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THE FIRST STEP NORMALIZATION FOR HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM OVER ORBIT CYLINDERS

GUILLERMO DÁVILA-RASCÓN, YURI VOROBIEV

Abstract. The near integrability property is studied for a class of perturbed Hamiltonian systems with two degrees of freedom on a phase space whose symplectic form depends non-uniformly on a small parameter.

1. Introduction

The recognition of a Hamiltonian system as a nearly integrable system is the first step of the normalization procedures in the framework of KAM theory or averaging methods. The question on the near integrability arises in the study of Hamiltonian dynamics near an invariant non-zero dimensional submanifold, typically, a periodic trajectory or a quasi-periodic torus (see, for example [2, 3, 4, 5, 6, 12]). In the present paper, we discuss the first step normalization for Hamiltonian systems with 2-degrees of freedom in the following setting which generalizes the case of a 2-submanifold of periodic trajectories (an orbit cylinder) [2, 3, 6, 13].

Let $M = (\mathbb{R}^1 \times \mathbb{S}^1) \times \mathbb{R}^2$ be the product manifold equipped with symplectic form non-uniformly depending on a parameter $\varepsilon > 0$,

$$\Omega^{\varepsilon} = \mathrm{d}s \wedge \mathrm{d}\varphi + \varepsilon\, \mathrm{d}p \wedge \mathrm{d}q, \tag{1.1}$$

where $(s, \varphi \mod 2\pi) \in \mathbb{R}^1 \times \mathbb{S}^1$ and $(p, q) \in \mathbb{R}^2$. Consider a Hamiltonian system on $(M, \Omega^{\varepsilon})$ given by a smooth Hamiltonian of the form

$$H_{\varepsilon} = f(s) + \varepsilon\, F(s, \varphi, p, q). \tag{1.2}$$

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The corresponding equations of motion read

$$\dot{s} = -\varepsilon \frac{\partial F}{\partial \varphi}, \tag{1.3}$$

$$\dot{\varphi} = \omega_1(s) + \varepsilon \frac{\partial F}{\partial s}, \tag{1.4}$$

$$\dot{p} = -\frac{\partial F}{\partial q}, \tag{1.5}$$

$$\dot{q} = \frac{\partial F}{\partial p}, \tag{1.6}$$

where $\omega_1(s) = \partial f(s)/\partial s > 0$. In general, this system is not completely integrable and our point is to study (1.3)-(1.6) for small ε, in the context of the Hamiltonian perturbation theory.

We remark that an alternative setting can be given on the standard phase space $(\mathbb{R}^4, \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2)$ by considering the following class of Hamiltonian systems on $\mathbb{R}^4$:

$$\mathcal{H} = \mathcal{H}_0(p_1, q_1) + \varepsilon\, \mathcal{H}_1(p_1, q_1, p_2/\varepsilon^{\mu}, q_2/\varepsilon^{1-\mu}), \tag{1.7}$$

where $\mu \in [0,1]$. If an open domain $\mathcal{C} \subset \mathbb{R}^2$ is foliated by periodic trajectories of $\mathcal{H}_0$, then as $\varepsilon \to 0$, the behavior of $\mathcal{H}$ in a region $\{(p_1, q_1) \in \mathcal{C},\, p_2 \sim \varepsilon^{\mu},\, q_2 \sim \varepsilon^{1-\mu}\}$ is described by a system like (1.3)-(1.6). In particular, when $\mu = 1/2$ and $\mathcal{H}_1$ is quadratic in p_2 and q_2, the Hamiltonian system $\mathcal{H}$ is independent of ε and has an orbit cylinder $\mathcal{C} \times \{0\}$.

Let $X_\varepsilon = X_{H_\varepsilon}$ be the Hamiltonian vector field of (1.3)-(1.6) viewed as a perturbed dynamical system on M. Then, $X_\varepsilon = X_0 + \varepsilon W$, where

$$X_0 = \omega_1(s)\frac{\partial}{\partial \varphi} + \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}, \tag{1.8}$$

corresponds to the unperturbed dynamics and

$$W = -\frac{\partial F}{\partial \varphi}\frac{\partial}{\partial s} + \frac{\partial F}{\partial s}\frac{\partial}{\partial \varphi}, \tag{1.9}$$

is a perturbation vector field. We assume that unperturbed system X_0 admits an additional integral of motion besides the trivial one, s. This means that the family of time-dependent Hamiltonian systems on $\mathbb{R}^2$ associated to X_0 is completely integrable. Nevertheless, in the context of near integrability of the perturbed system X_ε, the main difficulty is that the vector field X_0 (as an autonomous system) is not Hamiltonian relative to the original symplectic structure Ω^ε. This effect comes from the singular dependence of Ω^ε on the parameter ε (Ω^ε becomes degenerate at $\varepsilon = 0$). The idea is to search for a symplectic mapping Υ_ε (smoothly depending on ε) from M to a canonical model phase space N such that the transformed perturbed system $(\Upsilon_\varepsilon)_* H_\varepsilon$ is ε^2-close to a completely integrable Hamiltonian system on N. The existence of such a *twisting map* can be explained by the following observation [7, 8]: the unperturbed system X_0 is, in fact, Hamiltonian in a "deformed" non-canonical symplectic structure (see also [6, 13]). Here, we show that the normalizing transformation is represented as the composition $\Upsilon_\varepsilon = \Psi_\varepsilon \circ \mathcal{T}$ of an ε-independent mapping $\mathcal{T}$ and a near identity transformation Ψ_ε. The mapping $\mathcal{T}$ transforms X_0 to a system with parallel dynamics but it is is not symplectic. The mapping Ψ_ε is defined as the time-1 flow of a non-autonomous system and gives a near identity isomorphism between the transformed symplectic form $\mathcal{T}_*\Omega^\varepsilon$ and the canonical symplectic structure on N. Here we apply a parameter-dependent version of the Moser homotopy method [9]. In general, the transformation Ψ_ε is not infinitesimal, except for some particular cases, for example, when F in (1.2) is independent of s (see [2, 3, 5]). In the linear case, when F is quadratic in p, q, the unperturbed vector field X_0 describes the linearized dynamics at the orbit cylinder and the existence of an additional integral of motion is provided by the stability property of X_0. The mapping $\mathcal{T}$ corresponds to the reducibility transformation in the sense of the Floquet theory for linear periodic Hamiltonian systems [14].

2. Main Results

On the phase space $M = (\mathbb{R}^1 \times \mathbb{S}^1) \times \mathbb{R}^2$, consider the dynamical system of the unperturbed vector field X_0:

$$\dot{s} = 0, \tag{2.1}$$

$$\dot{\varphi} = \omega_1(s), \tag{2.2}$$

$$\dot{p} = -\frac{\partial F}{\partial q}(s, \varphi, p, q), \tag{2.3}$$

$$\dot{q} = \frac{\partial F}{\partial p}(s, \varphi, p, q). \tag{2.4}$$

Denote by $\pi : M \to \mathbb{R}^1 \times \mathbb{S}^1$ the canonical projection onto the first factor and consider M as the total space of the trivial symplectic vector bundle π over $\mathbb{R}^1 \times \mathbb{S}^1$ with fiberwise symplectic structure $\mathrm{d}p \wedge \mathrm{d}q$. The base is trivially foliated by the periodic orbits of subsystem (2.1),(2.2) which is viewed as a Hamiltonian system with one degree of freedom. Geometrically, system (2.1)-(2.4) belongs to the class of projectable systems, that is, the trajectories of X_0 project under π to the periodic orbits of (2.1),(2.2). To each function G on M, one can associate the *vertical Hamiltonian vector field*

$$V_G = \frac{\partial G}{\partial p}\frac{\partial}{\partial q} - \frac{\partial G}{\partial q}\frac{\partial}{\partial p}. \tag{2.5}$$

It is clear that the trajectory of V_G passing through a point $m \in M$ belongs to the fiber over $\pi(m)$. In other words, we can think of V_G as a family of autonomous Hamiltonian system on $\mathbb{R}^2$ whose Hamiltonian $G_{s,\varphi}$ depends parametrically on $(s, \varphi) \in \mathbb{R}^1 \times \mathbb{S}^1$.

We assume that the following integrability hypothesis holds:

(IH) There exists an open domain $\mathcal{M} \subset M$ and a smooth integral of motion $G : \mathcal{M} \to \mathbb{R}$ of X_0,

$$L_{X_0} G \equiv \omega_1 \frac{\partial G}{\partial \varphi} + \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} = 0, \tag{2.6}$$

such that $\mathcal{M}$ is foliated by the periodic trajectories of V_G, $\mathcal{M} \cap \pi^{-1}(b)$ is connected for every $b = (s, \varphi)$ and

$$\pi(\mathcal{M}) = \Delta \times \mathbb{S}^1, \tag{2.7}$$

where $\Delta \subset \mathbb{R}$ is an open interval.

Consider the second product manifold

$$N = (\mathbb{R}^1 \times \mathbb{S}^1) \times (\mathbb{R}^1 \times \mathbb{S}^1) = \big\{(s_1, \varphi_1 \mod 2\pi, s_2, \varphi_2 \mod 2\pi)\big\}, \tag{2.8}$$

with natural projection $\nu : N \to \mathbb{R}^1 \times \mathbb{S}^1$ onto the first factor. Then, N is the total space of the trivial symplectic bundle with fiberwise symplectic structure $\mathrm{d}s_2 \wedge \mathrm{d}\varphi_2$. Therefore, we have two trivial symplectic bundles π and ν over one and the same base $\mathbb{R}^1 \times \mathbb{S}^1$. We say that a subset $\mathcal{N} \subset N$ is a *simple toroidal domain* if $\nu(\mathcal{N}) = \Delta \times \mathbb{S}^1$ and $\iota(\mathcal{N}) = D_{\mathcal{N}} \times \mathbb{T}^2$, where $D = D_{\mathcal{N}} \subset \mathbb{R}^2$ is an open, connected and simply connected subset. Here $\iota : N \to \mathbb{R}^2 \times \mathbb{T}^2$ denotes the canonical identification, $\iota(s_1, \varphi_1, s_2, \varphi_2) = (s_1, s_2, \varphi_1, \varphi_2)$.

Proposition 2.1 (Reducibility)**.** *There exist a simple toroidal domain $\mathcal{N} \subset N$ and a fibered diffeomorphism $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ over $\Delta \times \mathbb{S}^1$,*

$$\mathcal{T}(s, \varphi, p, q) = \big(s, \varphi, \mathcal{T}_{s,\varphi}(p, q)\big), \tag{2.9}$$

which preserves the fiberwise symplectic structures,

$$(\mathcal{T}_{s,\varphi})_* \mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}s_2 \wedge \mathrm{d}\varphi_2, \tag{2.10}$$

and such that the dynamical system of the push-forward $\mathcal{T}_* X_0$ takes the form

$$\dot{s_1} = 0, \tag{2.11}$$

$$\dot{\varphi}_1 = \omega_1(s_1), \tag{2.12}$$

$$\dot{s_2} = 0, \tag{2.13}$$

$$\dot{\varphi}_2 = \omega_2(s_1, s_2), \tag{2.14}$$

where $\omega_2 = \omega_2(s_1, s_2)$ is a smooth function on $D_{\mathcal{N}}$.

As a consequence, we get that the domain $\mathcal{M}$ is trivially foliated by 2-tori

$$\Lambda_{c_1,c_2} = \mathcal{T}^{-1}(\mathbb{T}^2_{c_1,c_2}), \tag{2.15}$$

where (c_1, c_2) runs over D and $\mathbb{T}^2_{c_1,c_2} = \{s_1 = c_1,\ s_2 = c_2\}$. Each torus Λ_{c_1,c_2} is the level set of the integrals of motion s and G, carrying a quasi-periodic motion along the trajectories of X_0 with frequencies $\omega_1(c_1)$ and $\omega_2(c_1, c_2)$. However, if $\partial G/\partial\varphi \neq 0$, then Λ_{c_1,c_2} are not Lagrangian tori with respect to the symplectic structure (1.1). An interpretation of Λ_{c_1,c_2} as Liouville tori is related with a Hamiltonian formulation for (2.1)-(2.4) in a non-canonical symplectic structure on M [7, 8]. In Section 3, we give a construction of $\mathcal{T}$ which is based on the Poincaré–Cartan invariant. In Section 5, we show that in the particular case when X_0 corresponds to the linearized dynamics around the orbit cylinder, the mapping $\mathcal{T}$ is just the Floquet–Lyapunov transformation [11, 14].

Fix a diffeomorphism $\mathcal{T}$ in Proposition 2.1 and consider N as a model phase space equipped with non-uniform canonical symplectic form

$$\widetilde{\Omega}^{\varepsilon} = \mathrm{d}s_1 \wedge \mathrm{d}\varphi_1 + \varepsilon\, \mathrm{d}s_2 \wedge \mathrm{d}\varphi_2. \tag{2.16}$$

The following observation says that the reducibility map is not symplectic.

Proposition 2.2. *The original symplectic structure Ω^{ε} (1.1) is transformed under $\mathcal{T}$ to the following non-canonical symplectic form on $\mathcal{N}$,*

$$\mathcal{T}_* \Omega^{\varepsilon} = \widetilde{\Omega}^{\varepsilon} - \varepsilon\, \mathrm{d}Q, \tag{2.17}$$

where

$$Q = Q_1(s_1, \varphi_1, s_2, \varphi_2)\, \mathrm{d}s_1 + Q_2(s_1, \varphi_1, s_2, \varphi_2)\, \mathrm{d}\varphi_1, \tag{2.18}$$

is a horizontal 1-form.

The 2-form (2.17) gives a special deformation of $\widetilde{\Omega}^\varepsilon$ and belongs to the class of the so-called *weak coupling of symplectic structures* [9]. To complete the normalization procedure, we search for an isomorphism between $\mathcal{T}_*\Omega^\varepsilon$ and $\widetilde{\Omega}^\varepsilon$ in the class of near identity transformations.

Given Δ and $\mathcal{M}$ in (IH), we say that another open domain $\mathcal{M}_0 \subset M$, also satisfying the hypothesis (IH), is *admissible* if (2.7) holds for a certain open interval Δ_0 such that $\Delta_0 \subset \Delta$, the closure $\overline{\mathcal{M}}_0$ is compact and $\overline{\mathcal{M}}_0 \subset \mathcal{M}$. It is clear that an admissible domain always exists.

Now, we formulate our main result.

Theorem 2.3. *Let $\mathcal{M}_0 \subset \mathcal{M}$ be an admissible domain and $\mathcal{N}_0 = \mathcal{T}(\mathcal{M}_0)$. For sufficiently small $\varepsilon \geq 0$, there exists a diffeomorphism $\Psi_\varepsilon : \mathcal{N}_0 \to \mathcal{N}$ onto its image, smoothly depending on ε, with $\Psi_0 = \mathrm{id}$ and such that*

$$\Upsilon_\varepsilon = \Psi_\varepsilon \circ \mathcal{T}, \tag{2.19}$$

is a symplectic map,

$$(\Upsilon_\varepsilon)_*\Omega^\varepsilon = \widetilde{\Omega}^\varepsilon, \quad (\varepsilon > 0), \tag{2.20}$$

transforming the original Hamiltonian system (1.3)-(1.6) *into the normal form,*

$$\widetilde{H}_\varepsilon := (\Upsilon_\varepsilon)_* H_\varepsilon = f(s_1) + \varepsilon\, h(s_1, s_2) + O(\varepsilon^2), \tag{2.21}$$

where

$$\frac{\partial h}{\partial s_2}(s_1, s_2) = \omega_2(s_1, s_2). \tag{2.22}$$

In Section 4, the near identity transformation Ψ_ε is constructed by means of a parameter dependent version of the Moser homotopy method [9]. The transformed perturbed Hamiltonian $\widetilde{H}_\varepsilon$ is ε^2-close to the Hamiltonian $\widetilde{H}^{(0)}_\varepsilon = f(s_1) + \varepsilon\, h(s_1, s_2)$ which defines a completely integrable Hamiltonian system on $(N, \widetilde{\Omega}^\varepsilon)$. The invariant tori $\mathbb{T}^2_{c_1,c_2}$ of system (2.11)-(2.14) are now the Liouville tori of $\widetilde{H}^{(0)}_\varepsilon$ which carry the quasi-periodic motion with deformed frequencies $\omega_1(c_1) + \varepsilon\, \partial h(c_1, c_2)/\partial s_1$ and $\omega(c_1, c_2)$. If the frequencies satisfy some appropriate nondegeneracy condition (for example, in the sense of Kolmogorov or Rüssmann), then one can apply the KAM type results [4] to state the persistence of quasi-periodic tori $\Upsilon_\varepsilon^{-1}(\mathbb{T}^2_{c_1,c_2})$ for perturbed system (1.3)-(1.6) as $\varepsilon \to 0$.

Example 2.4. On the phase space $(\mathbb{R}^4 = \{(y, x, p, q)\}, \Omega^\varepsilon = \mathrm{d}y \wedge \mathrm{d}x + \varepsilon\, \mathrm{d}p \wedge \mathrm{d}q)$, consider the perturbed Hamiltonian system

$$H_\varepsilon = \frac{y^2}{2} + U_0(x) + \varepsilon\Big(\frac{p^2}{2} + \frac{U_0'(x)}{2x} q^2 + \frac{1}{x^2} U_1\left(\frac{q}{x}\right)\Big), \tag{2.23}$$

where U_0, U_1 are arbitrary smooth functions. Then, the corresponding unperturbed system

$$\dot{y} = -U_0'(x), \tag{2.24}$$

$$\dot{x} = y, \tag{2.25}$$

$$\dot{p} = -\frac{U_0'(x)}{x} q - \frac{1}{x^3} U_1'\left(\frac{q}{x}\right), \tag{2.26}$$

$$\dot{q} = p, \tag{2.27}$$

has the following integral of motion, [16, 17],

$$G = \frac{1}{2}\left(xp - yq\right)^2 + U_1'\left(\frac{q}{x}\right). \tag{2.28}$$

Suppose that the potentials U_0 and U_1 have local nondegenerate minima at some points x_0 and z_0, respectively. In this case, under passing to the standard action-angle variables $(y,x) \mapsto (s,\varphi)$ around $(0,x_0)$ associated with (2.24), (2.25), system (2.24)-(2.27) takes the form (2.1)-(2.4). Moreover, for a fixed (y,x), the equilibrium (yz_0, xz_0) of V_G is surrounded by periodic trajectories. Therefore, the integral of motion G in (2.28) satisfies the condition (IH) and we can apply Theorem 2.3.

In the linear case, when F in (1.2) is quadratic in p, q, the integrability hypothesis holds if the unperturbed system X_0 is *strongly stable*. The linear version of Theorem 2.3 is discussed in Section 5.

3. Reducibility

Here we describe an algorithm for the construction of the reducibility transformation $\mathcal{T}$ in Proposition 2.1, assuming that hypothesis (IH) holds and the corresponding data $(\Delta, \mathcal{M}, G)$ are given.

Consider the vector field X_0 of system (2.1)-(2.4) which can be rewritten in the form

$$X_0 = \omega_1 \frac{\partial}{\partial\varphi} + V_F, \tag{3.1}$$

where V_F is the vertical Hamiltonian vector field given by (2.5). Then, X_0 has two integrals of motion, namely, s and G. Let $J : \mathcal{M} \to \mathbb{R}^2$ be the corresponding "momentum" map, $J(s,\varphi,p,q) = \big(s, G(s,\varphi,p,q)\big)$. It follows from (IH) that $\left(\frac{\partial G}{\partial p}, \frac{\partial G}{\partial q}\right) \neq 0$ on $\mathcal{M}$ and hence, J is a surjective submersion onto its image. Moreover, $\mathcal{M}$ is foliated by the compact connected 2-manifolds

$$\Lambda_\xi = J^{-1}(\xi), \quad \big(\xi \in J(\mathcal{M})\big). \tag{3.2}$$

We have also the following properties of vector fields X_0 and V_G:

(a) X_0 and V_G are linear independent on $\mathcal{M}$;
(b) V_G is tangent to each fiber Λ_ξ;
(c) X_0 and V_G commute,

$$[X_0, V_G] = V_{L_{X_0}G} = 0. \tag{3.3}$$

It follows from here [1, 15] that every level set Λ_ξ is diffeomorphic to the 2-torus and carries a quasi-periodic motion along the trajectories of X_0. In particular, X_0 is a complete vector field on $\mathcal{M}$. However, as we mentioned above, X_0 is not Hamiltonian relative to the original symplectic structure and we can not directly apply the Arnold-Liouville theorem on the action-angle variables to construct $\mathcal{T}$. Our argument is based on the Poincaré–Cartan invariant for time-dependent Hamiltonian systems.

Denote by $\mathsf{Fl}^t_{X_0} : \mathcal{M} \to \mathcal{M}$ the flow of X_0. For a fixed $s \in \Delta$, one can associate to X_0 the time-dependent Hamiltonian system on $\mathbb{R}^2$ with Hamiltonian $F(s, \omega_1(s)\,t, p, q)$. Then, we have the following fact [1, 15]: for any closed curve $\Gamma \subset \{s\} \times \mathbb{S}^1 \times \mathbb{R}^2$ transversal to X_0, the integral

$$\oint_{\Gamma_t} \Big[p\,\mathrm{d}q - F(s,\varphi,p,q)\frac{\mathrm{d}\varphi}{\omega_1(s)}\Big] = \text{const}, \tag{3.4}$$

that is, does not depend on t. Here $\Gamma_t = \mathsf{Fl}^t_{X_0}(\Gamma)$.

For every $(s,\varphi) \in \Delta \times \mathbb{S}^1$, denote by $G_{s,\varphi} : \mathbb{R}^2 \to \mathbb{R}$ the function given by $G_{s,\varphi}(p,q) = G(s,\varphi,p,q)$. Let $\pi^{-1}(s,\varphi) = \{(s,\varphi)\} \times \mathbb{R}^2$ be the fiber of π over (s,φ). We have

$$\pi^{-1}(s,\varphi) \cap \mathcal{M} = \{(s,\varphi)\} \times \mathcal{U}_{s,\varphi}, \tag{3.5}$$

where $\mathcal{U}_{s,\varphi}$ is an open connected domain in $\mathbb{R}^2$. Observe that under varying (s,φ), the set of values of $G_{s,\varphi}$ on $\mathcal{U}_{s,\varphi}$ is independent of φ. This follows from (3.3) and the property that the flow $\mathsf{Fl}^t_{X_0}$ preserves the fibers of π. Taking into account that G can be renormalized by multiplication for any nonzero function of s, without lost of generality, we may assume that

$$G_{s,\varphi}(\mathcal{U}_{s,\varphi}) = (E_1, E_2), \tag{3.6}$$

for some constants $E_1 < E_2$.

Fix (s,φ) and consider the Hamiltonian system on $\mathbb{R}^2$ associated with the restriction of the vertical field V_G to the fiber over (s,φ):

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial G_{s,\varphi}}{\partial q}, \tag{3.7}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial G_{s,\varphi}}{\partial p}. \tag{3.8}$$

By the hypothesis, the level set

$$\gamma_{s,\varphi}(E) = \{(p,q) \in \mathcal{U}_{s,\varphi} : G_{s,\varphi}(p,q) = E\}, \tag{3.9}$$

is a periodic trajectory of system (3.7), (3.8) for every $E \in (E_1,E_2)$. It is clear that $\mathcal{U}_{s,\varphi}$ is trivially foliated by $\gamma_{s,\varphi}(E)$ over (E_1,E_2). Thus, one can introduce the standard action-angle variables on $\mathcal{U}_{s,\varphi}$ associated to this foliation. The point is to choose these coordinates to be smooth functions of the parameters (s,φ).

Let $g : \mathcal{M} \to (\Delta \times \mathbb{S}^1) \times (E_1,E_2)$ be the surjective submersion defined as $g = \pi \times G$. Then, the fibers

$$\Gamma_{s,\varphi}(E) := g^{-1}(s,\varphi,E) \equiv \{(s,\varphi)\} \times \gamma_{s,\varphi}(E), \tag{3.10}$$

are just the periodic trajectories of V_G in $\mathcal{M}$. It follows from (3.3) that the flow $\mathsf{Fl}^t_{X_0}$ is also a fiber preserving map with respect to g,

$$\mathsf{Fl}^t_{X_0}(\Gamma_{s,\varphi}(E)) = \Gamma_{s,\varphi+\omega_1(s)t}(E). \tag{3.11}$$

Putting $\Gamma = \Gamma_{s,\varphi}(E)$ into (3.4) and using (3.11), we get that the action along $\gamma_{s,\varphi}(E)$ is independent of φ,

$$a(s,E) = \frac{1}{2\pi} \oint_{\gamma_{s,\varphi}(E)} p \, \mathrm{d}q, \tag{3.12}$$

and defines a smooth function $a : \Delta \times (E_1,E_2) \to \mathbb{R}$. The period of $\gamma_{s,\varphi}(E)$ is given by

$$T(s,E) = \frac{\partial a}{\partial E}(s,E). \tag{3.13}$$

Let us define

$$A := a \circ J, \tag{3.14}$$

or, equivalently, $A(s,\varphi,p,q) = a(s, G(s,\varphi,p,q))$. It is clear that A is an integral of motion of X_0 which represents a parameter-dependent action variable of (3.7), (3.8). The construction of the corresponding angle variable (smoothly depending on s and φ) is related with the existence of a global section of the fibration g.

Lemma 3.1. *The domain $\mathcal{M}$ is trivially fibered by periodic trajectories $\Gamma_{s,\varphi}(E)$ of V_G, that is, there exists a smooth section $L : (\Delta \times \mathbb{S}^1) \times (E_1, E_2) \to \mathcal{M}$ of g, $g \circ L = \mathrm{id}$.*

Proof. Let $g_0 : \mathcal{M}_0 \to \Delta \times (E_1, E_2)$ be the restriction of g to the slice $\mathcal{M}_0 = \mathcal{M} \cap \{\varphi = 0\}$ which is a contractible open subset. Then, there exists a smooth section L_0 of g_0, $L_0(s, E) \in \Gamma_{s,0}(E)$. To extend L_0 to the whole $\mathcal{M}$, let us consider a vector field on $\mathcal{M}$ of the form

$$Y = \frac{1}{\omega_1} X_0 - \frac{\tau \circ J}{2\pi} V_G,$$

Here τ is a smooth function on $\Delta \times (E_1, E_2)$ which is defined in the following way. Fix (s, E) and consider the 2-torus $\Lambda_{s,E} = J^{-1}(s, E)$. Then, $\Lambda_{s,E}$ is the disjoint union of the periodic trajectories of V_G,

$$\Lambda_{s,E} = \cup_{\varphi \in (0,2\pi]} \Gamma_{s,\varphi}(E).$$

Pick a point $m = (s, \varphi, p, q) \in \Gamma_{s,\varphi}(E)$. It follows from (3.11) that the trajectory of X_0 starting at m stays on $\Lambda_{s,E}$ and meets again the trajectory $\Gamma_{s,\varphi}(E)$ at the point $\widetilde{m}$ after the time $t_0 = 2\pi/\omega_1(s)$,

$$\mathsf{F}l^{t_0}_{X_0}(m) = \widetilde{m} \in \Gamma_{s,\varphi}(E).$$

Let τ be the time along the trajectory of V_G from m to $\widetilde{m}$, $\mathsf{Fl}^{\tau}_{V_G}(m) = \widetilde{m}$. One can show that τ does not depend on the choice of m and $\tau = \tau(s, E)$ smoothly varies with (s, E). Taking into account that $[Y, V_G] = 0$, we derive the following properties: (i) the flow of Y is 2π-periodic, and (ii) $\mathsf{Fl}^t_Y(\Gamma_{s,\varphi}(E)) = \Gamma_{s,\varphi+t}(E)$. Finally, we put

$$L(s, \varphi, E) = \mathsf{Fl}^{\varphi}_Y(L^0(s, E)). \tag{3.15}$$

□

We will suppose that a section L in (3.15) is given. Consider now the product manifold $N = (\mathbb{R}^1 \times \mathbb{S}^1) \times (\mathbb{R}^1 \times \mathbb{S}^1)$ with coordinates $(s_1, \varphi_1 \mod 2\pi, s_2, \varphi_2 \mod 2\pi)$. Denote by $E(s_1, s_2)$ the solution of the equation $s_2 = a(s_1, E)$. Here (s_1, s_2) runs over the open domain

$$D = \cup_{s_1 \in \Delta} \{s_1\} \times D_{s_1} \subset \mathbb{R}^2, \tag{3.16}$$

where $D_{s_1} = \{s_2 = a(s_1, E),\ E \in (E_1, E_2)\}$. Define also

$$\mathcal{N} := \cup_{s_1 \in \Delta} \{s_1\} \times \mathbb{S}^1 \times D_{s_1} \times \mathbb{S}^1. \tag{3.17}$$

It is clear that $\mathcal{N}$ is a simple toroidal domain in N. Let $Z = (\mathcal{T} \circ J/2\pi) V_G$ be the infinitesimal generator of the trivial $\mathbb{S}^1$-action on $\mathcal{M}$ whose orbits are just the periodic trajectories $\Gamma_{s,\varphi}(E)$. Using the section L and the flow of Z, we define a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{M}$ as follows:

$$\mathcal{R}(s_1, \varphi_1, s_2, \varphi_2) := \mathsf{Fl}^{\varphi_2}_Z\big(L(s_1, \varphi_1, E(s_1, s_2))\big). \tag{3.18}$$

It follows that $\mathcal{R}$ is a fiber preserving diffeomorphism covering the identity, with the inverse $\mathcal{R}^{-1} : \mathcal{M} \to \mathcal{N}$ given by

$$s_1 \circ \mathcal{R}^{-1} = s, \quad \varphi_1 \circ \mathcal{R}^{-1} = \varphi, \tag{3.19}$$

$$s_2 \circ \mathcal{R}^{-1} = A, \quad \varphi_2 \circ \mathcal{R}^{-1} = \phi_0. \tag{3.20}$$

Therefore, $A = A(s,\varphi,p,q)$ and $\phi_0 = \phi_0(s,\varphi,p,q)$ in (3.20) are the standard action-angle coordinates of system (3.7), (3.8), parametrically depending on s and φ. It follows that

$$\{A,\phi_0\} := \frac{\partial A}{\partial p}\frac{\partial \phi_0}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial \phi_0}{\partial p} = 1. \tag{3.21}$$

Lemma 3.2. *Under the coordinate change $\mathcal{R}^{-1} : (s,\varphi,p,q) \mapsto (s_1,\varphi_1,s_2,\varphi_2)$, the equations of motion of $\mathcal{R}^* X_0$ take the form*

$$\dot{s_1} = 0, \tag{3.22}$$
$$\dot{\varphi}_1 = \omega_1(s_1), \tag{3.23}$$
$$\dot{s_2} = 0, \tag{3.24}$$
$$\dot{\varphi}_2 = \Theta(s_1,\varphi_1,s_2), \tag{3.25}$$

where Θ is 2π-periodic function in φ_1.

Proof. Equations (3.22)-(3.24) follow directly from the definition of $\mathcal{R}$. The time evolution of $\varphi_2 = \phi_0 \circ \mathcal{R}$, according to (2.1)-(2.4) is given by $\dot{\varphi}_2 = \theta \circ \mathcal{R}$, where

$$\theta = \omega_1(s)\frac{\partial \phi_0}{\partial \varphi} + \{F,\phi_0\}. \tag{3.26}$$

Observe that

$$\{\theta, A\} = 0. \tag{3.27}$$

Indeed, combining relations (3.21) and (3.26) with the Jacobi identity for the bracket $\{,\}$, we derive

$$\begin{aligned}
\{\theta,A\} &= \omega_1\big\{\frac{\partial \phi_0}{\partial \varphi}, A\big\} + \{\{F,\phi_0\},A\} \\
&= \omega_1\big\{\frac{\partial \phi_0}{\partial \varphi}, A\big\} - \{\{\phi_0,A\},F\} - \{\{A,F\},\phi_0\} \\
&= \omega_1\big\{\frac{\partial \phi_0}{\partial \varphi}, A\big\} - \{\phi_0,\{F,A\}\} \\
&= \omega_1\big\{\frac{\partial \phi_0}{\partial \varphi}, A\big\} + \omega_1\big\{\phi_0, \frac{\partial A}{\partial \varphi}\big\} \\
&= \omega_1 \frac{\partial}{\partial \varphi}\{\phi_0, A\} = 0.
\end{aligned}$$

Now, from (3.27) we have

$$\{\theta,A\}\circ\mathcal{R} = \{\theta\circ\mathcal{R}, A\circ\mathcal{R}\} = \{\theta\circ\mathcal{R}, s_2\} = -\frac{\partial}{\partial \varphi_2}(\theta\circ\mathcal{R}) = 0,$$

and hence $\theta = \Theta(s,\varphi,A(s,\varphi,p,q))$, where $\Theta = \Theta(s_1,\varphi_1,s_2)$ is a smooth function 2π-periodic in φ_1. ∎

Next, given an arbitrary smooth function $\chi = \chi(s_1,\varphi_1,s_2)$ which is 2π-periodic in φ_1, one can correct the angle variable as follows: $\phi_0 \mapsto \phi \equiv \phi_0 + \chi \circ \mathcal{A}$. Here, $\mathcal{A}(s,\varphi,p,q) = (s,\varphi,A(s,\varphi,p,q))$. It is clear that such a transformation preserves bracket relation (3.21). In order to eliminate the dependence on φ_1 in the right hand side of (3.25), we have to put

$$\chi = \omega_2(s_1,s_2)\varphi_1 - \int_0^{\varphi_1} \Theta(s_1,\varphi_1',s_2)\,\mathrm{d}\varphi_1', \tag{3.28}$$

where
$$\omega_2(s_1, s_2) = \frac{1}{2\pi} \int_0^{2\pi} \Theta(s_1, \varphi_1, s_2)\, \mathrm{d}\varphi_1, \tag{3.29}$$
for $(s_1, s_2) \in D$. Summarizing, we get that the transformation $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ satisfying the assertions of Proposition 2.1 is given by the formula
$$\mathcal{T}(s, \varphi, p, q) = \big(s, \varphi, A(s, \varphi, p, q), \phi(s, \varphi, p, q)\big), \tag{3.30}$$
with
$$A = a \circ J, \quad \phi = \varphi_2 \circ \mathcal{R}^{-1} + \chi \circ \mathcal{A}. \tag{3.31}$$

Remark 3.3. According to the standard time-dependent Hamiltonian approach [1, 15], one can associate to X_0 the s-parameter family of completely integrable Hamiltonian systems $\widetilde{F}_s = \omega_1(s)\,\eta + F(s, \varphi, p, q)$ on the phase space $(\mathbb{R}^1 \times \mathbb{S}^1) \times \mathbb{R}^2$ with canonical symplectic structure $\mathrm{d}\eta \wedge \mathrm{d}\varphi + \mathrm{d}p \wedge \mathrm{d}q$. Then, functions A and ϕ in (3.31) can be also derived form the action-angle variables associated to the trivial foliation by the Liouville 2-tori $\{\widetilde{F}_s = \mathrm{const}, G_s = \mathrm{const}\}$ (see, for example, [10]).

4. Constructing a Near Identity Transformation

Fix a section L in (3.15) and consider the corresponding reducibility map $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ given by (3.30). First we show that the symplectic structure Ω^{ε} is transformed under $\mathcal{T}$ by the rule (2.17). Pick a $(s, \varphi) \in \mathbb{R}^1 \times \mathbb{S}^1$ and consider the connected open domain $\mathcal{U}_{s,\varphi} \subset \mathbb{R}^2$ in (3.5). Let $\mathcal{T}_{s,\varphi} : \mathcal{U}_{s,\varphi} \to \mathbb{R}^1 \times \mathbb{S}^1$ be a mapping defined by $\mathcal{T}_{s,\varphi}(p, q) = \mathcal{T}(s, \varphi, p, q)$. Introduce the following 1-form on $\mathcal{U}_{s,\varphi}$,
$$\alpha_{s,\varphi} = \mathcal{T}^*_{s,\varphi}(s_2\, \mathrm{d}\varphi_2) - p\, \mathrm{d}q.$$
Denote by d_1 and d_2 the partial exterior derivatives on M along the factors $\mathbb{R}^1 \times \mathbb{S}^1$ and $\mathbb{R}^2$, respectively. Then, $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$ and $\mathrm{d}_1 \circ \mathrm{d}_2 + \mathrm{d}_2 \circ \mathrm{d}_1 = 0$. The following observation says that $\alpha_{s,\varphi}$ is exact on $\mathcal{U}_{s,\varphi} \subset \mathbb{R}^2$ and there exists a primitive which smoothly varies with (s, φ).

Lemma 4.1. *There exists a smooth function $K = K(s, \varphi, p, q)$ on $\mathcal{M} \subset (\mathbb{R}^1 \times \mathbb{S}^1) \times \mathbb{R}^2$ such that*
$$\alpha_{s,\varphi} = -\mathrm{d}_2 K, \tag{4.1}$$
on $\mathcal{U}_{s,\varphi}$.

Proof. For every (s, φ), the open domain $\mathcal{U}_{s,\varphi}$ is trivially foliated by periodic orbits $\gamma_{s,\varphi}(E)$ over (E_1, E_2) and it is isomorphic to the 1-cylinder. Taking into account that
$$\mathcal{T}^*_{s,\varphi}(s_2\, \mathrm{d}\varphi_2) = A(s, \varphi, p, q)\big[\frac{\partial \phi}{\partial p}\, \mathrm{d}p + \frac{\partial \phi}{\partial q}\, \mathrm{d}q\big],$$
we get
$$\begin{aligned} \oint_{\Gamma_{s,\varphi}(E)} \alpha_{s,\varphi} &= \oint_{\Gamma_{s,\varphi}(E)} \mathcal{T}^*_{s,\varphi}(s_2\, \mathrm{d}\varphi_2) - \oint_{\Gamma_{s,\varphi}(E)} p\, \mathrm{d}q \\ &= \int_0^{2\pi} a(s, E)\, \mathrm{d}\phi - 2\pi a(s, E) = 0. \end{aligned}$$

This means that $\alpha_{s,\varphi}$ is exact on $\mathcal{U}_{s,\varphi}$. Using the section L in (3.15) and fixing $E_0 \in (E_1, E_2)$, we define the primitive $-K_{s,\varphi}$ of $\alpha_{s,\varphi}$ smoothly depending on s and φ (as parameters) by

$$K_{s,\varphi}(p,q) = -\int_{(p^0,q^0)}^{(p,q)} \alpha_{s,\varphi}.$$

Here, the integral is taken over a curve joining any point $(p,q) \in \mathcal{U}_{s,\varphi}$ and the point $(p^0,q^0) = \big(p^0(s,\varphi,E_0), q^0(s,\varphi,E_0)\big)$ given by

$$L(s,\varphi,E_0) = \{(s,\varphi)\} \times \big\{\big(p^0(s,\varphi,E_0), q^0(s,\varphi,E_0)\big)\big\}.$$

□

We remark that in terms of A and ϕ, condition (4.1) is rewritten as follows

$$A\,\mathrm{d}_2\phi = p\,\mathrm{d}q - \mathrm{d}_2 K, \tag{4.2}$$

or, equivalently,

$$\frac{\partial K}{\partial p} = -A\,\frac{\partial \phi}{\partial p}, \quad \frac{\partial K}{\partial q} = -A\,\frac{\partial \phi}{\partial q} + p.$$

Lemma 4.2. *The pull-back of the symplectic form $\widetilde{\Omega}^\varepsilon$ under the transformation $\mathcal{T}$ is*

$$\mathcal{T}^*\widetilde{\Omega}^\varepsilon = \Omega^\varepsilon + \varepsilon\,\mathrm{d}P, \tag{4.3}$$

where $P = P_1\,\mathrm{d}s + P_2\,\mathrm{d}\varphi$ is a horizontal 1-form on $\mathcal{M}$ with coefficients

$$P_1 = A\,\frac{\partial \phi}{\partial s} + \frac{\partial K}{\partial s}, \tag{4.4}$$

$$P_2 = A\,\frac{\partial \phi}{\partial \varphi} + \frac{\partial K}{\partial \varphi}. \tag{4.5}$$

Proof. By (4.4), (4.5) we have

$$P = A\,\mathrm{d}_1\phi + \mathrm{d}_1 K. \tag{4.6}$$

From (4.2) and (4.6) we derive that $P = A\,\mathrm{d}\phi - p\,\mathrm{d}q + \mathrm{d}K$, which implies (4.3). □

As a consequence of (4.3), we get formula (2.17), where $Q = Q_1\,\mathrm{d}s_1 + Q_2\,\mathrm{d}\varphi_1$ is a horizontal 1-form on $\mathcal{N}$ with coefficients

$$Q_1 = \Big(A\,\frac{\partial \phi}{\partial s} + \frac{\partial K}{\partial s}\Big) \circ \mathcal{T}^{-1}, \tag{4.7}$$

$$Q_2 = \Big(A\,\frac{\partial \phi}{\partial \varphi} + \frac{\partial K}{\partial \varphi}\Big) \circ \mathcal{T}^{-1}. \tag{4.8}$$

This proves Proposition 2.2.

Using (4.7), (4.8), one can show that Q_1 and Q_2 are related by the following "zero curvature" equation [7, 13]:

$$\frac{\partial Q_2}{\partial s_1} - \frac{\partial Q_1}{\partial \varphi_1} + \{Q_1, Q_2\} = 0. \tag{4.9}$$

Let us denote $\widetilde{\Omega}^\varepsilon_Q = \widetilde{\Omega}^\varepsilon - \varepsilon\,\mathrm{d}Q$. It is clear that $\widetilde{\Omega}^\varepsilon_Q$ is nondegenerate on $\mathcal{N}$ for all $\varepsilon \neq 0$. Observe also that $\widetilde{\Omega}^\varepsilon_Q$ admits the following representation:

$$\widetilde{\Omega}^\varepsilon_Q = \mathrm{d}s_1 \wedge \mathrm{d}\varphi_1 + \frac{1}{\varepsilon}\,\Gamma^1 \wedge \Gamma^2, \tag{4.10}$$

where

$$\Gamma^1 = \varepsilon\big[\mathrm{d}s_2 + \frac{\partial Q_1}{\partial \varphi_2}\,\mathrm{d}s_1 + \frac{\partial Q_2}{\partial \varphi_2}\,\mathrm{d}\varphi_1\big], \tag{4.11}$$

$$\Gamma^2 = \varepsilon\big[\mathrm{d}\varphi_2 - \frac{\partial Q_1}{\partial s_2}\,\mathrm{d}s_1 - \frac{\partial Q_2}{\partial s_2}\,\mathrm{d}\varphi_1\big]. \tag{4.12}$$

From (4.9), (4.10) we derive that the Poisson bracket on $\mathcal{N}$ corresponding to $\widetilde{\Omega}^{\varepsilon}_{Q}$ is given by the relations:

$$\{s_1, \varphi_1\}_{\mathcal{N}} = 1, \tag{4.13}$$

$$\{s_1, s_2\}_{\mathcal{N}} = -\frac{\partial Q_2}{\partial \varphi_2}, \quad \{s_1, \varphi_2\}_{\mathcal{N}} = \frac{\partial Q_2}{\partial s_2}, \tag{4.14}$$

$$\{\varphi_1, s_2\}_{\mathcal{N}} = \frac{\partial Q_1}{\partial \varphi_2}, \qquad \{\varphi_1, \varphi_2\}_{\mathcal{N}} = -\frac{\partial Q_1}{\partial s_2}, \tag{4.15}$$

$$\{s_2, \varphi_2\}_{\mathcal{N}} = \frac{1}{\varepsilon} + \Big(-\frac{\partial Q_1}{\partial \varphi_2}\frac{\partial Q_2}{\partial s_2} + \frac{\partial Q_1}{\partial s_2}\frac{\partial Q_2}{\partial \varphi_2}\Big). \tag{4.16}$$

Now we proceed to the construction of a near identity symplectomorphism Ψ_ε and a proof of Theorem 2.3. Consider the original perturbed system (1.3)-(1.6). The push-forward of H_ε by $\mathcal{T}$ gives a Hamiltonian system on $(\mathcal{N}, \widetilde{\Omega}^{\varepsilon}_{Q})$ with Hamiltonian $H_\varepsilon \circ \mathcal{T}^{-1} = f + \varepsilon\, F \circ \mathcal{T}^{-1}$. By Proposition 2.1, the corresponding dynamical system on $\mathcal{N}$ is of the form

$$\dot{s_1} = O(\varepsilon), \tag{4.17}$$

$$\dot{\varphi}_1 = \omega_1(s_1) + O(\varepsilon), \tag{4.18}$$

$$\dot{s_2} = O(\varepsilon), \tag{4.19}$$

$$\dot{\varphi}_2 = \omega_2(s_1, s_2) + O(\varepsilon), \tag{4.20}$$

where ω_2 is given by (3.29).

The function K is uniquely determined by (4.1) up to adding an arbitrary function depending on s_1, φ_1. This means that we have certain freedom in choosing the 1-form Q. To fix Q, we use the following criterion.

Lemma 4.3. *One can choose Q in (4.7), (4.8) and a smooth function $h = h(s_1, s_2)$ on D such that*

$$\frac{\partial h(s_1, s_2)}{\partial s_2} = \omega_2(s_1, s_2), \tag{4.21}$$

and

$$F \circ \mathcal{T}^{-1} = -\omega_1\, Q_2 + h. \tag{4.22}$$

Proof. Suppose we are given some Q in (4.7), (4.8) and h satisfying (4.21). Computing the components of the Hamiltonian vector field of $H_\varepsilon \circ \mathcal{T}^{-1}$ relative to bracket (4.13)-(4.16) up to $O(\varepsilon)$ and comparing with the right hand side of (4.17)-(4.20), we obtain the following relationship between F, Q and h:

$$F \circ \mathcal{T}^{-1} = -\omega_1\, Q_2 + h + \mu \tag{4.23}$$

where $\mu = \mu(s_1, \varphi_1)$ is a smooth function, 2π-periodic in φ_1. Clearly, Q and h are uniquely determined up to the transformations

$$h \mapsto h + c_1, \quad Q \mapsto Q + \mathrm{d}_1 c_2,$$

for any smooth functions $c_1 = c_1(s_1)$ and $c_2 = c_2(s_1, \varphi_1) = c_2(s_1, \varphi_1 + 2\pi)$. To eliminate μ in (4.23), we take

$$c_1 = \frac{1}{2\pi}\int_0^{2\pi} \mu \, \mathrm{d}\varphi_1, \quad c_2 = \frac{1}{\omega_1}\Big(\int_0^{\varphi_1} \mu \, \mathrm{d}\varphi_1' - \varphi_1 c_1\Big).$$

□

We shall assume that h and Q satisfying (4.21) and (4.22) are given. Introduce the following curve of closed 2-forms $[0,1] \ni \lambda \mapsto \sigma_\lambda^\varepsilon$ joining $\widetilde{\Omega}_Q^\varepsilon$ and $\widetilde{\Omega}^\varepsilon$:

$$\begin{aligned}\sigma_\lambda^\varepsilon &= (1-\lambda)\,\widetilde{\Omega}_Q^\varepsilon + \lambda\,\widetilde{\Omega}^\varepsilon \\ &= \mathrm{d}s_1 \wedge \mathrm{d}\varphi_1 + \varepsilon\,\mathrm{d}s_2 \wedge \mathrm{d}\varphi_2 - \varepsilon\,(1-\lambda)\,\mathrm{d}Q.\end{aligned}$$

Using (4.9), by straightforward computations we show that $\sigma_\lambda^\varepsilon$ has the representation

$$\sigma_\lambda^\varepsilon = m\,\mathrm{d}s_1 \wedge \mathrm{d}\varphi_1 + \frac{1}{\varepsilon}\,\Gamma_\lambda^1 \wedge \Gamma_\lambda^2, \tag{4.24}$$

where

$$m = 1 - \lambda\,(1-\lambda)\varepsilon\Big(\frac{\partial Q_2}{\partial s_1} - \frac{\partial Q_1}{\partial \varphi_1}\Big),$$

and 1-forms Γ_λ^1, Γ_λ^2 are defined by formulas (4.11), (4.12) under replacing Q by $(1-\lambda)\,Q$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be an admissible domain $\mathcal{N}_0 = \mathcal{T}(\mathcal{M}_0) \subset \mathcal{N}$. Pick another open domain $\mathcal{W}$ in $\mathcal{N}$ such that $\mathcal{N}_0 \subset \mathcal{W}$ and $\overline{\mathcal{W}}$ is compact. Then, the functions Q_1 and Q_2 are bounded on $\overline{\mathcal{W}}$ and

$$\delta_0 = \frac{1}{4\max_{\overline{\mathcal{W}}} |\frac{\partial Q_2}{\partial s_1} - \frac{\partial Q_1}{\partial \varphi_1}|} > 0.$$

From here and (4.24), we derive the key property.

Lemma 4.4. *For any $\varepsilon \in (0, \delta_0)$ and $\lambda \in [0,1]$ the 2-form $\sigma_\lambda^\varepsilon$ is nondegenerate on $\mathcal{W}$.*

Therefore, for every $\varepsilon \in (0, \delta_0)$, we have the family $\{\sigma_\lambda^\varepsilon\}_{\lambda\in[0,1]}$ of symplectic structures on $\mathcal{W}$. According to a general scheme [9], an isomorphism between $\sigma_\lambda^\varepsilon$ and σ_0^ε is given by the flow $\mathsf{Fl}_{Z_\lambda}^\lambda$ of a time-dependent vector field Z_λ satisfying the homological equation

$$L_{Z_\lambda}\sigma_\lambda^\varepsilon - \frac{\mathrm{d}\sigma_\lambda^\varepsilon}{\mathrm{d}\lambda} = 0.$$

In this case, $(\mathsf{Fl}_{Z_\lambda}^\lambda)_*\sigma_\lambda^\varepsilon = \sigma_0^\varepsilon$. By standard arguments, finding Z_λ is reduced to solving the algebraic equation

$$\mathbf{i}_{Z_\lambda}\sigma_\lambda^\varepsilon = Q. \tag{4.25}$$

By (4.24), we derive that the dynamical system of the vector field Z_λ satisfying (4.25), is of the form

$$\frac{\mathrm{d}s_1}{\mathrm{d}\lambda} = -\frac{\varepsilon}{m}\,Q_2, \tag{4.26}$$

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}\lambda} = \frac{\varepsilon}{m}\,Q_1, \tag{4.27}$$

$$\frac{\mathrm{d}s_2}{\mathrm{d}\lambda} = \frac{\varepsilon(1-\lambda)}{m}\big[Q_2\frac{\partial Q_1}{\partial \varphi_2} - Q_1\frac{\partial Q_2}{\partial \varphi_2}\big], \tag{4.28}$$

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}\lambda} = \frac{\varepsilon(1-\lambda)}{m}\big[Q_1\frac{\partial Q_2}{\partial s_2} - Q_2\frac{\partial Q_1}{\partial s_2}\big], \tag{4.29}$$

Therefore, for every $\varepsilon \in [0, \delta_0)$ we have a time-dependent vector field Z_λ on $\mathcal{W}$ which vanishes at $\varepsilon = 0$. From this property and the compactness of $\mathcal{N}_0$ it follows that there is a $\delta_1 \in (0, \delta_0)$ such that the flow $\mathsf{Fl}^{\lambda}_{Z_\lambda}$ on $\mathcal{N}_0$ is well-defined for all $\varepsilon \in [0, \delta_1)$ and $\lambda \in [0, 1]$.

We arrive at the following result.

Lemma 4.5. *For every $\varepsilon \in [0, \delta_1)$, the time-1 flow $\Psi_\varepsilon = \mathsf{Fl}^1_{Z_\lambda} : \mathcal{N}_0 \to N$ of system* (4.26)-(4.29) *is a near identity symplectomorphism between $\widetilde{\Omega}^\varepsilon_Q$ and $\widetilde{\Omega}^\varepsilon$,*

$$\Psi_0 = \mathrm{id} \quad \textit{and} \quad (\Psi_\varepsilon)_* \widetilde{\Omega}^\varepsilon_Q = \widetilde{\Omega}^\varepsilon.$$

We remark that the inverse Ψ_ε^{-1} is defined as the time-1 flow of the following non-autonomous system

$$\begin{aligned}
\frac{\mathrm{d}s_1}{\mathrm{d}\lambda} &= \frac{\varepsilon}{m} Q_2, \\
\frac{\mathrm{d}\varphi_1}{\mathrm{d}\lambda} &= -\frac{\varepsilon}{m} Q_1, \\
\frac{\mathrm{d}s_2}{\mathrm{d}\lambda} &= \frac{\varepsilon\lambda}{m} \big[Q_1 \frac{\partial Q_2}{\partial \varphi_2} - Q_2 \frac{\partial Q_1}{\partial \varphi_2} \big], \\
\frac{\mathrm{d}\varphi_2}{\mathrm{d}\lambda} &= \frac{\varepsilon\lambda}{m} \big[Q_2 \frac{\partial Q_1}{\partial s_2} - Q_1 \frac{\partial Q_2}{\partial s_2} \big],
\end{aligned}$$

which corresponds to the vector field $\widetilde{Z}_\lambda = -Z_{1-\lambda}$. Finally, using (4.21) and (4.22), for $\widetilde{H}_\varepsilon = H_\varepsilon \circ \mathcal{T}^{-1}$, we compute

$$\begin{aligned}
\widetilde{H}_\varepsilon \circ \Psi_\varepsilon^{-1} &= \widetilde{H}_\varepsilon - L_{Z_\lambda} \widetilde{H}_\varepsilon + O(\varepsilon^2) \\
&= f + \varepsilon\,(-\omega_1 Q_2 + h) - \varepsilon\, L_{Z_\lambda} f + O(\varepsilon^2) \\
&= f + \varepsilon\, h + O(\varepsilon^2).
\end{aligned}$$

It follows that $\Upsilon_\varepsilon = \Psi_\varepsilon \circ T$ satisfies (2.20), (2.21). This completes the proof of Theorem 2.3.

Corollary 4.6. *If F in* (1.2) *is independent of s, $F = F(\varphi, p, q)$, then one can choose $Q_1 = 0$ and $Q_2 = Q_2(\varphi_1, s_2, \varphi_2)$. In this case, Ψ_ε is an infinitesimal transformation of the form*

$$\Psi_\varepsilon(s_1, \varphi_1, s_2, \varphi_2) = \big(s_1 - \varepsilon\, Q_2(\varphi_1, s_2, \varphi_2), \varphi_1, s_2, \varphi_2 \big).$$

Such type of transformations appear in the normalization of Hamiltonian systems near an individual periodic trajectory [2, 3, 5].

5. The Linear Case

As an illustration of above results, we consider the case when the function F in (1.2) is quadratic in coordinates p and q,

$$F = \frac{1}{2} \big(w_1 p^2 + 2\, w_2 pq + w_3 q^2 \big).$$

Here $w_i = w_i(s,\varphi)$ $(i = 1,2,3)$ are smooth functions, 2π-periodic in φ. Then, unperturbed system (2.1)-(2.4) takes the form

$$\dot{s} = 0, \tag{5.1}$$

$$\dot{\varphi} = \omega_1(s), \tag{5.2}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \mathbb{J}\, W(s,\varphi) \begin{pmatrix} p \\ q \end{pmatrix}, \tag{5.3}$$

where

$$W(s,\varphi) = \begin{bmatrix} w_1(s,\varphi) & w_2(s,\varphi) \\ w_2(s,\varphi) & w_3(s,\varphi) \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, (5.1)-(5.3) corresponds to a s-parameter family of linear time-periodic Hamiltonian systems on $\mathbb{R}^2$.

Recall that a linear periodic Hamiltonian system is said to be *stable* (in the sense of Lyapunov) if all solutions are bounded for $t \in (-\infty,\infty)$. Moreover, a stable linear T-periodic Hamiltonian system is called *strongly stable* (or *parametrically stable*), if all sufficiently small linear T-periodic Hamiltonian perturbations of this system are stable as well [11, 12, 14].

We assume that system (5.1)-(5.3) is *strongly stable for every* $s \in \Delta = (\Delta_1, \Delta_2)$. Let $\mathbb{F}(s,\varphi)$ be the fundamental solution of the corresponding linear problem,

$$\omega_1(s)\frac{\mathrm{d}\mathbb{F}}{\mathrm{d}\varphi} = \mathbb{J}\, W(s,\varphi)\, \mathbb{F},$$
$$\mathbb{F}(s,0) = I.$$

Then, $\mathbb{F}$ is a $\mathrm{Sp}(1,\mathbb{R})$-valued smooth function in s,φ with $\det \mathbb{F}(s,\varphi) = 1$. In terms of the monodromy matrix $\mathfrak{M}(s) = \mathbb{F}(s,2\pi)$, the strongly stability condition is formulated as follows [11, 14]: $-2 < \operatorname{tr}\mathfrak{M}(s) < 2$, for $s \in \Delta$. This means that the spectrum of the the monodromy matrix $\mathfrak{M}(s)$ is simple and belongs to the unit circle in the complex plane, $\operatorname{Spec}\mathfrak{M}(s) = \{\exp(\pm 2\pi\, i\, \beta(s))\}$, where $\beta(s) > 0$ is the Floquet exponent.

Let us associate to system (5.1)-(5.3) the following Riccati equation for a $\mathbb{C}$-valued function $(s,\varphi) \mapsto \mathfrak{D}(s,\varphi)$ (depending on s as a parameter):

$$\omega_1 \frac{\partial \mathfrak{D}}{\partial \varphi} + w_1 \mathfrak{D}^2 + 2w_2 \mathfrak{D} + w_3 = 0. \tag{5.4}$$

Proposition 5.1. [11] *If system* (5.1)-(5.3) *is strongly stable, then:*

(a) *There exists a unique smooth solution* $\mathfrak{D}(s,\varphi) = \mathfrak{D}_1(s,\varphi) + i\mathfrak{D}_2(s,\varphi)$ *of the Riccati equation* (5.4) *satisfying the following properties*

$$\mathfrak{D}_2(s,\varphi) > 0, \tag{5.5}$$

$$\mathfrak{D}_2(s,\varphi + 2\pi) = \mathfrak{D}_2(s,\varphi), \tag{5.6}$$

for all s, φ. *The Floquet exponent* $\beta = \beta(s)$ *is smoothly varying with* s *and has the representation*

$$\beta(s) = \frac{1}{2\pi\omega_1(s)} \int_0^{2\pi} w_1(s,\varphi)\, \mathfrak{D}_2(s,\varphi)\, \mathrm{d}\varphi.$$

(b) *System* (5.1)-(5.3) *admits an integral of motion* $G : (\Delta \times \mathbb{S}^1) \times \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$G(s,\varphi,p,q) = \frac{1}{2\,\mathfrak{D}_2}\big[(p - \mathfrak{D}_1\, q)^2 + (\mathfrak{D}_2\, q)^2\big]. \tag{5.7}$$

The dynamical system of the vertical Hamiltonian vector field V_G is easily integrated by reducing it to the harmonic oscillator under the change of variables

$$p \mapsto \frac{p - \mathfrak{D}_1 q}{\sqrt{\mathfrak{D}_2}}, \quad q \mapsto \sqrt{\mathfrak{D}_2}\, q$$

For every (s, φ) and $E > 0$, the level set $\gamma_{s,\varphi}(E)$ of $G_{s,\varphi}$ in $\mathbb{R}^2$ is an elliptic trajectory of V_G with period $T = 2\pi$. The action along $\gamma_{s,\varphi}(E)$ is $a(s, E) = E$. Let $\mathcal{U}_{s,\varphi}$ be the open domain defined as the union of $\gamma_{s,\varphi}(E)$, where E runs over $(0, \infty)$. Then, the hypothesis (IH) is satisfied for the domain $\mathcal{M} = \cup_{(s,\varphi)\in\Delta\times\mathbb{S}^1}\mathcal{U}_{s,\varphi}$ and G (5.7). Taking $L(s, \varphi, E) = \big(s, \varphi, \sqrt{2E\,\mathfrak{D}_2(s, \varphi)}, 0\big)$, and applying the algorithm for constructing A and ϕ in Section 3, we get that the inverse $\mathcal{T}^{-1}_{s_1,\varphi_1} : (s_2, \varphi_2) \mapsto (p, q)$ of the reducibility transformation $\mathcal{T}$ (3.30) is given by

$$p = \sqrt{2s_2\,\mathfrak{D}_2}\,\cos(\varphi_2 + \chi) + \mathfrak{D}_1\sqrt{\frac{2s_2}{\mathfrak{D}_2}}\,\sin(\varphi_2 + \chi), \quad q = \sqrt{\frac{2s_2}{\mathfrak{D}_2}}\,\sin(\varphi_2 + \chi).$$

Here $\mathfrak{D}_1 = \mathfrak{D}_1(s_1, \varphi_1), \mathfrak{D}_2 = \mathfrak{D}_2(s_1, \varphi_1)$ and

$$\chi(s_1, \varphi_1) = \frac{1}{\omega_1(s_1)}\Big[\omega_2(s_1)\varphi_1 - \int_0^{\varphi_1} w_1(s_1, \varphi_1')\,\mathfrak{D}_2(s_1, \varphi_1')\,\mathrm{d}\varphi_1'\Big],$$

$$\omega_2(s_1) = \omega_1(s_1)\,\beta(s_1) = \frac{1}{2\pi}\int_0^{2\pi} w_1(s_1, \varphi_1)\,\mathfrak{D}_2(s_1, \varphi_1)\,\mathrm{d}\varphi_1.$$

It follows that

$$\mathcal{T}(s, \varphi, p, q) = \Big(s, \varphi, G(s, \varphi, p, q), \operatorname{arctg}\big(\frac{\mathfrak{D}_2}{(p/q) - \mathfrak{D}_1}\big) + \chi\Big), \tag{5.8}$$

and $\mathcal{N} = \mathcal{T}(\mathcal{M}) = (\Delta \times S^1) \times (\mathbb{R}^1_+ \times S^1)$. It is easy to see that a function K in (4.1), can be chosen in the form $K = (p\,q/2) - \mathfrak{D}_1\, q^2$. Substituting the formulas for K and $\mathcal{T}$ into (4.7), (4.8) gives

$$Q_1 = \frac{s_2}{2\,\mathfrak{D}_2}\Big[\sin 2(\varphi_2 + \chi)\,\frac{\partial \mathfrak{D}_2}{\partial s_1} + (1 - \cos 2(\varphi_2 + \chi))\,\frac{\partial \mathfrak{D}_1}{\partial s_1} + \frac{\partial \chi}{\partial s_1}\Big], \tag{5.9}$$

$$Q_2 = \frac{s_2}{2\,\mathfrak{D}_2}\Big[\sin 2(\varphi_2 + \chi)\,\frac{\partial \mathfrak{D}_2}{\partial \varphi_1} + (1 - \cos 2(\varphi_2 + \chi))\,\frac{\partial \mathfrak{D}_1}{\partial \varphi_1} + \frac{\partial \chi}{\partial \varphi_1}\Big]. \tag{5.10}$$

Finally, we observe that if we take $h(s_1, s_2) = s_2\,\omega_2(s_1)$, then (4.22) holds. Fix some E_1 and E_2 such that $0 < E_1 < E_2 < \infty$, and consider the admissible domain $\mathcal{M}_0$ which is the union of the open elliptic rings $\mathcal{U}^0_{s,\varphi} = \cup_{E_1<E<E_2}\,\gamma_{s,\varphi}(E)$ in $\mathbb{R}^2$. Then, according to Theorem 2.3, for small enough ε, the symplectomorphism $\Upsilon_\varepsilon = \Psi_\varepsilon \circ \mathcal{T} : \mathcal{M}_0 \to \mathcal{N}$ transforms H_ε into the normal form $f(s_1) + \varepsilon\, s_2\,\omega_2(s_1) + O(\varepsilon^2)$. Here, $\mathcal{T}$ is the reducibility transformation in (5.8) and Ψ_ε is the time-1 flow of system (4.26)-(4.29) with Q_1 and Q_2 given by (5.9), (5.10).

Notice that the reducibility map (5.8) can be also derived from the Floquet theory. Let $\mathcal{F}_{s,\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ be the standard Floquet-Lyapunov transformation [14],

$$\mathcal{F}_{s,\varphi} = \exp\big(\frac{\varphi}{\omega_1(s)}\mathcal{K}(s)\big) \circ \mathbb{F}^{-1}(s, \varphi).$$

Here $\mathcal{K}(s) = \big[\omega_1(s)/2\pi\big]\ln \mathfrak{M}(s)$ and a real branch of the logarithm of $\mathfrak{M}(s)$ exists because of the stability assumption. Then, one can show that $\mathcal{T} = \mathcal{S} \circ \mathcal{F}$, where $\mathcal{S} : \mathcal{M} \to \mathcal{N}$ is a symplectic map uniquely determined by $\mathcal{K}(s)$.

We remark that for time-dependent harmonic oscillators, invariants like (5.7) were studied in [16].

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Guillermo Dávila-Rascón
Departamento de Matemáticas, Universidad de Sonora, Rosales y Blvd. Luis Encinas, Hermosillo, Sonora, 83000, Mexico
E-mail address: `davila@gauss.mat.uson.mx`

Yuri Vorobiev
Departamento de Matemáticas, Universidad de Sonora, Rosales y Blvd. Luis Encinas, Hermosillo, Sonora, 83000, Mexico
E-mail address: `yurimv@guaymas.uson.mx`