Electronic Journal of Differential Equations, Vol. 2009(2009), No. 26, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS TO BOUNDARY-VALUE PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. we discuss the existence, uniqueness and continuous dependence of solutions for a boundary value problem of nonlinear fractional differential equation.

1. Introduction

Fractional differential equations have gained considerable popularity and importance during the past three decades or so, due mainly to their varied applications in many fields of science and engineering. Analysis of fractional differential equations has been carried out by various authors. As for the research in solutions and also many real applications for factional differential equations, we refer to the book by Kilbas, Srivastava and Trujillo [4] and references therein. Boundary-value problems for fractional differential equations have been discussed in [1, 2, 3, 5, 7, 8, 9]. Bai and Lü [2] used fixed-point theorems on a cone to obtain the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2,$$
$$u(0) = u(1) = 0,$$

where $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and D^{α}_{0+} is the fractional derivative in the sense of Riemann-Liouville. However, as mentioned in [8], the Riemann-Liouville fractional derivative is not suitable for nonzero boundary values. Therefore, Zhang [8] investigated the existence and multiplicity of positive solutions for the problem

$$\mathbf{D}^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2,$$
$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0,$$

with the Caputo's fractional derivative $\mathbf{D}^{\alpha}_{0+}$ and a nonnegative continuous function f on $[0, 1]\times[0, \infty)$. The existence of solutions for the nonlinear fractional differential

2000 *Mathematics Subject Classification.* 34B05, 26A33.
Key words and phrases. Boundary value problem; fractional derivative; fixed-point theorem; Green's function; existence and uniqueness; continuous dependence.

Submitted June 25, 2008. Published February 3, 2009.

equation

$$
\begin{gathered}
{}_0^C D_t^\delta u(t) = g(t, u(t)), \quad 0 < t < 1,\ 1 < \delta < 2, \\
u(0) = \alpha \neq 0, \quad u(1) = \beta \neq 0
\end{gathered}
$$

has been discussed using the Laplace transform method in [9], where ${}_0^C D_t^\delta$ denotes the Caputo's fractional derivative and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. By means of Schauder fixed-point theorem, Su [7] proved an existence result for the problem

$$
\begin{gathered}
D^\alpha u(t) = f(t, v(t), D^\mu v(t)), \quad 0 < t < 1, \\
D^\beta v(t) = g(t, u(t), D^\nu u(t)), \quad 0 < t < 1, \\
u(0) = u(1) = v(0) = v(1) = 0,
\end{gathered}
$$

where $1 < \alpha,\ \beta < 2,\ \mu, \nu > 0,\ \alpha - \nu \geq 1,\ \beta - \mu \geq 1,\ f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and D is the standard Riemann-Liouville differentiation.

Motivated by the previous results, we present in this paper analysis of a boundary value problem for the fractional differential equation involving more general boundary conditions and a nonlinear term dependent on the fractional derivative of the unknown function

$$
\begin{gathered}
{}^C D_{0+}^\alpha u(t) = f(t, u(t), {}^C D_{0+}^\beta u(t)), \quad 0 < t < 1, \\
a_1 u(0) - a_2 u'(0) = A, b_1 u(1) + b_2 u'(1) = B,
\end{gathered}
\tag{1.1}
$$

where $1 < \alpha \leq 2,\ 0 < \beta \leq 1,\ a_i, b_i \geq 0,\ i = 1, 2,\ a_1 b_1 + a_1 b_2 + a_2 b_1 > 0,\ {}^C D_{0+}^\alpha$ and ${}^C D_{0+}^\beta$ are the Caputo's fractional derivatives and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. We impose a growth condition on the function f to prove an existence result for (1.1). For f Lipschitz in the second and third variables, the uniqueness of solution and the solution's dependence on the order α of the differential operator, the boundary values A and B, and the nonlinear term f are also discussed.

Throughout this work, we denote by I_{0+}^α and D_{0+}^α the Riemann-Liouville fractional integral and derivative respectively. The definitions and some properties of fractional integrals and fractional derivatives of different types can be found in [4, 6]. In order to proceed, we recall some fundamental facts of fractional calculus theory.

Remark 1.1. If $\alpha = n$ is an integer, the Riemann-Liouville fractional derivative of order α is the usual derivative of order n. The following properties are well known: $I_{0+}^\alpha I_{0+}^\beta f(t) = I_{0+}^{\alpha+\beta} f(t),\ D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t), \alpha > 0, \beta > 0, f \in L^1(0,1)$; $I_{0+}^\alpha : C[0,1] \to C[0,1],\ \alpha > 0$.

Remark 1.2. For $\alpha = n$, the Caputo's fractional derivative of order α becomes a conventional n-th derivative. The Caputo's fractional derivative is defined in [4] as follows: ${}^C D_{0+}^\alpha f(t) = D_{0+}^\alpha (f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k)$, provided that the right-side derivative exists. In particular, ${}^C D_{0+}^\alpha C = 0$ for any constant $C \in \mathbb{R},\ \alpha > 0$. Moreover, we can derive the following useful properties from [4, Lemmas 2.21 and 2.22]: ${}^C D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t), \alpha > 0, f(t) \in C[0,1]; I_{0+}^\alpha {}^C D_{0+}^\alpha f(t) = f(t) - f(0), 0 < \alpha \leq 1, f(t) \in C[0,1]$.

Similar composition relation below between I_{0+}^α and ${}^C D_{0+}^\alpha$ can be found in [8, Lemma 2.3], but the author did not point out the space to which $u(t)$ belongs. Besides, the subscript n of the coefficient c_n is wrong.

Lemma 1.3. *Assume that $u(t) \in C(0,1) \cap L^1(0,1)$ with a derivative of order n that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where n is the smallest integer greater than or equal to α.

We can also prove this lemma using [2, Lemma 2.2] and Remark 1.1. This proof is obvious and we omit it here.

2. Existence and uniqueness results

In this section, we first impose a growth condition on f which allows us to establish an existence result of solution, and then utilize the Lipschitz condition on f to prove a uniqueness theorem for the problem (1.1). Our approaches are based on the fixed-point theorems due to Schauder and Banach.

Let $I = [0,1]$ and $C(I)$ be the space of all continuous real functions defined on I. Define the space $X = \{u(t) \mid u(t) \in C(I) \text{ and } {}^{C}D_{0+}^{\beta} u(t) \in C(I), 0 < \beta \leq 1\}$ endowed with the norm $\|u\| = \max_{t \in I} |u(t)| + \max_{t \in I} |{}^{C}D_{0+}^{\beta} u(t)|$. Then by the method in [7, Lemma 3.2] and Remark 1.2 we can know that $(X, \|\cdot\|)$ is a Banach space.

Now we present the Green's function for boundary value problem of fractional differential equation.

Lemma 2.1. *Let $1 < \alpha \leq 2$. Assume that $g : [0,1] \to \mathbb{R}$ is a continuous function. Then the unique solution of*

$$\begin{gathered} {}^{C}D_{0+}^{\alpha} u(t) = g(t), \quad 0 < t < 1, \\ a_1 u(0) - a_2 u'(0) = 0, b_1 u(1) + b_2 u'(1) = 0, \end{gathered}$$

is $u(t) = \int_0^1 G(t,s) g(s) \mathrm{d}s$, where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}[(t-s)^{\alpha-1} - \frac{a_2 b_1}{l}(1-s)^{\alpha-1} - \frac{a_1 b_1}{l}(1-s)^{\alpha-1} t] \\ + \frac{1}{\Gamma(\alpha-1)}[-\frac{a_2 b_2}{l}(1-s)^{\alpha-2} - \frac{a_1 b_2}{l}(1-s)^{\alpha-2} t], & s \leq t, \\ \frac{1}{\Gamma(\alpha)}[-\frac{a_2 b_1}{l}(1-s)^{\alpha-1} - \frac{a_1 b_1}{l}(1-s)^{\alpha-1} t] \\ + \frac{1}{\Gamma(\alpha-1)}[-\frac{a_2 b_2}{l}(1-s)^{\alpha-2} - \frac{a_1 b_2}{l}(1-s)^{\alpha-2} t], & t \leq s, \end{cases}$$

here $l = a_1 b_1 + a_1 b_2 + a_2 b_1$.

This lemma can be proved using Lemma 1.3 and Remark 1.1. For details, we refer the reader to [8, Lemma 3.1].

Similarly, we can obtain the solution for the boundary-value problem with homogeneous equation and nonhomogeneous boundary conditions.

Lemma 2.2. *Let $1 < \alpha \leq 2$. Then the unique solution of*

$$\begin{gathered} {}^{C}D_{0+}^{\alpha} u(t) = 0, \quad 0 < t < 1, \\ a_1 u(0) - a_2 u'(0) = A, \quad b_1 u(1) + b_2 u'(1) = B, \end{gathered}$$

is $u(t) = \frac{(b_1 + b_2)A + a_2 B}{l} + \frac{a_1 B - b_1 A}{l} t$.

In the following discussion, we denote

$$\varphi(t) := \frac{(b_1+b_2)A + a_2B}{l} + \frac{a_1B - b_1A}{l}t,$$

and use the assumption

(H) $1 < \alpha \le 2$, $0 < \beta \le 1$, $a_i, b_i \ge 0$, $i = 1, 2$, $a_1b_1 + a_1b_2 + a_2b_1 > 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Lemma 2.3. *Assume that* (H) *holds. Then* (1.1) *is equivalent to the nonlinear integral equation*

$$u(t) = \int_0^1 G(t,s) f(s, u(s), {}^C D_{0^+}^{\beta} u(s)) \mathrm{d}s + \varphi(t). \tag{2.1}$$

In other words, every solution of (1.1) *is also a solution of* (2.1) *and vice versa.*

Proof. Let $u \in X$ be a solution of (1.1), applying the method used to prove Lemma 2.1, we can obtain that u is a solution of (2.1).

Conversely, let $u \in X$ be a solution of (2.1). We denote the right-hand side of the equation (2.1) by $w(t)$; i.e.,

$$\begin{aligned} w(t) &= I_{0^+}^{\alpha} f(t, u(t), {}^C D_{0^+}^{\beta} u(t)) + \frac{-a_2b_1 - a_1b_1t}{l} I_{0^+}^{\alpha} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) \\ &\quad + \frac{-a_2b_2 - a_1b_2t}{l} I_{0^+}^{\alpha-1} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) + \varphi(t). \end{aligned}$$

Using Remarks 1.1 and 1.2, we have

$$\begin{aligned} w'(t) &= D_{0^+}^{1} I_{0^+}^{1} I_{0^+}^{\alpha-1} f(t, u(t), {}^C D_{0^+}^{\beta} u(t)) - \frac{a_1b_1}{l} I_{0^+}^{\alpha} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) \\ &\quad - \frac{a_1b_2}{l} I_{0^+}^{\alpha-1} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) + \frac{a_1B - b_1A}{l} \\ &= I_{0^+}^{\alpha-1} f(t, u(t), {}^C D_{0^+}^{\beta} u(t)) - \frac{a_1b_1}{l} I_{0^+}^{\alpha} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) \\ &\quad - \frac{a_1b_2}{l} I_{0^+}^{\alpha-1} f(1, u(1), {}^C D_{0^+}^{\beta} u(1)) + \frac{a_1B - b_1A}{l}, \end{aligned}$$

$$\begin{aligned} {}^C D_{0^+}^{\alpha} w(t) &= D_{0^+}^{\alpha}(w(t) - w(0^+) - w'(0^+)t) = D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t, u(t), {}^C D_{0^+}^{\beta} u(t)) \\ &= f(t, u(t), {}^C D_{0^+}^{\beta} u(t)), \end{aligned}$$

namely, ${}^C D_{0^+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0^+}^{\beta} u(t))$. One can verify easily that $a_1u(0) - a_2u'(0) = A, b_1u(1) + b_2u'(1) = B$. Therefore, u is a solution of (1.1), which completes the proof. □

Lemma 2.3 indicates that the solution of the problem (1.1) coincides with the fixed point of the operator T defined as

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s), {}^C D_{0^+}^{\beta} u(s)) \mathrm{d}s + \varphi(t).$$

Now, we give the main results of this section.

Theorem 2.4. *Let the assumption* (H) *be satisfied. Suppose further that*

(H1)
$$\lim_{(|x|+|y|)\to\infty}\frac{\max_{t\in I}|f(t,x,y)|}{|x|+|y|}<\frac{l\Gamma(\alpha)\Gamma(2-\beta)}{(2l+a_2b_2)\Gamma(2-\beta)+2l}=:K.$$

Then there exists at least one solution $u(t)$ to the boundary-value problem (1.1).

Proof. For any $t\in I$, we find
$$\begin{aligned}
&\int_0^1|G(t,s)|\mathrm{d}s\\
&\le\frac{1}{\Gamma(\alpha)}\Big(\int_0^t(t-s)^{\alpha-1}\mathrm{d}s+\frac{a_2b_1}{l}\int_0^1(1-s)^{\alpha-1}\mathrm{d}s+\frac{a_1b_1t}{l}\int_0^1(1-s)^{\alpha-1}\mathrm{d}s\Big)\\
&\quad+\frac{1}{\Gamma(\alpha-1)}\Big(\frac{a_2b_2}{l}\int_0^1(1-s)^{\alpha-2}\mathrm{d}s+\frac{a_1b_2t}{l}\int_0^1(1-s)^{\alpha-2}\mathrm{d}s\Big)\\
&=\frac{1}{\Gamma(\alpha+1)}\Big(t^\alpha+\frac{a_2b_1+a_1b_1t}{l}\Big)+\frac{1}{\Gamma(\alpha)}\frac{a_2b_2+a_1b_2t}{l}\\
&\le\frac{1}{\Gamma(\alpha)}\Big(1+\frac{a_2b_1+a_1b_1}{l}+\frac{a_2b_2+a_1b_2}{l}\Big)\\
&=\frac{2l+a_2b_2}{l\Gamma(\alpha)}
\end{aligned}$$
and
$$\begin{aligned}
&\int_0^1|G_t'(t,s)|\mathrm{d}s\\
&\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\mathrm{d}s+\frac{a_1b_1}{l\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\mathrm{d}s+\frac{a_1b_2}{l\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\mathrm{d}s\\
&=\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{a_1b_1}{l\Gamma(\alpha+1)}+\frac{a_1b_2}{l\Gamma(\alpha)}\\
&\le\frac{1}{\Gamma(\alpha)}\Big(1+\frac{a_1b_1}{l}+\frac{a_1b_2}{l}\Big)\le\frac{2}{\Gamma(\alpha)}.
\end{aligned}$$
Therefore, $|G(t,\cdot)|$ and $|G_t'(t,\cdot)|$ are integrable for any $t\in I$.

Denote $h(x,y)=\max_{t\in I}|f(t,x,y)|$ and Choose $\varepsilon=\frac{1}{2}(K-\lim_{(|x|+|y|)\to\infty}\frac{h(x,y)}{|x|+|y|})$. It follows from the condition (H1) that there exists a constant $d_1>0$ such that $h(x,y)\le(K-\varepsilon)(|x|+|y|)$ for $|x|+|y|\ge d_1$. Let $M=\max\{h(x,y):|x|+|y|\le d_1\}$ and choose $d_2>d_1$ such that $M/d_2\le K-\varepsilon$. Then we get $h(x,y)\le(K-\varepsilon)d_2$, $|x|+|y|\le d_2$. Therefore, $h(x,y)\le(K-\varepsilon)c$, $|x|+|y|\le c$ for any $c\ge d_2$.

Let $k_1=\max_{t\in I}|\varphi(t)|$, $k_2=\max_{t\in I}|\varphi'(t)|$, $k=\max\{k_1,k_2/\Gamma(2-\beta)\}$, $d_3=2Kk/\varepsilon$ and $d=\max\{d_2,d_3\}$. Define
$$U=\{u(t):u(t)\in X,\|u(t)\|\le d,\ t\in I\}.$$
Then U is a convex, closed and bounded subset of X. Moreover, for any $u\in U$, $h(u(t),{}^CD^\beta_{0+}u(t))\le(K-\varepsilon)d$.

Now we prove that the operator T maps U to itself. For any $u\in U$, we can get
$$\begin{aligned}
|Tu(t)|&\le|\varphi(t)|+\int_0^1|G(t,s)h(u(s),{}^CD^\beta_{0+}u(s))|\mathrm{d}s\\
&\le k_1+d(K-\varepsilon)\int_0^1|G(t,s)|\mathrm{d}s
\end{aligned}$$

$$\leq k_1 + d(K-\varepsilon)\frac{2l + a_2b_2}{l\Gamma(\alpha)},$$

$$\begin{aligned}
&|{}^CD^{\beta}_{0^+}(Tu)(t)| \\
&= \Big|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\mathrm{d}s\Big| \\
&\leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_s(s,\tau)f(\tau,u(\tau),{}^CD^{\beta}_{0^+}u(\tau))|\mathrm{d}\tau + |\varphi'(s)|\Big)\mathrm{d}s \\
&\leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(|\varphi'(s)| + \int_0^1 |G'_s(s,\tau)h(u(\tau),{}^CD^{\beta}_{0^+}u(\tau))|\mathrm{d}\tau\Big)\mathrm{d}s \\
&\leq \frac{k_2}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\mathrm{d}s + d(K-\varepsilon)\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_s(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s \\
&\leq \frac{k_2}{\Gamma(2-\beta)} + d(K-\varepsilon)\frac{2}{\Gamma(2-\beta)\Gamma(\alpha)}
\end{aligned}$$

for $0 < \beta < 1$, and

$$\begin{aligned}
|(Tu)'(t)| &= \Big|\int_0^1 G'_t(t,s)f(s,u(s),{}^CD^{\beta}_{0^+}u(s))\mathrm{d}s + \varphi'(t)\Big| \\
&\leq |\varphi'(t)| + \int_0^1 |G'_t(t,s)h(u(s),{}^CD^{\beta}_{0^+}u(s))|\mathrm{d}s \\
&\leq k_2 + d(K-\varepsilon)\int_0^1 |G'_t(t,s)|\mathrm{d}s \leq k_2 + d(K-\varepsilon)\frac{2}{\Gamma(\alpha)}
\end{aligned}$$

for $\beta = 1$. Hence,

$$\|Tu\| \leq 2k + d(K-\varepsilon)\frac{(2l+a_2b_2)\Gamma(2-\beta)+2l}{l\Gamma(\alpha)\Gamma(2-\beta)} \leq d\frac{\varepsilon}{K} + d(K-\varepsilon)\frac{1}{K} = d.$$

Note also that

$$\begin{aligned}
Tu(t) &= I^{\alpha}_{0^+}f(t,u(t),{}^CD^{\beta}_{0^+}u(t)) + \frac{-a_2b_1 - a_1b_1t}{l}I^{\alpha}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) \\
&\quad + \frac{-a_2b_2 - a_1b_2t}{l}I^{\alpha-1}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) + \varphi(t),
\end{aligned}$$

$$\begin{aligned}
(Tu)'(t) &= I^{\alpha-1}_{0^+}f(t,u(t),{}^CD^{\beta}_{0^+}u(t)) - \frac{a_1b_1}{l}I^{\alpha}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) \\
&\quad - \frac{a_1b_2}{l}I^{\alpha-1}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) + \frac{a_1B - b_1A}{l},
\end{aligned}$$

$$\begin{aligned}
{}^CD^{\beta}_{0^+}(Tu)(t) &= I^{1-\beta}_{0^+}(Tu)'(t) \\
&= I^{\alpha-\beta}_{0^+}f(t,u(t),{}^CD^{\beta}_{0^+}u(t)) - \frac{a_1b_1}{l}I^{\alpha-\beta+1}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) \\
&\quad - \frac{a_1b_2}{l}I^{\alpha-\beta}_{0^+}f(1,u(1),{}^CD^{\beta}_{0^+}u(1)) + I^{1-\beta}_{0^+}\frac{a_1B - b_1A}{l}.
\end{aligned}$$

It is easy to see $Tu(t), {}^CD^{\beta}_{0^+}(Tu)(t) \in C(I)$. Therefore, $T : U \to U$.

Claim: T is a continuous operator. In fact, for u_n, $n = 0, 1, 2, \dots$ and $u \in U$ such that $\lim_{n\to\infty}\|u_n - u\| \to 0$, we have

$$|Tu_n(t) - Tu(t)|$$

$$
\begin{aligned}
&= \Big|\int_0^1 G(t,s)\Big(f(s,u_n(s),{}^C D_{0+}^{\beta}u_n(s)) - f(s,u(s),{}^C D_{0+}^{\beta}u(s))\Big)\mathrm{d}s\Big| \\
&\le \max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|\int_0^1 |G(t,s)|\mathrm{d}s \\
&\le \frac{2l+a_2b_2}{l\Gamma(\alpha)}\max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|,
\end{aligned}
$$

$$
\begin{aligned}
&|{}^C D_{0+}^{\beta}(Tu_n)(t) - {}^C D_{0+}^{\beta}(Tu)(t)| \\
&= \Big|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}((Tu_n)'(s)-(Tu)'(s))\mathrm{d}s\Big| \\
&\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \Big|G_s'(s,\tau)\Big(f(\tau,u_n(\tau),{}^C D_{0+}^{\beta}u_n(\tau)) \\
&\quad - f(\tau,u(\tau),{}^C D_{0+}^{\beta}u(\tau))\Big)\Big|\mathrm{d}\tau\Big)\mathrm{d}s \\
&\le \max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))| \\
&\quad \times \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G_s'(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s \\
&\le \max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|\frac{2}{\Gamma(1-\beta)\Gamma(\alpha)}\int_0^t (t-s)^{-\beta}\mathrm{d}s \\
&\le \frac{2}{\Gamma(2-\beta)\Gamma(\alpha)}\max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|
\end{aligned}
$$

for $0 < \beta < 1$, and

$$
\begin{aligned}
&|(Tu_n)'(t) - (Tu)'(t)| \\
&= \Big|\int_0^1 G_t'(t,s)\Big(f(s,u_n(s),{}^C D_{0+}^{\beta}u_n(s)) - f(s,u(s),{}^C D_{0+}^{\beta}u(s))\Big)\mathrm{d}s\Big| \\
&\le \max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|\int_0^1 |G_t'(t,s)|\mathrm{d}s \\
&\le \frac{2}{\Gamma(\alpha)}\max_{t\in I}|f(t,u_n(t),{}^C D_{0+}^{\beta}u_n(t)) - f(t,u(t),{}^C D_{0+}^{\beta}u(t))|
\end{aligned}
$$

for $\beta = 1$. Then in view of the uniform continuity of the function f on $I\times[-d,d]\times[-d,d]$, we obtain that T is continuous.

The last step is to prove that T is a completely continuous operator. Let $t,\tau\in I$ be such that $t<\tau$ and $N = \max_{t\in I,u\in U}|f(t,u(t),{}^C D_{0+}^{\beta}u(t))| + 1$. Then we have

$$
\begin{aligned}
&|Tu(t) - Tu(\tau)| \\
&= \Big|\int_0^1 (G(t,s)-G(\tau,s))f(s,u(s),{}^C D_{0+}^{\beta}u(s))\mathrm{d}s + \varphi(t) - \varphi(\tau)\Big| \\
&\le N\Big(\int_0^t |G(t,s)-G(\tau,s)|\mathrm{d}s + \int_t^\tau |G(t,s)-G(\tau,s)|\mathrm{d}s \\
&\quad + \int_\tau^1 |G(t,s)-G(\tau,s)|\mathrm{d}s\Big) + |\varphi(t)-\varphi(\tau)|
\end{aligned}
$$

$$
\begin{aligned}
&\le N\Big[\int_0^t \Big(\frac{(\tau-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+(\tau-t)\Big(\frac{a_1b_1}{l}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\quad +\frac{a_1b_2}{l}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\Big)\mathrm{d}s\\
&\quad +\int_t^\tau \Big(\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}+(\tau-t)\Big(\frac{a_1b_1}{l}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{a_1b_2}{l}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\Big)\mathrm{d}s\\
&\quad +\int_\tau^1 (\tau-t)\Big(\frac{a_1b_1}{l}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{a_1b_2}{l}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\mathrm{d}s\Big]+|\varphi(t)-\varphi(\tau)|\\
&= N\Big[\int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+(\tau-t)\Big(\frac{a_1b_1}{l}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\\
&\quad +\frac{a_1b_2}{l}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s\Big)\Big]+(\tau-t)\frac{|a_1B-b_1A|}{l}\\
&= N\Big[\frac{\tau^\alpha-t^\alpha}{\Gamma(\alpha+1)}+(\tau-t)\Big(\frac{a_1b_1}{l\Gamma(\alpha+1)}+\frac{a_1b_2}{l\Gamma(\alpha)}\Big)\Big]+(\tau-t)\frac{|a_1B-b_1A|}{l},
\end{aligned}
$$

$$
\begin{aligned}
&|{}^CD_{0^+}^\beta(Tu)(t)-{}^CD_{0^+}^\beta(Tu)(\tau)|\\
&=\frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (t-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta+\varphi'(s)\Big)\mathrm{d}s\\
&\quad -\int_0^\tau (\tau-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta+\varphi'(s)\Big)\mathrm{d}s\Big|\\
&\le \frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (t-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta\Big)\mathrm{d}s\\
&\quad -\int_0^t (\tau-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta\Big)\mathrm{d}s\Big|\\
&\quad +\frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (\tau-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta\Big)\mathrm{d}s\\
&\quad -\int_0^\tau (\tau-s)^{-\beta}\Big(\int_0^1 G'_s(s,\theta)f(\theta,u(\theta),{}^CD_{0^+}^\beta u(\theta))\mathrm{d}\theta\Big)\mathrm{d}s\Big|\\
&\quad +\frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (t-s)^{-\beta}\varphi'(s)\mathrm{d}s-\int_0^t (\tau-s)^{-\beta}\varphi'(s)\mathrm{d}s\Big|\\
&\quad +\frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (\tau-s)^{-\beta}\varphi'(s)\mathrm{d}s-\int_0^\tau (\tau-s)^{-\beta}\varphi'(s)\mathrm{d}s\Big|\\
&\le \frac{2N}{\Gamma(1-\beta)\Gamma(\alpha)}\int_0^t ((t-s)^{-\beta}-(\tau-s)^{-\beta})\mathrm{d}s+\frac{2N}{\Gamma(1-\beta)\Gamma(\alpha)}\int_t^\tau (\tau-s)^{-\beta}\mathrm{d}s\\
&\quad +\frac{|a_1B-b_1A|}{l\Gamma(1-\beta)}\int_0^t ((t-s)^{-\beta}-(\tau-s)^{-\beta})\mathrm{d}s+\frac{|a_1B-b_1A|}{l\Gamma(1-\beta)}\int_t^\tau (\tau-s)^{-\beta}\mathrm{d}s\\
&\le \Big(\frac{2N}{\Gamma(2-\beta)\Gamma(\alpha)}+\frac{|a_1B-b_1A|}{l\Gamma(2-\beta)}\Big)(\tau^{1-\beta}-t^{1-\beta}+2(\tau-t)^{1-\beta})
\end{aligned}
$$

for $0<\beta<1$, and

$$
|(Tu)'(t)-(Tu)'(\tau)|
$$

$$
\begin{aligned}
&= \Big| \int_0^1 G_t'(t,s) f(s,u(s),{}^C D_{0+}^{\beta} u(s)) \mathrm{d}s + \varphi'(t) \\
&\quad - \int_0^1 G_\tau'(\tau,s) f(s,u(s),{}^C D_{0+}^{\beta} u(s)) \mathrm{d}s - \varphi'(\tau) \Big| \\
&\le \frac{N}{\Gamma(\alpha-1)} \Big(\int_0^t ((t-s)^{\alpha-2} - (\tau-s)^{\alpha-2}) \mathrm{d}s + \int_t^\tau (\tau-s)^{\alpha-2} \mathrm{d}s \Big) \\
&\le \frac{N}{\Gamma(\alpha)} (\tau^{\alpha-1} - t^{\alpha-1} + 2(\tau-t)^{\alpha-1})
\end{aligned}
$$

for $\beta = 1$.

Now, using the fact that the functions $\tau^\alpha - t^\alpha, \tau^{\alpha-1} - t^{\alpha-1}$ and $\tau^{1-\beta} - t^{1-\beta}$ are uniformly continuous on the interval I, we conclude that TU is an equicontinuous set. Obviously it is uniformly bounded since $TU \subseteq U$. Thus, T is completely continuous. The Schauder fixed-point theorem asserts the existence of solution in U for the problem (1.1) and the theorem is proved. □

The following corollary is obvious.

Corollary 2.5. *Let the assumption* (H) *be satisfied. Suppose further that there exist two nonnegative functions* $a(t), b(t) \in C[0,1]$ *such that* $|f(t,x,y)| \le a(t)|x|^\rho + b(t)|y|^\theta$*, where* $0 < \rho, \theta < 1$*. Then there exists at least one solution for the boundary value problem* (1.1)*.*

Example 2.6. Consider the problem

$$
\begin{gathered}
{}^C D_{0+}^{3/2} u = (t - \frac{1}{2})^3 (u(t) + {}^C D_{0+}^{1/2} u(t)), \quad 0 < t < 1, \\
a_1 u(0) - a_2 u'(0) = A, b_1 u(1) + b_2 u'(1) = B.
\end{gathered}
$$

Using $\Gamma(1/2) = \sqrt{\pi}$, a simple computation shows $K = (l\pi)/(2(2l + a_2 b_2)\sqrt{\pi} + 8l)$. Since $|f(t,x,y)| = |t - \frac{1}{2}|^3 |x+y| \le \frac{|x|+|y|}{8}$,

$$
\lim_{(|x|+|y|)\to\infty} \frac{\frac{|x|+|y|}{8}}{|x|+|y|} = \frac{1}{8},
$$

then, if $1/8 < (l\pi)/(2(2l + a_2 b_2)\sqrt{\pi} + 8l)$ (for example, we choose $a_1 = b_2 = 1, a_2 = b_1 = 0$, then $K \approx 0.2082 > 1/8$), Theorem 2.4 ensures the existence of solution for this problem.

Theorem 2.7. *Let the assumption* (H) *be satisfied. Furthermore, let the function* f *fulfill a Lipschitz condition with respect to the second and third variables; i.e.,* $|f(t,x,y) - f(t,u,v)| \le L(|x-u| + |v-y|)$ *with a Lipschitz constant* L *such that* $0 < L < K$*, where* K *is as the same as that in Theorem 2.4. Then the boundary value problem* (1.1) *has a unique solution* $u(t) \in X$*.*

Proof. We have shown in Theorem 2.4 that $Tu(t), {}^C D_{0+}^{\beta}(Tu)(t) \in C(I)$; i.e., $T : X \to X$. To apply the Banach fixed-point theorem, we need to verify that T is a contraction mapping. For any $u, v \in X$, we have

$$
\begin{aligned}
&|Tu(t) - Tv(t)| \\
&= \Big| \int_0^1 G(t,s) \Big(f(s,u(s),{}^C D_{0+}^{\beta} u(s)) - f(s,v(s),{}^C D_{0+}^{\beta} v(s)) \Big) \mathrm{d}s \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le L\|u-v\|\int_0^1 |G(t,s)|\mathrm{d}s\\
&\le \frac{(2l+a_2b_2)L}{l\Gamma(\alpha)}\|u-v\|,
\end{aligned}
$$

$$
\begin{aligned}
&|{}^CD_{0+}^{\beta}(Tu)(t)-{}^CD_{0+}^{\beta}(Tv)(t)|\\
&=\Big|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}((Tu)'(s)-(Tv)'(s))\mathrm{d}s\Big|\\
&\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \big|G_s'(s,\tau)\big(f(\tau,u(\tau),{}^CD_{0+}^{\beta}u(\tau))\\
&\quad - f(\tau,v(\tau),{}^CD_{0+}^{\beta}v(\tau))\big)\big|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le L\|u-v\|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G_s'(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le \frac{2L}{\Gamma(2-\beta)\Gamma(\alpha)}\|u-v\|
\end{aligned}
$$

for $0<\beta<1$, and

$$
\begin{aligned}
&|(Tu)'(t)-(Tv)'(t)|\\
&=\Big|\int_0^1 G_t'(t,s)\Big(f(s,u(s),{}^CD_{0+}^{\beta}u(s))-f(s,v(s),{}^CD_{0+}^{\beta}v(s))\Big)\mathrm{d}s\Big|\\
&\le L\|u-v\|\int_0^1 |G_t'(t,s)|\mathrm{d}s \le \frac{2L}{\Gamma(\alpha)}\|u-v\|
\end{aligned}
$$

for $\beta=1$. Thus, $\|Tu-Tv\|\le L(\frac{2l+a_2b_2}{l\Gamma(\alpha)}+\frac{2}{\Gamma(2-\beta)\Gamma(\alpha)})\|u-v\|=\frac{L}{K}\|u-v\|$. Hence, the Banach fixed-point theorem yields that T has a unique fixed point which is the unique solution of the problem (1.1). The proof is therefore complete. □

3. Dependence on the parameters

The present section is devoted to the study of the dependence of solution on the parameters α, A and B, and f for the problem (1.1), provided that the function $f(t,x,y)$ is Lipschitz with respect to x and y.

Theorem 3.1. *Suppose that the conditions of Theorem 2.7 hold. Let $u_1(t),u_2(t)$ be the solutions, respectively, of the problems* (1.1) *and*

$$
\begin{gathered}
{}^CD_{0+}^{\alpha-\varepsilon}u(t)=f(t,u(t),{}^CD_{0+}^{\beta}u(t)),\quad 0<t<1,\\
a_1u(0)-a_2u'(0)=A,\quad b_1u(1)+b_2u'(1)=B,
\end{gathered}
\tag{3.1}
$$

where $1<\alpha-\varepsilon<\alpha\le 2$. Then $\|u_1-u_2\|=O(\varepsilon)$.

Proof. Let $G_1(t,s)=G(t,s)$ and

$$
G_2(t,s)=\begin{cases}
\frac{1}{\Gamma(\alpha-\varepsilon)}[(t-s)^{\alpha-\varepsilon-1}-\frac{a_2b_1}{l}(1-s)^{\alpha-\varepsilon-1}-\frac{a_1b_1}{l}(1-s)^{\alpha-\varepsilon-1}t]\\
+\frac{1}{\Gamma(\alpha-\varepsilon-1)}[-\frac{a_2b_2}{l}(1-s)^{\alpha-\varepsilon-2}-\frac{a_1b_2}{l}(1-s)^{\alpha-\varepsilon-2}t], & s\le t,\\
\frac{1}{\Gamma(\alpha-\varepsilon)}[-\frac{a_2b_1}{l}(1-s)^{\alpha-\varepsilon-1}-\frac{a_1b_1}{l}(1-s)^{\alpha-\varepsilon-1}t]\\
+\frac{1}{\Gamma(\alpha-\varepsilon-1)}[-\frac{a_2b_2}{l}(1-s)^{\alpha-\varepsilon-2}-\frac{a_1b_2}{l}(1-s)^{\alpha-\varepsilon-2}t], & t\le s,
\end{cases}
$$

be the Green's function of (3.1). Then

$$\begin{aligned} u_1(t) &= \int_0^1 G_1(t,s) f(t, u_1(s), {}^C D_{0+}^{\beta} u_1(s)) \mathrm{d}s + \varphi(t), \\ u_2(t) &= \int_0^1 G_2(t,s) f(t, u_2(s), {}^C D_{0+}^{\beta} u_2(s)) \mathrm{d}s + \varphi(t). \end{aligned}$$

First we show that

$$\int_0^1 |G_1(t,s) - G_2(t,s)| \mathrm{d}s = O(\varepsilon), \quad \int_0^1 |G'_{1t}(t,s) - G'_{2t}(t,s)| \mathrm{d}s = O(\varepsilon). \tag{3.2}$$

Observing that

$$\begin{aligned} \int_0^1 |G_1(t,s) - G_2(t,s)| \mathrm{d}s \le & \int_0^t \Big| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)} \Big| \mathrm{d}s \\ & + \Big(\frac{a_2 b_1}{l} + \frac{a_1 b_1 t}{l}\Big) \int_0^1 \Big| \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)} \Big| \mathrm{d}s \\ & + \Big(\frac{a_2 b_2}{l} + \frac{a_1 b_2 t}{l}\Big) \int_0^1 \Big| \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-\varepsilon-2}}{\Gamma(\alpha-\varepsilon-1)} \Big| \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |G'_{1t}(t,s) - G'_{2t}(t,s)| \mathrm{d}s \\ & \le \int_0^t \Big| \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-\varepsilon-2}}{\Gamma(\alpha-\varepsilon-1)} \Big| \mathrm{d}s + \frac{a_1 b_1}{l} \int_0^1 \Big| \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)} \Big| \mathrm{d}s \\ & \quad + \frac{a_1 b_2}{l} \int_0^1 \Big| \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-\varepsilon-2}}{\Gamma(\alpha-\varepsilon-1)} \Big| \mathrm{d}s, \end{aligned}$$

without loss of generality, we only estimate one of the integrals in the right-hand side of the two inequalities above.

$$\begin{aligned} & \int_0^t \Big| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)} \Big| \mathrm{d}s \\ & \le \int_0^t \Big| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha)} \Big| \mathrm{d}s + \int_0^t \Big| \frac{(t-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)} \Big| \mathrm{d}s \\ & = \frac{1}{\Gamma(\alpha)} \int_0^t \big| x^{\alpha-1} - x^{\alpha-\varepsilon-1} \big| \mathrm{d}x + \Big| \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha-\varepsilon)} \Big| \int_0^t (t-s)^{\alpha-\varepsilon-1} \mathrm{d}s \\ & \le \frac{1}{\Gamma(\alpha)} \Big(\frac{1}{\alpha-\varepsilon} - \frac{1}{\alpha} \Big) + \Big| \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha-\varepsilon)} \Big| \frac{1}{\alpha-\varepsilon} \\ & = \varepsilon \Big(\frac{1}{\alpha(\alpha-\varepsilon)\Gamma(\alpha)} + \frac{|\Gamma'(\alpha-\varepsilon+\theta\varepsilon)|}{(\alpha-\varepsilon)\Gamma(\alpha)\Gamma(\alpha-\varepsilon)} \Big), \end{aligned}$$

for some θ such that $0 < \theta < 1$. So we arrive at the relations in (3.2). Furthermore,

$$\begin{aligned} & |u_1(t) - u_2(t)| \\ & = \Big| \int_0^1 G_1(t,s) f(t, u_1(s), {}^C D_{0+}^{\beta} u_1(s)) \mathrm{d}s - \int_0^1 G_2(t,s) f(t, u_2(s), {}^C D_{0+}^{\beta} u_2(s)) \mathrm{d}s \Big| \\ & \le \int_0^1 \big| G_1(t,s) (f(t, u_1(s), {}^C D_{0+}^{\beta} u_1(s)) \mathrm{d}s - f(t, u_2(s), {}^C D_{0+}^{\beta} u_2(s)) \big| \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \big|G_1(t,s)f(t,u_2(s),{}^C D_{0^+}^{\beta}u_2(s))\mathrm{d}s - G_2(t,s)f(t,u_2(s),{}^C D_{0^+}^{\beta}u_2(s))\big|\mathrm{d}s\\
&\le L\|u_1-u_2\|\int_0^1 |G_1(t,s)|\mathrm{d}s + |||f|||\int_0^1 |G_1(t,s)-G_2(t,s)|\mathrm{d}s\\
&\le \frac{(2l+a_2b_2)L}{l\Gamma(\alpha)}\|u_1-u_2\| + |||f|||\int_0^1 |G_1(t,s)-G_2(t,s)|\mathrm{d}s,
\end{aligned}
$$

where $|||f||| = \sup_{0<\varepsilon<\alpha-1}\{\max_{t\in I}|f(t,u_2(t),{}^C D_{0^+}^{\beta}u_2(t))|\}$.

$$
\begin{aligned}
&|{}^C D_{0^+}^{\beta}u_1(t) - {}^C D_{0^+}^{\beta}u_2(t)|\\
&\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \big|G'_{1s}(s,\tau)f(\tau,u_1(\tau),{}^C D_{0^+}^{\beta}u_1(\tau))\\
&\quad - G'_{2s}(s,\tau)f(\tau,u_2(\tau),{}^C D_{0^+}^{\beta}u_2(\tau))\big|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \big|G'_{1s}(s,\tau)f(\tau,u_1(\tau),{}^C D_{0^+}^{\beta}u_1(\tau))\\
&\quad - G'_{1s}(s,\tau)f(\tau,u_2(\tau),{}^C D_{0^+}^{\beta}u_2(\tau))\big|\mathrm{d}\tau\\
&\quad + \int_0^1 \big|G'_{1s}(s,\tau)f(\tau,u_2(\tau),{}^C D_{0^+}^{\beta}u_2(\tau)) - G'_{2s}(s,\tau)f(\tau,u_2(\tau),{}^C D_{0^+}^{\beta}u_2(\tau))\big|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le L\|u_1-u_2\|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_{1s}(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\quad + |||f|||\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_{1s}(s,\tau)-G'_{2s}(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le \frac{2L}{\Gamma(2-\beta)\Gamma(\alpha)}\|u_1-v_2\|\\
&\quad + |||f|||\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_{1s}(s,\tau)-G'_{2s}(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s
\end{aligned}
$$

for $0<\beta<1$, and

$$
\begin{aligned}
|u'_1(t)-u'_2(t)| &\le L\|u_1-u_2\|\int_0^1 |G'_{1t}(t,s)|\mathrm{d}s + |||f|||\int_0^1 |G'_{1t}(t,s)-G'_{2t}(t,s)|\mathrm{d}s\\
&\le \frac{2L}{\Gamma(\alpha)}\|u_1-u_2\| + |||f|||\int_0^1 |G'_{1t}(t,s)-G'_{2t}(t,s)|\mathrm{d}s
\end{aligned}
$$

for $\beta=1$. It follows that

$$
\begin{aligned}
&\|u_1-u_2\|\\
&\le \frac{1}{1-L/K}\Big[|||f|||\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |G'_{1s}(s,\tau)-G'_{2s}(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\quad + |||f|||\int_0^1 |G_1(t,s)-G_2(t,s)|\mathrm{d}s\Big]
\end{aligned}
$$

for $0<\beta<1$ and

$$
\begin{aligned}
&\|u_1-u_2\|\\
&\le \frac{1}{1-L/K}\Big(|||f|||\int_0^1 |G'_{1t}(t,s)-G'_{2t}(t,s)|\mathrm{d}s + |||f|||\int_0^1 |G_1(t,s)-G_2(t,s)|\mathrm{d}s\Big)
\end{aligned}
$$

for $\beta = 1$. Thus, in accordance with (3.2), we obtain $\|u_1 - u_2\| = O(\varepsilon)$, which completes the proof. $\square$

Theorem 3.2. *Assume the conditions of Theorem 2.7 are valid. Let $u_1(t), u_2(t)$ be the solutions, respectively, of the problems*

$$
\begin{gathered}
{}^C D_{0+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0+}^{\beta} u(t)), \quad 0 < t < 1, \\
a_1 u(0) - a_2 u'(0) = A, b_1 u(1) + b_2 u'(1) = B,
\end{gathered}
$$

and

$$
\begin{gathered}
{}^C D_{0+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0+}^{\beta} u(t)), \quad 0 < t < 1, \\
a_1 u(0) - a_2 u'(0) = A + \varepsilon_1, \quad b_1 u(1) + b_2 u'(1) = B + \varepsilon_2,
\end{gathered}
$$

Then $\|u_1 - u_2\| = O(\max\{\varepsilon_1, \varepsilon_2\})$.

Proof. Let

$$
\begin{gathered}
\varphi_1(t) = \frac{(b_1 + b_2)A + a_2 B}{l} + \frac{a_1 B - b_1 A}{l} t, \\
\varphi_2(t) = \frac{(b_1 + b_2)(A + \varepsilon_1) + a_2(B + \varepsilon_2)}{l} + \frac{a_1(B + \varepsilon_2) - b_1(A + \varepsilon_1)}{l} t.
\end{gathered}
$$

Then

$$
\begin{gathered}
u_1(t) = \int_0^1 G(t,s) f(s, u_1(s), {}^C D_{0+}^{\beta} u_1(s)) \mathrm{d}s + \varphi_1(t), \\
u_2(t) = \int_0^1 G(t,s) f(s, u_2(s), {}^C D_{0+}^{\beta} u_2(s)) \mathrm{d}s + \varphi_2(t).
\end{gathered}
$$

So we obtain

$$
\begin{aligned}
|u_1(t) - u_2(t)| &\le L\|u_1 - u_2\| \int_0^1 |G(t,s)| \mathrm{d}s + |\varphi_1(t) - \varphi_2(t)| \\
&\le \frac{L(2l + a_2 b_2)}{l\Gamma(\alpha)} \|u_1 - u_2\| + \frac{(b_1 + b_2)\varepsilon_1 + a_2 \varepsilon_2}{l} + \frac{|a_1 \varepsilon_2 - b_1 \varepsilon_1|}{l},
\end{aligned}
$$

$$
\begin{aligned}
&|{}^C D_{0+}^{\beta} u_1(t) - {}^C D_{0+}^{\beta} u_2(t)| \\
&\le L\|u_1 - u_2\| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \Big(\int_0^1 |G_s'(s,\tau)| \mathrm{d}\tau \Big) \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} |\varphi_1'(s) - \varphi_2'(s)| \mathrm{d}s \\
&\le \frac{2L}{\Gamma(2-\beta)\Gamma(\alpha)} \|u_1 - u_2\| + \frac{1}{\Gamma(2-\beta)} \frac{|a_1 \varepsilon_2 - b_1 \varepsilon_1|}{l}
\end{aligned}
$$

for $0 < \beta < 1$, and

$$
\begin{aligned}
|u_1'(t) - u_2'(t)| &\le L\|u_1 - u_2\| \int_0^1 |G_t'(t,s)| \mathrm{d}s + |\varphi_1'(t) - \varphi_2'(t)| \\
&\le \frac{2L}{\Gamma(\alpha)} \|u_1 - u_2\| + \frac{|a_1 \varepsilon_2 - b_1 \varepsilon_1|}{l}
\end{aligned}
$$

for $\beta = 1$. Thus,

$$
\|u_1 - u_2\|
$$

$$
\begin{aligned}
&\le \frac{1}{1-L/K}\Big(\frac{(b_1+b_2)\varepsilon_1+a_2\varepsilon_2}{l}+\frac{|a_1\varepsilon_2-b_1\varepsilon_1|}{l}+\frac{1}{\Gamma(2-\beta)}\frac{|a_1\varepsilon_2-b_1\varepsilon_1|}{l}\Big)\\
&\le \frac{\max\{\varepsilon_1,\varepsilon_2\}}{1-L/K}\Big(\frac{b_1+b_2+a_2}{l}+\frac{a_1+b_1}{l}+\frac{1}{\Gamma(2-\beta)}\frac{a_1+b_1}{l}\Big).
\end{aligned}
$$

Therefore, the conclusion of the theorem follows. □

Theorem 3.3. *Suppose the conditions of Theorem 2.7 are satisfied. Let $u_1(t), u_2(t)$ be the solutions, respectively, of the problems*

$$
\begin{gathered}
{}^CD^{\alpha}_{0^+}u(t)=f(t,u(t),{}^CD^{\beta}_{0^+}u(t)),\quad 0<t<1,\\
a_1u(0)-a_2u'(0)=A, b_1u(1)+b_2u'(1)=B,
\end{gathered}
$$

and

$$
\begin{gathered}
{}^CD^{\alpha}_{0^+}u(t)=f(t,u(t),{}^CD^{\beta}_{0^+}u(t))+\varepsilon,\quad 0<t<1,\\
a_1u(0)-a_2u'(0)=A, b_1u(1)+b_2u'(1)=B.
\end{gathered}
$$

Then $\|u_1-u_2\|=O(\varepsilon)$.

Proof. Note that $u_1(t)=\int_0^1 G(t,s)f(s,u_1(s),{}^CD^{\beta}_{0^+}u_1(s))\mathrm{d}s+\varphi(t)$, $u_2(t)=\int_0^1 G(t,s)(f(s,u_2(s),{}^CD^{\beta}_{0^+}u_2(s))+\varepsilon)\mathrm{d}s+\varphi(t)$. Thus,

$$
\begin{aligned}
|u_1(t)-u_2(t)| &\le L\|u_1-u_2\|\int_0^1|G(t,s)|\mathrm{d}s+\varepsilon\int_0^1|G(t,s)|\mathrm{d}s\\
&\le \frac{L(2l+a_2b_2)}{l\Gamma(\alpha)}\|u_1-u_2\|+\frac{\varepsilon(2l+a_2b_2)}{l\Gamma(\alpha)},
\end{aligned}
$$

$$
\begin{aligned}
&|{}^CD^{\beta}_{0^+}u_1(t)-{}^CD^{\beta}_{0^+}u_2(t)|\\
&\le L\|u_1-u_2\|\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\Big(\int_0^1|G'_s(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\quad+\frac{\varepsilon}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\Big(\int_0^1|G'_s(s,\tau)|\mathrm{d}\tau\Big)\mathrm{d}s\\
&\le \frac{2L}{\Gamma(2-\beta)\Gamma(\alpha)}\|u_1-u_2\|+\frac{2\varepsilon}{\Gamma(2-\beta)\Gamma(\alpha)}
\end{aligned}
$$

for $0<\beta<1$, and

$$
\begin{aligned}
|u_1'(t)-u_2'(t)| &\le L\|u_1-u_2\|\int_0^1|G'_t(t,s)|\mathrm{d}s+\varepsilon\int_0^1|G'_t(t,s)|\mathrm{d}s\\
&\le \frac{2L}{\Gamma(\alpha)}\|u_1-u_2\|+\frac{2\varepsilon}{\Gamma(\alpha)}
\end{aligned}
$$

for $\beta=1$. Then,

$$
\|u_1-u_2\|\le\frac{\varepsilon}{1-L/K}\Big(\frac{2l+a_2b_2}{l\Gamma(\alpha)}+\frac{2}{\Gamma(2-\beta)\Gamma(\alpha)}\Big),
$$

and we get the desired result. □

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