$.

Let $\beta(t) = \alpha(u(t))$, where α was defined in Lemma 3.5, and $Y(x) = \int_{-\infty}^x (1 - \partial_z^2)y(z) dz$, where y was defined in (4.3). Then define

$$N(t) = \int_{-\infty}^{\infty} Y(x - \beta(t))u(x, t)dx, \tag{4.7}$$

which will serve as a Lyapunov functional. First, it will be shown that $N(t)$ is finite.

Lemma 4.4. *There is a positive constant D such that $|N(t)| \leq D(1 + t^\zeta)$ for $0 \leq t < t_1$, where $0 < \zeta < 1$ is defined in Theorem 2.4.*

Proof. Let \mathcal{H} be the Heaviside function, and define $\gamma = \int_{-\infty}^{\infty} y(x) dx$, and $F(x) = \int_{-\infty}^x y(\xi) d\xi$. Then the following equality appears after integration by parts.

$$N(t) = \int_{-\infty}^{\infty} \left[F(x - \beta(t)) - \gamma \mathcal{H}(x - \beta(t)) \right] u(x, t) dx + \int_{-\infty}^{\infty} y(x - \beta(t)) u_x(x, t) dx + \gamma \int_{\beta(t)}^{\infty} u(x, t) dx.$$

Using the Cauchy-Schwarz inequality on the first and second integrals, and applying Theorem 2.4 to the last integral, an upper bound for $|N(t)|$ is estimated as follows.

$$|N(t)| \leq \left(\|F - \gamma \mathcal{H}\|_{L^2(\mathbb{R})} + \|y\|_{L^2(\mathbb{R})} \right) \|u(t)\|_{H^1(\mathbb{R})} + |\gamma| C(1 + t^\zeta). \tag{4.8}$$

Next, $F - \gamma \mathcal{H}$ can be shown to belong to $L^2(\mathbb{R})$, as follows. First of all, note that

$$F(x) - \gamma \mathcal{H}(x) = \begin{cases} F(x), & \text{if } x < 0 \\ F(x) - \gamma, & \text{if } x \geq 0. \end{cases}$$

Thus, to investigate $\|F - \gamma \mathcal{H}\|_{L^2(\mathbb{R})}$, it is expedient to consider two cases $x < 0$ and $x > 0$ separately. When $x < 0$, Minkowski's inequality can be used to show that

$$\begin{aligned} \|F - \gamma \mathcal{H}\|_{L^2(\mathbb{R})} &= \|F(x)\|_{L^2(-\infty, 0)} \\ &= \left(\int_{-\infty}^0 \left\{ \int_{-\infty}^x y(\xi) d\xi \right\}^2 dx \right)^{1/2} \\ &\leq \int_{-\infty}^0 \sqrt{|\xi|} |y(\xi)| d\xi. \end{aligned}$$

Since y is defined in terms of $d\Phi_c/dc$ and χ_c , both of which have exponential decay as $|\xi| \rightarrow \infty$, it is immediate that the last term in the above string of inequalities is finite. An analogous argument holds for $x > 0$.

By the exponential decay of $d\Phi_c/dc$ and χ_c , it is clear that $\|y\|_{L^2(\mathbb{R})}$ and $\gamma = \int_{-\infty}^{\infty} y(x) dx$ are finite. Furthermore $\|u(t)\|_{H^1(\mathbb{R})}$ is constant because it is given by the invariant integral $V(u(\cdot, t))$. Therefore the inequality (4.8) can be written as

$$|N(t)| \leq D(1 + t^\zeta),$$

with the positive constant $D = \left(\|F - \gamma \mathcal{H}\|_{L^2(\mathbb{R})} + \|y\|_{L^2(\mathbb{R})} \right) \|u_0\|_{H^1(\mathbb{R})} + |\gamma| C$, where C and ζ were defined in the statement of Theorem 2.4. \square

An estimate of the derivative of N is given in the next lemma.

Lemma 4.5. *If $d''(c) < 0$, there is a positive constant m such that $|N'(t)| > m$, for all $t \in [0, t_1)$.*

Proof. We have

$$N'(t) = -\beta'(t) \langle (1 - \partial_x^2) y(\cdot - \beta(t)), u(\cdot, t) \rangle + \langle Y(\cdot - \beta(t)), u_t(\cdot, t) \rangle.$$

Since $\beta'(t) = \langle \alpha'(u), u_t \rangle$, this derivative is equal to

$$\left\langle - \langle (1 - \partial_x^2) y(\cdot - \beta(t)), u(\cdot, t) \rangle \alpha'(u), u_t \right\rangle + \langle Y(\cdot - \beta(t)), u_t(\cdot, t) \rangle.$$

Since $1 - \partial_x^2$ is self-adjoint, this derivative can be written in the form

$$\left\langle - \langle y(\cdot - \beta(t)), (1 - \partial_x^2)u(\cdot, t) \rangle \alpha'(u) + Y(\cdot - \beta(t)), u_t \right\rangle.$$

In view of (2.1), and the fact that $E'(u) = u + u^p$, this derivative turns out to be

$$\left\langle - \langle y(\cdot - \beta(t)), (1 - \partial_x^2)u(\cdot, t) \rangle \alpha'(u) + Y(\cdot - \beta(t)), -\partial_x(1 - \partial_x^2)^{-1}E'(u) \right\rangle.$$

Using integration by parts together with the fact that $(1 - \partial_x^2)^{-1}$ is self-adjoint and ∂_x is skew-adjoint, this expression is equal to

$$\left\langle - \langle y(\cdot - \beta(t)), (1 - \partial_x^2)u(\cdot, t) \rangle \partial_x(1 - \partial_x^2)^{-1}\alpha'(u) + y(\cdot - \beta(t)), E'(u) \right\rangle.$$

In view of the definition of B , it is clear that $N'(t)$ has the compact expression

$$N'(t) = \langle B(u), E'(u) \rangle. \tag{4.9}$$

Recall that for $t \in [0, t_1)$, the solution $u(\cdot, t) \in U_\varepsilon$ is not a translation of Φ_c since its initial solution is not. However, $V(u(t)) = V(\Phi_c)$ since both are equal to $V(u_0)$. On the other hand, Lemma 4.1 together with Lemma 4.2 imply that

$$0 < E(\Phi_c) - E(u_0) = E(\Phi_c) - E(u(t)) < \Lambda(u(t)) \langle E'(u(t)), B(u(t)) \rangle. \tag{4.10}$$

Using the continuity of Λ and the fact that $\Lambda(\Phi_c) = 0$, which follow from the construction of the functional Λ in Lemma 4.2, and recalling the assumption that $u(t) \in U_\varepsilon$, for $t \in [0, t_1)$, we may assume that $|\Lambda(u(t))| < 1$, possibly by choosing ε smaller if necessary. Therefore, in view of equations (4.9) and (4.10), we have

$$|N'(t)| = \left| \langle E'(u(t)), B(u(t)) \rangle \right| > [E(\Phi_c) - E(u(t))] = E(\Phi_c) - E(u_0) = m.$$

for all $t \in [0, t_1)$. □

Finally, we are in a position to complete the proof of Theorem 4.3. In view of Lemma 4.4 and Lemma 4.5, it turns out that

$$2D(1 + t^\zeta) \geq |N(t)| + |N(0)| \geq \int_0^t |N'(s)| ds > \int_0^t m ds = mt,$$

for $t \in [0, t_1)$. However, since $\zeta < 1$, the rate of growth of the curve $f(t) = 2D(1 + t^\zeta)$ is less than the rate of growth of the line $l(t) = mt$. Therefore, t_1 must be the point where these two curves meet, and thus $t_1 < \infty$. □

Figure 3 shows a perturbation of an unstable negative solitary wave with velocity $c = -0.1 > c_4^- = -0.2612$, $p = 4$ and amplitude $\max_x |\Phi_{-0.1}| = 1.4010$, propagating to the left. The instability manifests itself in a slow disintegration of the solitary wave over time.

5. PROOF OF STABILITY

The stability theory will be presented in this section. The key element in the proof is the conditional coercivity of the bilinear form $\langle \mathcal{L}_c y, y \rangle$. This is established in the following lemma.

Lemma 5.1. *Assume $d''(c) > 0$. There is a constant $\beta > 0$, such that for any nonzero $y \in H^1(\mathbb{R})$ satisfying $\langle y, V'(\Phi_c) \rangle = 0$ and $\langle y, \Phi'_c \rangle = 0$, the estimate $\langle \mathcal{L}_c y, y \rangle \geq \beta \|y\|_{H^1}^2$ holds.*

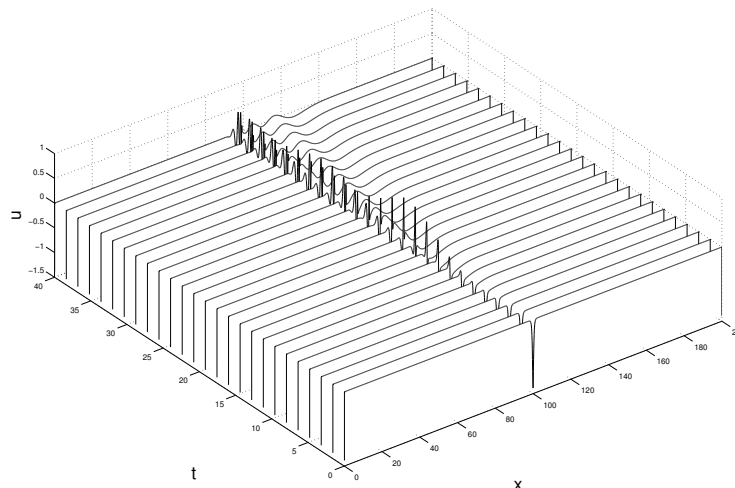


FIGURE 3. Unstable negative solitary wave with velocity $c = -0.1$ in the case where $p = 4$.

Proof. The proof follows the ideas in [8], [12] and [13]. First of all, we will show that $\langle \mathcal{L}_c y, y \rangle$ is positive as follows. Using equation (4.5), and Lemma 3.6, it can be seen that

$$d''(c) = -\langle \mathcal{L}_c(d\Phi_c/dc), d\Phi_c/dc \rangle. \quad (5.1)$$

Next, using the spectral decomposition of \mathcal{L}_c delineated in Proposition 3.7, we can write $d\Phi_c/dc = a_0\chi_c + b_0\Phi'_c + p_0$, where p_0 is in the positive subspace of \mathcal{L}_c . Also recall that $\mathcal{L}_c\chi_c = -\lambda^2\chi_c$ with $\lambda > 0$, and $\mathcal{L}_c\Phi'_c = 0$. A short computation then transforms (5.1) into the equation

$$\langle \mathcal{L}_c p_0, p_0 \rangle = a_0^2 \lambda^2 - d''(c). \quad (5.2)$$

Now, since y is assumed to be orthogonal to $V'(\Phi_c) = \mathcal{L}_c \frac{d\Phi_c}{dc}$, there appears

$$\langle \mathcal{L}_c(d\Phi_c/dc), y \rangle = 0. \quad (5.3)$$

Furthermore, since it is assumed that $\langle y, \Phi'_c \rangle = 0$, y can be decomposed into the sum $y = a_1\chi_c + p$, with p in the positive subspace of \mathcal{L}_c . Using this decomposition of y and $\frac{d\Phi_c}{dc}$ in equation (5.3), yields

$$\langle \mathcal{L}_c p_0, p \rangle = a_0 a_1 \lambda^2. \quad (5.4)$$

On the other hand, using the generalized Cauchy-Schwarz inequality, the decomposed form of y also implies that

$$\langle \mathcal{L}_c y, y \rangle = -a_1^2 \lambda^2 + \langle \mathcal{L}_c p, p \rangle \geq -a_1^2 \lambda^2 + \langle \mathcal{L}_c p, p_0 \rangle^2 / \langle \mathcal{L}_c p_0, p_0 \rangle. \quad (5.5)$$

Next, using equations (5.2) and (5.4) in the inequality (5.5), the positivity of $\langle \mathcal{L}_c y, y \rangle$ is finally revealed, because

$$\langle \mathcal{L}_c y, y \rangle \geq -a_1^2 \lambda^2 + \frac{(a_0 a_1 \lambda^2)^2}{a_0^2 \lambda^2 - d''(c)} = a_1^2 P, \tag{5.6}$$

where $P = \lambda^2 d''(c) / \langle \mathcal{L}_c p_0, p_0 \rangle$ is a positive constant since $d''(c) > 0$.

The next step in the proof is to show that there is a positive constant ν , such that

$$\langle \mathcal{L}_c y, y \rangle \geq \nu \|y\|_{L^2}. \tag{5.7}$$

To this end, denote by S the set of all nonzero $z \in H^1(\mathbb{R})$, satisfying $\langle z, V'(\Phi_c) \rangle = 0$, $\langle z, \Phi'_c \rangle = 0$, and $\|z\|_{L^2} = 1$. If (5.7) is not true for any positive ν , we must have $\inf_{z \in S} \langle \mathcal{L}_c z, z \rangle = 0$, and there is a sequence $\{z_n\}$ in S , such that

$$\langle \mathcal{L}_c z_n, z_n \rangle \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Again using the spectral decomposition described in Proposition 3.7, z_n can be written as $z_n = a_n \chi_c + p_n$, where p_n is in the positive subspace of \mathcal{L}_c . Thus there appears

$$-a_n^2 \lambda^2 + \langle \mathcal{L}_c p_n, p_n \rangle \rightarrow 0, \text{ as } n \rightarrow \infty.$$

In view of the inequality (5.6), we have

$$0 < a_n^2 P \leq \langle \mathcal{L}_c z_n, z_n \rangle \rightarrow 0, \text{ as } n \rightarrow \infty,$$

so that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Consequently, $\lim_{n \rightarrow \infty} \langle \mathcal{L}_c p_n, p_n \rangle \rightarrow 0$, and since $\langle \mathcal{L}_c p_n, p_n \rangle > \rho_0 \|p_n\|_{L^2}^2$, it turns out that $\lim_{n \rightarrow \infty} \|p_n\|_{L^2} = 0$. On the other hand, using the decomposed form of z_n , we also conclude that $\lim_{n \rightarrow \infty} \|p_n\|_{L^2} = \lim_{n \rightarrow \infty} \|z_n\|_{L^2} = 1$, and this is a contradiction. Thus, (5.7) is proved for z with $\|z\|_{L^2} = 1$, and it follows for general y by letting $z = y/\|y\|_{L^2}$.

Finally, the statement (5.7) is also true if the L^2 -norm is replaced by the H^1 -norm as follows. Directly from the definition of \mathcal{L}_c in equation (3.4), we see that

$$\begin{aligned} \langle \mathcal{L}_c y, y \rangle &= -c \int_{-\infty}^{\infty} y_x^2 dx + \int_{-\infty}^{\infty} (-c + 1 + p\Phi_c^{p-1}) y^2 dx \\ &\geq -c \int_{-\infty}^{\infty} y_x^2 dx + \min_x (-c + 1 + p\Phi_c^{p-1}) \int_{-\infty}^{\infty} y^2 dx \\ &= -c \int_{-\infty}^{\infty} y_x^2 dx + (-c + 1 + pA^{p-1}) \int_{-\infty}^{\infty} y^2 dx, \end{aligned}$$

by the definition of Φ_c , and since p is even. By definition of A in the Introduction, it appears that

$$\langle \mathcal{L}_c y, y \rangle \geq -c \int_{-\infty}^{\infty} y_x^2 dx + r \int_{-\infty}^{\infty} y^2 dx, \tag{5.8}$$

where $r = (c - 1) \left[\frac{p(p+1)}{2} - 1 \right]$ is a *negative* constant because $c < 0$ and $p \geq 2$. Now for some $\theta \in (0, 1)$, write

$$\begin{aligned} \langle \mathcal{L}_c y, y \rangle &= \theta \langle \mathcal{L}_c y, y \rangle + (1 - \theta) \langle \mathcal{L}_c y, y \rangle \\ &\geq -c\theta \|y_x\|_{L^2}^2 + r\theta \|y\|_{L^2}^2 + (1 - \theta)\nu \|y\|_{L^2}^2 \\ &= -c\theta \|y_x\|_{L^2}^2 + [r\theta + (1 - \theta)\nu] \|y\|_{L^2}^2. \end{aligned}$$

Choosing θ between 0 and $\nu/(\nu-r)$ yields $\langle \mathcal{L}_c y, y \rangle \geq \beta \|y\|_{H^1}$ for a positive constant β . □

Before the stability of Φ_c can be proved, another lemma is needed.

Lemma 5.2. *Let β be the constant found in Lemma 5.1, and let α be as defined in Lemma 3.5. If $d''(c) > 0$, there exists an $\varepsilon > 0$, such that*

$$E(u) - E(\Phi_c) \geq \frac{\beta}{4} \|u(\cdot + \alpha(u)) - \Phi_c\|_{H^1}^2$$

for all $u \in U_\varepsilon$ which satisfy $V(u) = V(\Phi_c)$.

Proof. Let ε be chosen so small that $\|u(\cdot + \alpha(u)) - \Phi_c\|_{H^1} < 1$, for any $u \in U_\varepsilon$, and let $u \in U_\varepsilon$ be such that $V(\Phi_c) = V(u)$. Define $v = u(\cdot + \alpha(u)) - \Phi_c$, and write v in the form $v = aV'(\Phi_c) + y$, where a is a scalar, and y is a nonzero element in $H^1(\mathbb{R})$ for which $\langle y, V'(\Phi_c) \rangle = 0$. Then we claim that

$$\langle y, \Phi'_c \rangle = 0.$$

This can be seen as follows. First note that

$$\langle y, \Phi'_c \rangle = \langle u(\cdot + \alpha(u)) - \Phi_c - aV'(\Phi_c), \Phi'_c \rangle.$$

Since $u(\cdot + \alpha(u))$ is orthogonal to Φ'_c , and $V'(\Phi_c) = \Phi_c - \Phi''_c$, it appears that

$$\langle y, \Phi'_c \rangle = -(1+a)\langle \Phi_c, \Phi'_c \rangle + a\langle \Phi'_c, \Phi''_c \rangle = 0.$$

Next, recall that $V(u) = V(u(\cdot + \alpha(u))) = V(v + \Phi_c)$, so that $V(\Phi_c) = V(v + \Phi_c)$. From the definition of V in equation (2.7) it can be seen that after an integration by parts that

$$\begin{aligned} V(\Phi_c) &= \frac{1}{2} \int_{-\infty}^{\infty} (\Phi_c^2 + \Phi_c'^2) d\xi + \int_{-\infty}^{\infty} (\Phi_c - \Phi_c'')v d\xi + \frac{1}{2} \int_{-\infty}^{\infty} (v^2 + v'^2) d\xi \\ &= V(\Phi_c) + \langle V'(\Phi_c), v \rangle + \frac{1}{2} \|v\|_{H^1}^2. \end{aligned}$$

Using the form $v = aV'(\Phi_c) + y$ together with the fact that y is orthogonal to $V'(\Phi_c)$, this equation is equal to

$$V(\Phi_c) + a\|V'(\Phi_c)\|_{L^2}^2 + \frac{1}{2} \|v\|_{H^1}^2.$$

Finally, it is inferred that

$$a = -\frac{1}{2\|V'(\Phi_c)\|_{L^2}^2} \|v\|_{H^1}^2 = -k\|v\|_{H^1}^2 < 0, \tag{5.9}$$

where $k = 1/(2\|V'(\Phi_c)\|_{L^2}^2)$ is a positive constant. Now let $\Delta V = V(\Phi_c + v) - V(\Phi_c)$, and note that $\Delta V = 0$. However, according to the definition of V in equation (2.7), we also have

$$\Delta V = \frac{1}{2} \int_{-\infty}^{\infty} (v^2 + v'^2 + 2\Phi_c v + 2\Phi_c' v') d\xi. \tag{5.10}$$

Defining ΔE in a similar way, it can be seen that

$$\Delta E = E(\Phi_c + v) - E(\Phi_c) = \int_{-\infty}^{\infty} \left\{ \Phi_c v + \frac{1}{2} v^2 + \frac{1}{1+p} \sum_{n=1}^{1+p} \binom{1+p}{n} \Phi_c^{1+p-n} v^n \right\} d\xi, \tag{5.11}$$

where the binomial coefficient is defined as

$$\binom{m}{n} = \begin{cases} \frac{m!}{n!(m-n)!} & 0 \leq n \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, since $\Delta V = 0$, we may write

$$\Delta E = \Delta E - c\Delta V.$$

Therefore, using (5.10), (5.11), and an integration by parts, there appears the expression

$$\begin{aligned} \Delta E &= \int_{-\infty}^{\infty} \{ \Phi_c - c\Phi_c + \Phi_c^p + c\Phi_c'' \} v \, d\xi \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \{ -cv'^2 + (-c + 1 + p\Phi_c^{p-1})v^2 \} d\xi \\ &\quad + \frac{1}{1+p} \int_{-\infty}^{\infty} \left\{ \sum_{n=3}^{1+p} \binom{1+p}{n} \Phi_c^{1+p-n} v^n \right\} d\xi. \end{aligned}$$

Note that the first and second integral of this expression can be regarded as the first and second variation of E , respectively. Observe that the first variation of E vanishes identically since Φ_c satisfies equation (1.2). On the other hand, the second variation of E has the compact form $\frac{1}{2} \langle \mathcal{L}_c v, v \rangle$. Therefore,

$$\Delta E = \frac{1}{2} \langle \mathcal{L}_c v, v \rangle + \frac{1}{1+p} \int_{-\infty}^{\infty} \left\{ \sum_{n=3}^{1+p} \binom{1+p}{n} \Phi_c^{1+p-n} v^n \right\} d\xi.$$

Using the form $v = aV'(\Phi_c) + y$ and since \mathcal{L}_c is self-adjoint, this equation is equal to

$$\begin{aligned} &\frac{1}{2} \langle \mathcal{L}_c y, y \rangle + \frac{1}{2} a^2 \langle \mathcal{L}_c V'(\Phi_c), V'(\Phi_c) \rangle + a \langle \mathcal{L}_c V'(\Phi_c), y \rangle \\ &\quad + \frac{1}{1+p} \int_{-\infty}^{\infty} \left\{ \sum_{n=3}^{1+p} \binom{1+p}{n} \Phi_c^{1+p-n} v^{n-2} \right\} v^2 d\xi. \end{aligned}$$

Now using Lemma 5.1 on the first term and on the third term, and applying the Cauchy-Schwarz inequality together with inequality $\|y\|_{L^2} \leq \|v\|_{L^2}$, there appears the estimate

$$\begin{aligned} \Delta E &\geq \frac{\beta}{2} \|y\|_{H^1}^2 + \frac{1}{2} a^2 \langle \mathcal{L}_c V'(\Phi_c), V'(\Phi_c) \rangle + a \|\mathcal{L}_c V'(\Phi_c)\|_{L^2} \|v\|_{H^1} \\ &\quad - \frac{1}{1+p} \sum_{n=3}^{1+p} \binom{1+p}{n} \left(\sup_{\xi \in \mathbb{R}} |\Phi_c| \right)^{1+p-n} \left(\sup_{\xi \in \mathbb{R}} |v| \right)^{n-2} \int_{-\infty}^{\infty} v^2 d\xi, \end{aligned}$$

where the fact that a is negative was used. Because

$$\begin{aligned} \|y\|_{H^1}^2 &= \|v - aV'(\Phi_c)\|_{H^1}^2 \\ &\geq \left(\|v\|_{H^1} - |a| \|V'(\Phi_c)\|_{H^1} \right)^2 \\ &\geq \|v\|_{H^1}^2 + 2a \|v\|_{H^1} \|V'(\Phi_c)\|_{H^1}, \end{aligned}$$

it is inferred that

$$\begin{aligned} \Delta E &\geq \frac{\beta}{2} \|v\|_{H^1}^2 + a \|v\|_{H^1} \left(\beta \|V'(\Phi_c)\|_{H^1} + \|\mathcal{L}_c V'(\Phi_c)\|_{L^2} \right) \\ &\quad + \frac{1}{2} a^2 \langle \mathcal{L}_c V'(\Phi_c), V'(\Phi_c) \rangle - \frac{1}{\sqrt{2}(1+p)} \sum_{n=3}^{1+p} \binom{1+p}{n} \|\Phi_c\|_{H^1}^{1+p-n} \|v\|_{H^1}^n. \end{aligned}$$

Here we used the Sobolev estimate $\sup_{\xi \in \mathbb{R}} |v(\xi)| \leq \frac{1}{\sqrt{2}} \|v\|_{H^1}$. In view of the expression for a in equation (5.9) together with the condition $\|v\|_{H^1} < 1$, there appears the estimate

$$\begin{aligned} \Delta E &\geq \frac{\beta}{2} \|v\|_{H^1}^2 - k \|v\|_{H^1}^3 \left(\beta \|V'(\Phi_c)\|_{H^1} + \|\mathcal{L}_c V'(\Phi_c)\|_{L^2} \right) \\ &\quad - \frac{1}{2} k^2 \|v\|_{H^1}^3 \left| \langle \mathcal{L}_c V'(\Phi_c), V'(\Phi_c) \rangle \right| - \frac{1}{\sqrt{2}(1+p)} \|v\|_{H^1}^3 \sum_{n=3}^{1+p} \binom{1+p}{n} \|\Phi_c\|_{H^1}^{1+p-n} \\ &= \frac{\beta}{2} \|v\|_{H^1}^2 - k_1 \|v\|_{H^1}^3 = \|v\|_{H^1}^2 \left(\frac{\beta}{2} - k_1 \|v\|_{H^1} \right), \end{aligned}$$

where

$$\begin{aligned} k_1 &= k \left(\beta \|V'(\Phi_c)\|_{H^1} + \|\mathcal{L}_c V'(\Phi_c)\|_{L^2} \right) \\ &\quad + \frac{1}{2} k^2 \left| \langle \mathcal{L}_c V'(\Phi_c), V'(\Phi_c) \rangle \right| + \frac{1}{\sqrt{2}(1+p)} \sum_{n=3}^{1+p} \binom{1+p}{n} \|\Phi_c\|_{H^1}^{1+p-n}, \end{aligned}$$

is a positive constant. Therefore, if $\|v\|_{H^1}$ is sufficiently small, say $\|v\|_{H^1} < \beta/4k_1$ by choosing $\varepsilon < \min(\beta/4k_1, 1)$, we obtain

$$\Delta E \geq \frac{\beta}{4} \|v\|_{H^1}^2.$$

□

Proof of Stability. Finally, we close this section by showing a sufficient condition for stability of the solitary wave.

Theorem 5.3. *If $d''(c) > 0$ then the solitary-wave Φ_c is stable.*

Proof. The proof is based on the techniques in [8, 12]. In particular, the theorem will be proved by contradiction as follows. Suppose Φ_c is not stable, then there exists an $\varepsilon > 0$, and a sequence of initial data $u_n^0 \in H^1(\mathbb{R})$ and corresponding solutions $u_n \in \mathcal{C}([0, \infty); H^1)$ with $u_n(\cdot, 0) = u_n^0$, such that

$$\lim_{n \rightarrow \infty} \|u_n^0 - \Phi_c\|_{H^1} = 0, \tag{5.12}$$

but

$$\sup_{t>0} \inf_{s \in \mathbb{R}} \|u_n(\cdot, t) - \tau_s \Phi_c(\cdot)\|_{H^1} \geq \frac{1}{2} \varepsilon,$$

for large enough n . If $\tilde{\varepsilon} = \min(\varepsilon, \beta/4k_1, 1)$ for β , and k_1 is defined in Lemma 5.1 and Lemma 5.2, respectively, this statement is still valid if we replace ε by $\tilde{\varepsilon}$. Now since $u_n \in \mathcal{C}([0, \infty); H^1)$, we can pick the first time t_n so that

$$\inf_{s \in \mathbb{R}} \|u_n(\cdot, t_n) - \tau_s \Phi_c(\cdot)\|_{H^1} = \frac{1}{2} \varepsilon. \tag{5.13}$$

Since V is continuous in $H^1(\mathbb{R})$ and invariant under time evolution, we have $\lim_{n \rightarrow \infty} V(u_n^0) = V(\Phi_c)$, and consequently

$$\lim_{n \rightarrow \infty} V(u_n(\cdot, t_n)) = V(\Phi_c). \tag{5.14}$$

Choose a sequence $w_n \in H^1(\mathbb{R})$, such that $V(w_n) = V(\Phi_c)$ and $\lim_{n \rightarrow \infty} \|w_n - u_n(\cdot, t_n)\|_{H^1} = 0$. The sequence defined by $w_n = (\|\Phi_c\|_{H^1} / \|u_n\|_{H^1}) u_n(\cdot, t_n)$ will do the job.

Note that by H^1 -continuity of E , and time invariance,

$$\lim_{n \rightarrow \infty} [E(w_n) - E(\Phi_c)] = 0.$$

Also note that $w_n \in U_\varepsilon$ for large n . On the other hand, so long as ε is small enough, says $\varepsilon = \tilde{\varepsilon}$, Lemma 5.2 shows that

$$E(w_n) - E(\Phi_c) \geq \frac{\beta}{4} \|w_n(\cdot + \alpha(w_n)) - \Phi_c\|_{H^1}^2,$$

where β is the constant defined in Lemma 5.1. Therefore, since $\alpha(u)$ is a continuous function, it appears that

$$\lim_{n \rightarrow \infty} \|u_n(\cdot, t_n) - \Phi_c(\cdot - \alpha(u_n(\cdot, t_n)))\|_{H^1} = 0.$$

But this is a contradiction to (5.13). □

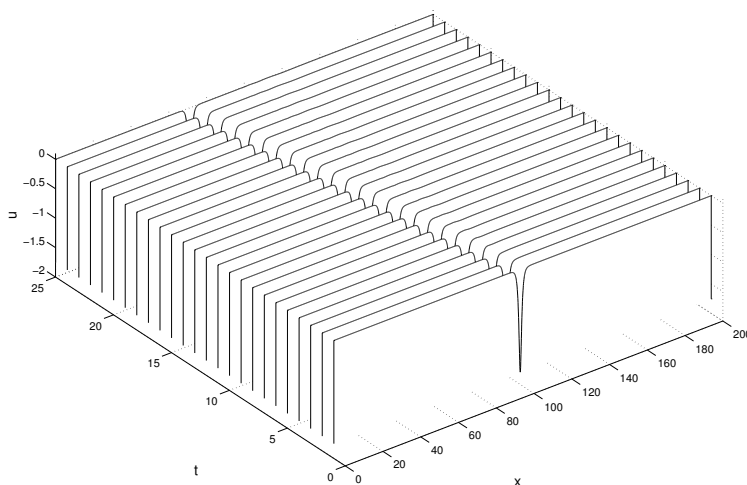


FIGURE 4. Stable negative solitary wave with velocity $c = -1.2$ in the case $p = 4$.

Figure 4 shows a stable negative solitary wave with velocity $c = -1.2 < c_4^- = -0.2612$, $p = 4$ and amplitude $\max_x |\Phi_{-1.2}| = 1.7652$, propagating to the left.

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